Lec 14

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Last time System of First-Order ODE's

$$\left.\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y + \dots + a_{1n}y_n + f_1(x) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_2 + f_2(x) \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n + f_n(x) \end{aligned}\right\} \iff \vec{y}' = A\vec{y} + \vec{f}(x)$$

where $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $\vec{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$

$A = [a_{ij}]$ and $\vec{f}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$

$\vec{y}' = A\vec{y} + \vec{f}(x)$ (NH)

The general solution to (NH) is $\vec{y} = \vec{y}_H + \vec{y}_P$ where

- $\vec{y}_H$ is the general solution to the corresponding Homogeneous system $\vec{y}' = A\vec{y}$ (H)

- $\vec{y}_P$ is one particular solution to (NH)

In this course, we only deal with systems $\vec{y}' = A\vec{y} + \vec{f}(x)$ where A is a 2×2 matrix

Solving homogeneous system

$\underline{\vec{y}' = A\vec{y}}$ (H)

$\underline{(A \text{ is a } 2\times2 \text{ matrix})}$

Step 1: Find the eigenvalues of A: $\det(A - \lambda I) = 0$

Step 2: The general solution to (H) depends on the nature of the eigenvalues found in step 1.

Case 1: 2 real distinct eigen values λ_1, λ_2

In this case, Find a basis $\vec{V}_1$ For E_{λ_1}, a basis $\vec{V}_2$ For E_{λ_2} the general solution for (H) is

$$\boxed{y = C_1 \vec{V}_1 e^{\lambda_1 x} + C_2 \vec{V}_2 e^{\lambda_2 x}}$$

Case 2; Repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$

In this case, Find a basis $\vec{V}$ of E_λ. The general Solution to (H) is:

$$\boxed{\vec{y} = C_1 \vec{V} e^{\lambda x} + C_2 \left(x\vec{V} + \vec{P}\right) e^{\lambda x}}$$

where $\vec{P}$ is a generalized eigenvector

This means, $\vec{P}$ is one solution to the system

$$\left[A - \lambda I \,\middle|\, \vec{V}\right]$$

Case 3: The complex conjugate eigenvalues

$$\lambda_1 = \alpha + i\beta \ , \ \lambda_2 = \alpha - i\beta$$

In this case, Find a basis $\vec{V}$ for E_{λ_1},

Write the vector

$\vec{V} e^{\lambda_1 x}$ as $\vec{y}_1 + i\,\vec{y}_2$ where $\vec{y}_1$ and $\vec{y}_2$ are real Vectors. Then the general solution to (H) is

$$\boxed{\vec{y} = C_1 \vec{y}_1 + C_2 \vec{y}_2}$$

Ex/ Solve the IVP

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = y_1 + y_2 \end{cases} ; \quad y_1(0) = -1 \ , \ y_2(0) = 2$$

Sol $\quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$; $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

The system is written as

$\vec{y}' = A\vec{y} \qquad (1)$

Eigenvalue of A:

$$\det(A-\lambda I) = 0 \Rightarrow \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)^2 - (-1) = 0$$

$$\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4-4(2)}}{2} = \frac{2 \pm \sqrt{i^2(4)}}{2} = 1 \pm i$$

two complex conjugate eigenvalues $\lambda_1 = 1+i$, $\lambda_2 = 1-i$

Next, Find a basis for E_{λ_1} : $[A - \lambda_1 I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 1-(1+i) & -1 & 0 \\ 1 & 1-(1+i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \underset{\sim}{R_1 \Leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -i & 0 \\ -i & -1 & 0 \end{array}\right]$$

$$\underset{\sim}{iR_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$ is free

$x_1 - ix_2 = 0 \Rightarrow x_1 - it = 0$

$$E_{\lambda_1} = \left\{ \begin{bmatrix} it \\ t \end{bmatrix} ; t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} t ; t \in \mathbb{R} \right\} = \text{span}\left\{ \underbrace{\begin{bmatrix} i \\ 1 \end{bmatrix}}_{\vec{v}} \right\}$$

Take $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is a basis for E_{λ_1}

$$\vec{v} e^{\lambda_1 x} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1+i)x} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^x \cdot \underbrace{e^{ix}}_{\cos x + i \sin x} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^x (\cos x + i \sin x)$$

$i \times i = -1$

$$= \begin{bmatrix} ie^x \cos x - e^x \sin x \\ e^x \cos x + ie^x \sin x \end{bmatrix} = \begin{bmatrix} -e^x \sin x \\ e^x \cos x \end{bmatrix} + i \begin{bmatrix} e^x \cos x \\ e^x \sin x \end{bmatrix},$$

$$= \begin{bmatrix} ie^{x}\cos x - e^{x}\sin x \\ e^{x}\cos x + ie^{x}\sin x \end{bmatrix} = \underbrace{\begin{bmatrix} -e^{x}\sin x \\ e^{x}\cos x \end{bmatrix}}_{\vec{y}_1} + i \underbrace{\begin{bmatrix} e^{x}\cos x \\ e^{x}\sin x \end{bmatrix}}_{\vec{y}_2}$$

The general solution to (H) is $\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2$

$$\vec{y} = c_1 \begin{bmatrix} -e^{x}\sin x \\ e^{x}\cos x \end{bmatrix} + c_2 \begin{bmatrix} e^{x}\cos x \\ e^{x}\sin x \end{bmatrix} = \begin{bmatrix} -c_1 e^{x}\sin x + c_2 e^{x}\cos x \\ c_1 e^{x}\cos x + c_2 e^{x}\sin x \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 2 \\ c_2 = -1 \end{matrix}$$

Ex 2 / $\vec{y}' = A\vec{y}$; $\vec{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ where

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Sol the eigenValues of A:

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 2-\lambda & 0 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - (-1)\times 0 = 0$$

$$= (2-\lambda)^2 = 0$$

$\lambda_1 = \lambda_2 = 2$: Repeated real eigenValue

Next, we find a basis for E_2: $\left[A - 2I \,\middle|\, \vec{0}\right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$ $-R_2 \to R_2$, $R_2 \Leftrightarrow R_1$

$$= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_2 = t \quad \text{is free} \\ x_1 = 0 \end{array}$$

$$E_2 = \left\{ \begin{bmatrix} 0 \\ t \end{bmatrix}, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} t \; ; t \in \mathbb{R} \right\} = \text{Span}\underbrace{\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}_{\vec{V}}$$

$\vec{V} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, a basis for E_2

The general solution is $\vec{y} = C_1 \vec{V} e^{2x} + C_2 (x\vec{V} + \vec{p}) e^{2x}$

$\vec{p}$ is a solution to the system: $[A - 2I \mid \vec{V}]$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 0 & 1 \end{array}\right] \underset{R_2 \leftrightarrow R_1}{\overset{R_2(-1) \to R_2}{\sim}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$X_2 = t$ is free

$X_1 = -1$

The general solution is $\begin{bmatrix} -1 \\ t \end{bmatrix}$, $t \in \mathbb{R}$

Note: take any value for t to get $\vec{p}$

$t = 0$, $\vec{p} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

The general solution is:

$$\vec{y} = C_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x} + C_2 \left(x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{2x}$$

$$\vec{y} = \begin{bmatrix} -C_2 e^{2x} \\ C_1 e^{2x} + C_2 x e^{2x} \end{bmatrix} \begin{array}{l} \text{—} y_1 \\ \text{—} y_2 \end{array}$$

$$\vec{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -C_2 \\ C_1 \end{bmatrix} \Rightarrow \begin{array}{l} C_1 = 1 \\ C_2 = -2 \end{array}$$

$$\vec{y} = \begin{bmatrix} 2e^{2x} \\ e^{2x} - 2xe^{2x} \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 2e^{2x} \\ e^{2x} - 2xe^{2x} \end{bmatrix}$$

Ex 3/ Find the general solution to the system

$$\begin{cases} y_1' = y_1 + y_2 \\ y_2' = 4y_1 + y_2 \end{cases} \Rightarrow \vec{y}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{y} \quad (H)$$

$$\det(A - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda+1)(\lambda-3) = 0$$

$\lambda_1 = -1$, $\lambda_2 = 3$: 2 distinct real eigen values

$\underline{E_{-1}}$ $\left[A - (-1)I \,\middle|\, \vec{0}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \; -2R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \underset{\sim}{\tfrac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_2 = t \\ x_1 = -\tfrac{1}{2}t \end{array}$$

$$E_{-1} = \left\{ \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix}, t \in \mathbb{R} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t \right\} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}, \vec{V_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\underline{E_3}$ $\left[A - 3I \,\middle|\, \vec{0}\right] = \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \underset{\sim}{2R_1 + R_2} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x_2 = t$ is free, $x_1 = \frac{1}{2}t$

$$E_3 = \left\{ \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix}, t \in \mathbb{R} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$$

$$= \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, \text{ take } \vec{V_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The general solution is:

$$\vec{y} = C_1 \vec{V_1} e^{-x} + C_2 \vec{V_2} e^{3x}$$

$$\vec{y} = C_1 V_1 e^{-x} + C_2 V_2 e^{3x}$$

$$= C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-x} + C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3x} = \begin{bmatrix} C_1 e^{-x} + C_2 e^{3x} \\ -2C_1 e^{-x} + 2C_2 e^{3x} \end{bmatrix}$$