Lec 13

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We start with some review of certain topics:

Definition: Given a $n \times n$ matrix A, we say that the number λ (real or complex) is an eigenvalue of A if there exists a non 0 vector $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v}$

$\boxed{A\vec{v} = \lambda \vec{v}}$ In this case the vector $\vec{v}$ is called an eigenvector corresponding to λ.

Note that $A\vec{v} = \lambda\vec{v} \Rightarrow A\vec{v} - \lambda\vec{v} = \vec{0}$

$\Rightarrow \underbrace{(A - \lambda I_n)}_{n \times n} \vec{v} = \vec{0} \Rightarrow A - \lambda I_n$ is not

invertible $\Rightarrow \boxed{\det(A - \lambda I_n) = 0}$ This is called the Char. Eq. of A

If λ is an eigenvalue of A. The set of all eigenvectors (plus $\vec{0}$) of λ is the set of solutions to the homogeneous system $[A - \lambda I \mid 0]$

The set of all eigenvectors corresponding to the eigenvalue λ (plus $\vec{0}$) is called the eigenspace

Corresponding to λ, denoted by E_λ

$$E_\lambda = \left\{ \vec{V} \in \mathbb{R}^n \,\middle|\, (A - \lambda I_n)\vec{V} = \vec{0} \right\}$$

Ex 1/ Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

Find the eigenvalues of A. For each eigenvalue, find a basis for the corresponding eigenspace!

Sol/ The char. Eq. of A is $\det(A - \lambda I_2) = 0$

$$A - \lambda I_2 = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = (1-\lambda)^2 - 4 \Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow (\lambda+1)(\lambda-3) = 0 \Rightarrow \underbrace{\lambda = -1 \ , \ \lambda = 3}_{\text{eigenvalues of } A}$$

<u>for $\lambda = -1$</u> $\quad [A - \lambda I \mid \vec{0}] = [A + I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 1+1 & 1 & 0 \\ 4 & 1+1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \underset{\sim\sim\sim}{\text{Row reduction}} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

x_2 is Free

$x_2 = t$

$x_1 + \frac{1}{2}t = 0$

$x_1 = -\frac{1}{2}t$

$$\text{So } E_{-1} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix} ; t \in \mathbb{R} \right\}$$

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t ; t \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$$

$\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ forms a basis of E_{-1}

For $\lambda = 3$

$$\left[A - 3I_2 \mid \vec{0}\right]$$

$$\left[\begin{array}{cc|c} 1-3 & 1 & 0 \\ 4 & 1-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right]$$ Row reduction

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 - \frac{1}{2}t = 0$

$x_1 = \frac{1}{2}t$

$$\text{So } E_3 = \left\{ \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix}, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} t, t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ form a basis of E_3

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

$$\underbrace{\begin{bmatrix} \cdots & \cdots \\ 4 & 1-\lambda \end{bmatrix}}_{A-\lambda I} = 0$$

$$(1-\lambda)^2 - (4)(-1) = \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-(4)(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{i^2 16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

For $\lambda_1 = 1+2i$

$$[A - \lambda_1 I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 1-(1+2i) & -1 & 0 \\ 4 & 1-(1+2i) & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -2i & -1 & 0 \\ 4 & -2i & 0 \end{array}\right] \begin{array}{l} \frac{1}{4}R_2 \to R_2 \\ R_1 \leftrightarrow R_2 \end{array}$$

$$= \left[\begin{array}{cc|c} 1 & -\frac{1}{2}i & 0 \end{array}\right. \quad 2i R_1 + R_2 \to R_2$$

$$= \left[\begin{array}{cc|c} 1 & -\frac{1}{2}i & 0 \\ -2i & -1 & 0 \end{array}\right] \quad 2iR_1 + R_2 \to R_2 \quad \sim$$

$$-\frac{1}{\cancel{2}}i\,(\cancel{2}i) = -i^2 = -(-1) = 1$$

$$= \left[\begin{array}{cc|c} 1 & -\frac{1}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 is free variable

$x_2 = t$

$x_1 - \frac{1}{2}it = 0$

$x_1 = \frac{1}{2}it$

$$E_{\lambda_1} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}it \\ t \end{bmatrix} ;\ t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \underbrace{\begin{bmatrix} \frac{1}{2}i \\ 1 \end{bmatrix}}_{\text{a basis for } E_{\lambda_1}} \right\}$$

For $\lambda_2 = 1 - 2i$

$$\left[A - \lambda_2 I_2 \,\middle|\, \vec{0} \right]$$

$$= \left[\begin{array}{cc|c} 1-(1-2i) & -1 & 0 \\ 4 & 1-(1-2i) & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 2i & -1 & 0 \\ 4 & 2i & 0 \end{array}\right] \begin{array}{l} \underbrace{R_2/4 \to R_2} \\ R_2 \leftrightarrow R_1 \end{array}$$

$$= \left[\begin{array}{cc|c} 1 & \frac{1}{2}i & 0 \\ 2i & -1 & 0 \end{array}\right] \quad \underbrace{-2i}R_1 + R_2 \to R_2$$

$$-\cancel{2}i\left(\frac{1}{\cancel{2}}i\right) = -i^2 = -(-1) = 1$$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 + \frac{1}{2}it = 0$

$x_1 = -\frac{1}{2}it$

$$E_{x_2} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} i \\ 1 \end{bmatrix} t \; ; \; t \in \mathbb{R} \right\}$$

<u>Remark</u>: Any complex vector $\vec{V}$ can be written as $\vec{V} = \vec{V_1} + i \vec{V_2}$ where $\vec{V_1}, \vec{V_2}$ are real vectors!

Ex/

$$\vec{V} = \begin{bmatrix} -3 + 2i \\ 5 + 5i \end{bmatrix} = \underbrace{\begin{bmatrix} -3 \\ 5 \end{bmatrix}}_{\vec{V_1}} + i \underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}_{\vec{V_2}}$$

$$\vec{V_1} = \underset{\text{real part}}{\mathrm{Re}}(\vec{V}) , \quad \vec{V_2} = \underset{\text{Imaginary Part}}{\mathrm{Im}}(\vec{V})$$

<u>Definition</u>: If $\alpha, \beta \in \mathbb{R}$:

$$e^{\alpha + i\beta} = e^{\alpha} \cos\beta + i e^{\alpha} \sin\beta$$

In this chapter we look at systems of first order ODE's of the form:

$$
\begin{cases}
y_1' = a_{11} y_1 + a_{12} y_2 + \dots + a_{1n} y_n + f_1(x) \\
y_2' = a_{21} y_1 + a_{22} y_2 + \dots + a_{2n} y_n + f_2(x) \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
y_n' = a_{n1} y_1 + a_{n2} y_2 + \dots + a_{nn} y_n + f_n(x)
\end{cases}
$$

Where a_{ij} is a constant and

$f_1(x), \dots, f_n(x)$ are given functions.

Ex/

$$
\begin{cases}
y_1' = 2y_1 - 3y_2 + y_3 + e^x \\
y_2' = y_3 + \sin x \\
y_3' = -2y_1 + 2y_2 - 4y_3 + \ln x
\end{cases}
$$

Solving system (S) means finding a function

$y_1, y_2, \dots, y_n$ satisfying all the equations in (S).

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, then $\vec{y}' \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}; \quad \vec{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

Then the system (S) can be written as.

$$\vec{y}\,' = A\vec{y} + \vec{f}(x)$$

- If $\vec{f(x)} = \vec{0}$, the system is called <u>homogeneous</u>
- If $\vec{f}(x) \neq 0$, " " <u>Non-homogeneous</u>

As we saw in the previous chapter, the general solution to the system (NH) is of the form:

$$\vec{y} = \vec{y}_H + \vec{y}_P \quad \text{where}$$

- $\vec{y}_H$ is a solution to the corresponding Homogeneous system: $\vec{y}\,' = A\vec{y}$ (H)

- $\vec{y}_P$ is a particular solution to (NH)

Homogeneous systems

In this course, we only deal with systems where A is a 2×2 matrix.

$$y' = A\vec{y} \qquad (H)$$

Try $\vec{y} = \vec{v}e^{\lambda x}$ where $\vec{v}$ is a constant vector

$$\vec{y}\,' = \lambda \vec{v} e^{\lambda x} \qquad \text{back to } (H)$$

$\vec{y}' = \lambda \vec{V} e^{\lambda x}$ back to (H)

$\lambda \vec{V} \cancel{e^{\lambda x}} = A \vec{V} \cancel{e^{\lambda x}} \Rightarrow A\vec{V} = \lambda \vec{V}$

λ is an eigenvalue of A and $\vec{V}$ is an eigen Vector