Lec 12 Ch.4 Var. of Param.

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Last time: Variation of parameters

$y_p = u_1 y_1 + u_2 y_2 + \dots + u_n y_n$ with:

$u_1' y_1 + u_2' y_2 + \dots + u_n' y_n = 0$

$u_1' y_1' + u_2' y_2' + \dots + u_n' y_n' = 0$

$u_1' y_1'' + u_2' y_2'' + \dots + u_n' y_n'' = 0$

$\vdots$

$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \dots + u_n' y_n^{(n-1)} = r(x)$

Is found after dividing the entire system by the coefficient of the highest order. the resulting right side is $r(x)$.

~~Ex 2~~ Solve the following IVP:

$$x^3 y''' + x^2 y'' - 2x y' + 2y = x^3 \ln x, \quad x > 0 \qquad y(1) = \frac{-7}{32}, \quad y'(1) = \frac{15}{32}, \quad y''(1) = \frac{21}{16}$$

Sol/ The corresponding homogeneous equation is

$$x^3 y''' + x^2 y'' - 2x y' + 2y = 0 \qquad (H)$$

Euler-couchy of order 3.

we look for solution of the type $y = x^m$

$y' = m x^{m-1}, \quad y'' = m(m-1) y^{m-2}, \quad y''' = m(m-1)(m-2) y^{m-3}$

Back in (H):

$$x^3 m(m-1) x^{m-3} + x^2 m(m-1) x^{m-2} - 2x m x^{m-1} + 2x^m = 0$$

$$m(m-1)(m-2) + m(m-1) - 2(m-1) = 0$$

$$(m-1)(m^2 - m - 2) = 0 \Rightarrow (m-1)(m+1)(m-2) = 0$$

$m_1 = -1 \qquad m_2 = 1 \qquad m_3 = 2$ three distinct real roots

$y_1 = x^{-1}, \quad y_2 = x, \quad y_3 = x^2$ is a basis of sol of (H)

The general solution for (H) is

$y = C_1 x^{-1} + C_2 x + C_3 x^2$

For the particular sol y_p, we use the Variation of parameters.

$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$ such that

$$\begin{cases} u_1' y_1 + u_2' y_2 + u_3' y_3 = 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = r(x) \end{cases} \Rightarrow \begin{cases} u_1' x^{-1} + u_2' x + u_3' x^2 = 0 \\ -u_1' x^{-2} + u_2' + 2u_3' x = 0 \\ 2u_1 x^{-3} + 0 + 2u_3' = \ln x \end{cases}$$

$\Leftarrow$ after dividing the ODE with x^3

$$\begin{cases} u_1' x^{-1} + u_2' x + u_3' x^2 = 0 & (1) \\ -u_1' x^{-2} + u_2' + 2u_3' x = 0 & (2) \\ 2u_1 x^{-3} + 0 + 2u_3' = \ln x & (3) \end{cases}$$

$-x\,(2) + (1) \Rightarrow 2u_1' x^{-1} - x^2 u_3' = 0$

$\Rightarrow u_3' = 2u_1' x^{-3}$

$(3) \Rightarrow 2u_1' x^{-3} + 4u_1' x^{-3} = \ln x \Rightarrow 6u_1' x^{-3} = \ln x$

$\Rightarrow u_1' = \frac{1}{6} x^3 \ln x$

So $u_3' = 2u_1' x^{-3} = \frac{1}{3} \ln x$

$(2) \Rightarrow u_2' = u_1' x^{-2} - 2x u_3' = \left(\frac{1}{6} x^3 \ln x\right) x^{-2} - 2x \frac{1}{3} \ln x$

$\Rightarrow u_2' = \frac{1}{6} x \ln x - \frac{2}{3} x \ln x \Rightarrow u_2' = -\frac{1}{2} x \ln x$

$u_1' = \frac{1}{6} x^3 \ln x \Rightarrow u_1 = \frac{1}{6} \int x^3 \ln x \, dx$

By parts: $u = \ln x \qquad v' = x^3$

$u' = \frac{1}{x} \qquad v = \frac{1}{4} x^4$

$\int x^3 \ln x \, dx = \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 \, dx$

$$\boxed{u_1 = \frac{1}{24}x^4\ln x - \frac{1}{96}x^4}$$

$$u_2' = -\frac{1}{2}x\ln x \Rightarrow u_2 = -\frac{1}{2}\int x\ln x\,dx$$

By parts: $u = \ln x \quad v' = x$

$u' = \frac{1}{x} \quad v = \frac{x^2}{2}$

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\underbrace{\int x\,dx}_{\frac{x^2}{2}}$$

$$\frac{x^2}{2}\ln x - \frac{1}{4}x^2$$

$$u_2 = -\frac{1}{2}\int x\ln x\,dx = -\frac{1}{2}\left(\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right)$$

$$= -\frac{x^2}{4}\ln x + \frac{1}{8}x^2$$

$$u_3' = \frac{1}{3}\ln x \Rightarrow u_3 = \frac{1}{3}\int \ln x\,dx$$

$$= \frac{1}{3}\left[x\ln x - x\right] = \frac{x}{3}\ln x - \frac{x}{3}$$

$$\boxed{u_3 = \frac{x}{3}\ln x - \frac{x}{3}}$$

So $y_p = u_1y_1 + u_2y_2 + u_3y_3$

$$y_p = \left(\frac{1}{24}x^4\ln x - \frac{1}{96}x^4\right)x^{-1} + \left(-\frac{1}{4}x^2\ln x + \frac{1}{8}x^2\right)x + \left(\frac{1}{3}x\ln x - \frac{1}{3}x\right)x^2$$

$$= \frac{1}{24}x^3\ln x - \frac{1}{96}x^3 - \frac{1}{4}x^3\ln x + \frac{1}{8}x^3 + \frac{1}{3}x^3\ln x - \frac{1}{3}x^3$$

$$\boxed{y_p = \frac{1}{8}x^3\ln x - \frac{7}{32}x^3}$$

So the general solution for (NH) is $y = y_H + y_p$

$$\boxed{y = C_1 x^{-1} + C_2 x + C_3 x^2 + \frac{1}{8} x^3 \ln x - \frac{7}{32} x^3}$$

$$y' = -C_1 x^{-2} + C_2 + 2C_3 x + \frac{3}{8} x^2 \ln x + \frac{1}{8} \cdot x^2 - \frac{21}{32} x^2$$

$$y'' = 2C_1 x^{-3} + 2C_3 + \frac{3}{4} x \ln x + \frac{3}{8} x + \frac{1}{4} x - \frac{21}{16} x$$

$$y(1) = -\frac{7}{32} \Rightarrow C_1 + C_2 + C_3 - \cancel{\frac{7}{32}} = \cancel{-\frac{7}{32}}$$

$$\Rightarrow C_1 + C_2 + C_3 = 0$$

$$y'(1) = \frac{15}{32} \Rightarrow -C_1 + C_2 + 2C_3 + \frac{1}{8} - \frac{21}{32} = \frac{15}{32}$$

$$\Rightarrow -C_1 + C_2 + 2C_3 = \frac{17}{32} + \frac{15}{32} = 1$$

$$\Rightarrow -C_1 + C_2 + 2C_3 = 1$$

$$y''(1) = \frac{21}{16} \Rightarrow 2C_1 + 2C_3 + \frac{3}{8} + \frac{1}{4} - \frac{21}{16} = \frac{21}{16}$$

$$\Rightarrow 2C_1 + 2C_3 = 2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & -1 & 0 & 1 \end{array}\right]$$

$$\vdots$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$C_1 = 0$

$C_2 = -1$

$C_3 = 1$

The solution to the IVP is

$$\boxed{y = -x + x^2 + \frac{1}{8}x^3 \ln x - \frac{7}{32}x^3}$$

Ex3 Solve the IVP:

$$y'' - 6y' + 9y = \frac{e^{3x}}{x^3} \quad (N.H), \quad y(1) = \frac{e^3}{2}, \quad y'(1) = -3e^3$$

Sol The corresponding Homogeneous Eq. is

$$y'' - 6y' + 9y = 0 \quad (H)$$

Char. Eq. is

$$\lambda^2 - 6\lambda + 9 = 0$$

$(\lambda-3)^2 = 0 \Rightarrow \lambda = 3$ is a double real root:

$y_1 = e^{3x}$, $y_2 = xe^{3x}$ is a basis of sol (H)

So $y_H = C_1 e^{3x} + C_2 x e^{3x}$

For the particular solution of (NH), we use the variation of parameters.

$y_p = u_1 y_1 + u_2 y_2$ Such that.

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = r(x) \end{cases} \Rightarrow \begin{cases} u_1' e^{3x} + u_2' x e^{3x} = 0 \\ 3u_1' e^{3x} + u_2'(e^{3x} + 3xe^{3x}) = \frac{e^{3x}}{x^3} \end{cases}$$

$$\Rightarrow \begin{cases} u_1' + x u_2' = 0 & (1) \\ 3u_1' + u_2'(1 + 3x) = \frac{1}{x^3} & (2) \end{cases}$$

$$-3(1) + (2) \Rightarrow u_2' = \frac{1}{x^3}$$

$$(1) \Rightarrow u_1' = -x u_2' = -\frac{1}{x^2}$$

$u_1' = \frac{-1}{x^2} \Rightarrow u_1 = -\int \frac{1}{x^2}dx = \frac{1}{x}$

$\boxed{u_1 = \frac{1}{x}}$

$u_2' = \frac{1}{x^3} \Rightarrow u_2 = \int \frac{1}{x^3}dx$

$\boxed{u_2 = \frac{-1}{2x^2}}$

So $y_p = u_1 y_1 + u_2 y_2 = \frac{1}{x}e^{3x} - \frac{1}{2x^2}(xe^{3x}) = \frac{1}{x}e^{3x} - \frac{1}{2x}e^{3x} = \frac{e^{3x}}{2x}$

So the general sol of (NH) is $y = y_H + y_p$

$y = C_1 e^{3x} + C_2 x e^{3x} + \frac{e^{3x}}{2x}$

$y' = 3C_1 e^{3x} + C_2 e^{3x} + 3C_2 x e^{3x} + \frac{6xe^{3x} - 2e^{3x}}{4x^2}$

$y(1) = \frac{e^3}{2} \Rightarrow C_1 e^3 + C_2 e^3 + \frac{e^3}{2} = \frac{e^3}{2}$

$\Rightarrow \boxed{C_1 + C_2 = 0}$ ①

$y'(1) = -3e^3 \Rightarrow 3C_1 e^3 + C_2 e^3 + 3C_2 e^3 + e^3 = -3e^3$

$\Rightarrow \boxed{3C_1 + 4C_2 = -4}$ ②

$C_2 = -C_1$

$3C_1 + 4(-C_1) = -4$

$-C_1 = -4$

$C_1 = 4$

$C_2 = -4$

The solution to the IVP is

$y = 4e^{3x} - 4xe^{3x} + \frac{e^{3x}}{2x}$