Last time - The undetermined coefficients Method to solve non homogeneous ODE'S.

EX/ Solve the IVP:

$$y''' - 2y'' + y' - 2y = 10\cos x + 20e^{3x} + 6x^2 + 1 \quad (NH)$$

$$y(0) = 1;\ y'(0) = -1;\ y''(0) = 2$$

Sol The corresponding homogeneous Eq is

$$y''' - 2y'' + y' - 2y = 0 \quad (H)$$

It's ch.eq. is $\lambda^3 - 2\lambda^2 + \lambda - 2 = 0$

$\Rightarrow \lambda^2(\lambda - 2) + (\lambda - 2) = 0$

$(\lambda - 2)(\lambda^2 + 1) = 0$

- $\lambda = 2$ simple real root $\longrightarrow y_1 = e^{2x}$

- $\lambda = \pm i$ 2 complex conjugate roots $\longrightarrow y_2 = \cos x$, $y_3 = \sin x$

A basis of solutions to (H) is $y_1 = e^{2x}$, $y_2 = \cos x$, $y_3 = \sin x$

The general solution to (H) is $\boxed{y_H = C_1 e^{2x} + C_2\cos x + C_3 \sin x}$

Next we find a particular solution y_P to (NH) we use the undetermined coefficient method:

note: Use "the undetermined coefficient method" Sheet in your emails.

These are already bases of the solution to (H)

- For $10\cos x \longrightarrow y_P = A\cos x + B\sin x \xrightarrow{\text{modify}}$

$= x(A\cos x + B\sin x)$

$= Ax\cos x + Bx\sin x$ ✓

- For $20e^{3x} \longrightarrow y_P = Ce^{3x}$ ✓

- For $6x^2+1 \longrightarrow y_P = Dx^2 + Ex + F$ ✓

By sum rule:

$$y_P = Ax\cos x + Bx\sin x + Ce^{3x} + Dx^2 + Ex + F$$

$y_p' = A\cos x - Ax\sin x + B\sin x + Bx\cos x + 3Ce^{3x} + 2Dx + E$

$y_p'' = -A\sin x - A\sin x - Ax\cos x + B\cos x + B\cos x - Bx\sin x + 9Ce^{3x} + 2D$

$= -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x + 9Ce^{3x} + 2D$

$y_p''' = -2A\cos x - A\cos x + Ax\sin x - 2B\sin x - B\sin x - Bx\cos x + 27Ce^{3x}$

$= -3A\cos x + Ax\sin x - 3B\sin x - Bx\cos x + 27Ce^{3x}$

Back into

$$-3A\cos x + Ax\sin x - 3B\sin x - Bx\cos x + 27Ce^{3x} - 2(-2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x + 9Ce^{3x} + 2D)$$
$$+ A\cos x - Ax\sin x + B\sin x + Bx\cos x + 3Ce^{3x} + 2Dx + E$$
$$- 2(Ax\cos x + Bx\sin x + Ce^{3x} + Dx^2 + Ex + F)$$
$$= 10\cos x + 20e^{3x} + 6x^2 + 1$$

Expand and simplify gives:

$$\underline{(-2A-4B)}\cos x + \underline{(4A-2B)}\sin x + \underline{10C}e^{3x} \underline{-2D}x^2 + \underline{(2D-2E)}x \underline{-4D-2F+E}$$
$$= \underline{10}\cos x + \underline{20}e^{3x} + \underline{6}x^2 + \underline{1} \underline{+0\sin x} + 0x$$

$-2A - 4B = 10$ (1)

$4A - 2B = 0$ (2)

$10C = 20$ (3)

$-2D = 6$ (4)

$2D - 2E = 0$ (5)

$-4D - 2F + E = 1$ (6)

2(1) + (2) $\Rightarrow -10B = 20 \Rightarrow \boxed{B = -2}$

(2) $\Rightarrow 4A = -4 \Rightarrow \boxed{A = -1}$

(3) $\Rightarrow \boxed{C = 2}$

(4) $\Rightarrow \boxed{D = -3}$

(5) $\Rightarrow \boxed{E = D = -3}$

(6) $\Rightarrow 2F = -4D + E - 1 \Rightarrow 12 - 3 - 1 = \boxed{4 = F}$

⑥ $\Rightarrow 2F = -4D + E - 1 \Rightarrow 12 - 3 - 1 = \boxed{4 = F}$

So $\boxed{y_p = -x\cos x - 2x\sin x + 2e^{3x} - 3x^2 - 3x + 4}$

The general solution to (NH) is $y = y_H + y_p$

$$\boxed{y = C_1 e^{2x} + C_2\cos x + C_3 \sin x - x\cos x - 2x\sin x + 2e^{3x} - 3x^2 - 3x + 4}$$

② The Method of Variation of Parameters

The undetermind coefficients Method has its Limitations!

In this section, we look at a Second (more general) method to find y_p.

The variation of parameters method suggests a particular solution of the form:

$y_p = u_1 y_1 + u_2 y_2 + \dots + u_n y_n$ where:

1) $y_1, y_2, \dots, y_n$ are a basis of solutions for the corresponding homogeneous Equation (H)

2) $u_1, u_2, \dots, u_n$ are functions satisfying the following system of equations

$$\begin{cases} u_1' y_1 + u_2' y_2 + \dots + u_n' y_n = 0 \\ u_1' y_1' + u_2' y_2' + \dots + u_n' y_n' = 0 \\ u_1' y_1'' + u_2' y_2'' + \dots + u_n' y_n'' = 0 \\ \vdots \qquad \vdots \qquad \vdots \\ u_1' y^{(n-1)} + u_2' y^{(n-1)} + \dots + u_n' y^{(n-1)} = r(x) \end{cases}$$

We need the same amount of equations as the order of the ODE.

Remark: The $r(x)$ in the last equation of the above system is based on the fact that the coefficient of $y^{(n)}$ is (NH) is 1.

Ex 1 Solve the IVP:

$$x^3 y''' - 3x^2 y'' + 6x y' - 6y = 2x^3$$

$$y(1) = \frac{3}{2} \; ; \; y'(1) = -\frac{7}{2} \; ; \; y''(1) = 4$$

Sol The coresponding homogeneous eq. is:

$$x^3 y''' - 3x^2 y'' + 6x y' - 6y = 0 \quad (H)$$

↳ Euler cauchy of order 3

we look for solutions of type $y = x^m \Rightarrow y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$ and $y''' = m(m-1)(m-2) x^{m-3}$

Back in (H)

$$x^3 m(m-1)(m-2) x^{m-3} - 3x^2 m(m-1) x^{m-2} + 6x m x^{m-1} - 6x^m = 0$$

$$m(m-1)(m-2) - 3m(m-1) + 6(m-1) = 0 \Rightarrow$$

$$(m-1)(m^2 - 2m - 3m + 6) = 0 \Rightarrow (m-1)(m^2 - 5m + 6) = 0$$

$$(m-1)(m-2)(m-3) = 0$$

The roots are 1, 2 and 3: three distinct real roots:

$y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ is a basis of solutions for (H)

The general solution for (H) is

$$y_H = C_1 x + C_2 x^2 + C_3 x^3$$

For y_p, we use Variation of parameter!

For y_p, we use Variation of parameter!

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

with u_1, u_2, u_3 satisfy:

$$\begin{cases} u_1' y_1 + u_2' y_2 + u_3' y_3 = 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' = 0 \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = 2 \end{cases}$$

(divide the ODE by x^3 first to make the coefficient of the highest order 1)

$$\begin{cases} u_1' x + u_2' x^2 + u_3' x^3 = 0 \\ u_1' + 2u_2' x + 3u_3' x^2 = 0 \\ 0 + 2u_2' + 6u_3' x = 2 \end{cases}$$

$$u_1' + x u_2' + x^2 u_3' = 0 \quad (1)$$

$$u_1' + 2x u_2' + 3x^2 u_3' = 0 \quad (2)$$

$$2u_2' + 6x u_3' = 2 \quad (3)$$

$$-(1) + (2) = x u_2' + 2x^2 u_3' = 0 \Rightarrow u_2' + 2x u_3' = 0 \quad (4)$$

$$(3) - 2(4) = 2x u_3' = 2 \Rightarrow u_3' = \frac{1}{x}$$

Replace in equation (4)

$$u_2' + 2x\left(\frac{1}{x}\right) = 0 \Rightarrow u_2' = -2$$

$$(1) \Rightarrow u_1' = -x u_2' - x^2 u_3' \Rightarrow$$

$$u_1' = 2x - x^2 \frac{1}{x} = x$$

$$u_1' = x \Rightarrow \boxed{u_1 = \frac{x^2}{2}}$$

$u_1' = x \Rightarrow \boxed{u_1 = \frac{x^2}{2}}$

$u_2' = -2 \Rightarrow \boxed{u_2 = -2x}$

$u_3' = \frac{1}{x} \Rightarrow \boxed{u_3 = \ln x}$

So $y_p = \frac{1}{2}x^2(\underset{y_1}{\underline{x}}) + (-2x)\underset{y_2}{\underline{x^2}} + \ln x\,(\underset{y_3}{\underline{x^3}})$

$= \frac{x^3}{2} - 2x^3 + x^3 \ln x$

$= -\frac{3}{2}x^3 + x^3 \ln x$

The general solution of (NH) is

$y = y_H + y_p$

$$\boxed{y = C_1 x + C_2 x^2 + C_3 x^3 - \frac{3}{2}x^3 + x^3 \ln x}$$