October 9, 2018 8:28 AM

chapter, we look at ODE's of the form..

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \dots + a_1(x)\,y' + a_0(x)\,y = r(x)$$

(NH)

The solution to (NH) is based on the Following:

**Theorem**: The general solution of (NH) is of the form

$$y = y_H + y_P \quad \text{where:}$$

- $y_H$ is the general solution to the corresponding homogeneous ODE:

$$a_n(x)\,y^{(n)} + \dots + a_1(x)\,y' + a_0(x)\,y = 0 \qquad (H)$$

- $y_P$ is (any) solution of (NH)

**Ex/** Consider the ODE:

$$y'' - 3y' + 2y = 6x^2 - 6x - 10 \qquad (NH)$$

① Verify that $y_P = 3x^2 + 6x + 1$ is a particular solution of (NH).

② Write the general solution of (NH)

Sol

$y_p' = 6x+6$ , $y_p'' = 6$

Substitute back into the ODE:

$y_p'' - 3y_p' + 2y_p = 6 - 3(6x+6) + 2(3x^2+6x+1)$

$= 6 - 18x - 18 + 6x^2 + 12x + 2$

$= 6x^2 - 6x - 10$

So $y_p$ is a particular solution of (NH)

(2) The corresponding Homogeneous ODE

$y'' - 3y' + 2y = 0$ (H)

The Char. Eq. is $\lambda^2 - 3\lambda + 2 = 0$

$(\lambda-1)(\lambda-2) = 0$

$\lambda_1 = 1$ , $\lambda_2 = 2$ Two distinct real roots

$y_1 = e^x$ , $y_2 = e^{2x}$ a basis of solutions.

So the general solution to (H) is $y_H = C_1 e^x + C_2 e^{2x}$

$$\boxed{y = C_1 e^x + C_2 e^{2x} + 3x^2 + 6x + 1}$$

The hard part is to find $y_p$. We are going to explore 2 methods to find $y_p$:

① The undetermind coefficients method

1) Left coefficient on the left hand side of (NH) are constants

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = r(x)$$

2) $r(x)$ is a polynomial, Sinusoidal, exponential or a sum or a product of these functions.

Ex/ Solve the IVP:

$$y'' - 4y' = 3e^{3x} + 4x + 2 \; ; \quad y(0) = 1, \; y'(0) = 2$$

Sol the corresponding Homogeneous ODE is

$$y'' - 4y' = 0 \qquad (H)$$

The char. eq. is $\lambda^2 - 4\lambda = 0$

$\lambda = 0 \; , \; \lambda = 4$

2 distinct real roots

$y_1 = e^{0x} = 1 \qquad y_2 = e^{4x}$ is a basis of Solution for (H)

The general solution of (H) is

$$\boxed{y_H = C_1 + C_2 e^{4x}}$$

For $y_p$, we use the method of undetermined coefficients:

appears as a basis

- For $3e^{4x} \rightarrow y_p = Ae^{4x} \xrightarrow{\text{modify}} y_p = Axe^{4x}$

1 appears as a basis

- For $4x+2 \rightarrow y_p = Bx + C \xrightarrow{\text{modify}} y_p = Bx^2 + Cx$

By the sum Rule,

$$y_p = Axe^{4x} + Bx^2 + Cx$$

Next, we find A, B, and C

$$y_p = Axe^{4x} + Bx^2 + Cx$$

$$y_p' = Ae^{4x} + 4Axe^{4x} + 2Bx + C$$

$$y_p'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x} + 2B$$

$$= 8Ae^{4x} + 16Axe^{4x} + 2B$$

Go back to (NH)

$$8Ae^{4x} + 16Axe^{4x} + 2B - 4(Ae^{4x} + 4Axe^{4x} + 2Bx + c) = 3e^{4x} + 4x + 2$$

$$8Ae^{4x} + 16Axe^{4x} + 2B - 4Ae^{4x} - 16Axe^{4x} - 8Bx - 4C = 3e^{4x} + 4x + 2$$

$4Ae^{4x} - 8Bx + 2B - 4C = 3e^{4x} + 4x + 2$

$4A = 3 \implies A = \frac{3}{4}$

$-8B = 4 \implies -\frac{1}{2}$

$2B - 4C = 2 \implies 2C = B - 1 \implies C = \frac{B-1}{2} = -\frac{3}{4}$

So $y_p = \frac{3}{4} x e^{4x} - \frac{1}{2} x^2 - \frac{3}{4} x$

So the general solution of (NH) is $y = y_H + y_p$

$$\boxed{y = C_1 + C_2 e^{4x} + \frac{3}{4} x e^{4x} - \frac{1}{2} x^2 - \frac{3}{4} x}$$

$y' = 4C_2 e^{4x} + \frac{3}{4} e^{4x} + 3x e^{4x} - x - \frac{3}{4}$

$y(0) = 1 = C_1 + C_2 = 1$

$y'(0) = 2 = 4C_2 + \frac{3}{4} - \frac{3}{4}$

$C_2 = \frac{1}{2}$

$C_1 = 1 - C_2 = \frac{1}{2}$

So the solution to the IVP is

$$\boxed{y = \frac{1}{2} + \frac{1}{2} e^{4x} + \frac{3}{4} x e^{4x} - \frac{1}{2} x^2 - \frac{3}{4} x}$$

Ex 2/ Find the general solution of the ODE:

$$y''' + 6y'' + 12y' + 8y = 6e^{-2x} + 8x - 24 \quad (NH)$$

Sol/ The corresponding homogeneous Equation:

$$y''' + 6y'' + 12y' + 8 = 0$$

The ch. eq. is $\underbrace{\lambda^3 + 6\lambda^2 + 12\lambda + 8}_{(\lambda+2)^3} = 0$

one root $\lambda = -2$ real root of multiplicity 3:

$y_1 = e^{-2x}$, $y_2 = xe^{-2x}$, $y_3 = x^2 e^{-2x}$ is a basis of solutions

$$y_H = C_1 e^{-2x} + C_2 x e^{-2x} + C_3 x^2 e^{-2x}$$

Use the undetermined coefficient's Method for $y_p$:

- For $6e^{-2x}$ ; $y_p = Ae^{-2x}$ modify →
  $= Axe^{-2x}$ modify →
  $= Ax^2 e^{-2x}$ modify →
  $= Ax^3 e^{-2x}$ ✓

• For $8x-24$ : $y_p = Bx + c$ ✓

By the sum rule,

$$y_p = Ax^3 e^{-2x} + Bx + C$$

$$y_p' = 3Ax^2 e^{-2x} - 2Ax^3 e^{-2x} + B$$

$$y_p'' = 6Axe^{-2x} - 6Ax^2 e^{-2x} - 6Ax^2 e^{-2x} + 4Ax^3 e^{-2x}$$

$$= 6Axe^{-2x} - 12Ax^2 e^{-2x} + 4Ax^3 e^{-2x}$$

$$y_p''' = 6Ae^{-2x} - 12Axe^{-2x} - 24Axe^{-2x} + 24Ax^2 e^{-2x} + 12Ax^2 e^{-2x} - 8Ax^3 e^{-2x}$$

$$= 6Ae^{-2x} - 36Axe^{-2x} + 36Ax^2 e^{-2x} - 8Ax^3 e^{-2x}$$

Back to (NH)

$$6Ae^{-2x} - 36Axe^{-2x} + 36Ax^2 e^{-2x} - 8Ax^3 e^{-2x} + 6(6Axe^{-2x} - 12Ax^2 e^{-2x} + 4Ax^3 e^{-2x})$$
$$+ 12(3Ax^2 e^{-2x} - 2Ax^3 e^{-2x} + B) + 8(Ax^3 e^{-2x} + Bx + C)$$
$$= 6e^{-2x} + 8x - 24$$

$$6Ae^{-2x} + 8Bx + 12B + 8C = 6e^{-2x} + 8x - 24$$

$6A = 6 \Rightarrow A = 1$

$8B = 8 \Rightarrow B = 1$

$12B + 8C = -24 \Rightarrow C = -\frac{9}{2}$

$$\boxed{y_p = x^3 e^{-2x} + x - \frac{9}{2}}$$

The general solution to (NH) is $y = y_H + y_p$

$$y = C_1 e^{-2x} + C_2 x e^{-2x} + C_3 x^2 e^{-2x} + x^3 e^{-2x} + x - \frac{9}{2}$$