Last time Higher order homogeneous linear ODE's with constant coefficients

Ex3/ Solve the IVP:

$$y''' + 6y'' + 12y' + 8y = 0, \quad y(0) = 1, \quad y'(0) = -5, \quad y''(0) = 2$$

Sol Third order linear with constant coefficients.

The Char. Eq is:

$$\lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0$$

$\lambda = -2$ is a root $\Longrightarrow$ $\lambda + 2$ is a factor

$$
\begin{array}{l|l}
\lambda^3 + 6\lambda^2 + 12\lambda + 8 & \lambda + 2 \\ \hline
-\lambda^3 - 2\lambda^2 & \lambda^2 + 4\lambda + 4 \\ \hline
4\lambda^2 + 12\lambda + 8 & \\
-4\lambda^2 - 8\lambda & \\ \hline
4\lambda + 8 & \\
-4\lambda - 8 & \\ \hline
0 &
\end{array}
$$

So $\lambda^3 + 6\lambda^2 + 12\lambda + 8 = (\lambda+2)\underbrace{(\lambda^2 + 4\lambda + 4)}_{(\lambda+2)^2}$

$$= (\lambda + 2)^3$$

$(\lambda+2)^3 = 0$ The roots:

$\lambda = -2$ Real root of multiplicity 3.

$y_1 = e^{-2x}$, $y_2 = xe^{-2x}$, $y_3 = x^2 e^{-2x}$ basis of solutions!

The general solution is $y = C_1 e^{-2x} + C_2 x e^{-2x} + C_3 x^2 e^{-2x}$

$$y' = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x} + 2C_3 x e^{-2x} - 2C_3 x^2 e^{-2x}$$

$y'' = 4c_1 e^{-2x} - 2c_2 e^{-2x} - 2c_2 e^{-2x} + 4c_2 x e^{-2x} + 2c_3 e^{-2x} - 4c_3 x e^{-2x} - 4c_3 x e^{-2x} + 4c_3 x^2 e^{-2x}$

$y'' = 4c_1 e^{-2x} - 4c_2 e^{-2x} + 4c_2 x e^{-2x} + 2c_3 e^{-2x} - 8c_3 x e^{-2x} + 4c_3 x^2 e^{-2x}$

$y(0) = 1 \implies c_1 = 1$

$y'(0) = -5 \implies -2c_1 + c_2 = -5 \implies c_2 = -3$

$y''(0) = 2 \implies 4c_1 - 4c_2 + 2c_3 = 2 \implies 2c_3 = 2 - 4(1) + 4(-3) = -14$

$c_3 = -7$

The solution to the IVP is

$$\boxed{y = e^{-2x} - 3xe^{-2x} - 7x^3 e^{-2x}}$$

Ex/ $y^{(4)} + 4y''' + 7y'' + 6y' + 2y = 0$

CE: $\lambda^4 + 4\lambda^3 + 7\lambda^2 + 6\lambda + 2 = 0$

$\lambda = -1$ is a root which means that $(\lambda + 1)$ is a factor

$$\begin{array}{l|l}
\lambda^4 + 4\lambda^3 + 7\lambda^2 + 6\lambda + 2 & \lambda + 1 \\ \hline
-\underline{\lambda^4 - \lambda^3} & \lambda^3 + 3\lambda^2 + 4\lambda + 2 \\
3\lambda^3 + 7\lambda^2 + 6\lambda + 2 & \\
-\underline{3\lambda^3 - 3\lambda^2} & \\
4\lambda^2 + 6\lambda + 2 & \\
-\underline{4\lambda^2 - 4\lambda} & \\
2\lambda + 2 & \\
-\underline{2\lambda - 2} & \\
0 &
\end{array}$$

So $\lambda^4 + 4\lambda^3 + 7\lambda^2 + 6\lambda + 2 =$

$(\lambda+1)\underbrace{(\lambda^3 + 3\lambda^2 + 4\lambda + 2)}$

$\lambda = -1$ is a root

$\lambda + 1$ is a factor

$$\begin{array}{l|l}
\lambda^3 + 3\lambda^2 + 4\lambda + 2 & \lambda + 1 \\ \hline
-\underline{\lambda^3 - \lambda^2} & \lambda^2 + 2\lambda + 2 \\
2\lambda^2 + 4\lambda + 2 & \\
-\underline{2\lambda^2 - 2\lambda} & \\
2\lambda + 2 & \\
-\underline{2\lambda - 2} & \\
0 &
\end{array}$$

So $\lambda^4+4\lambda^3+7\lambda^2+6\lambda+2=$

$(\lambda+1)(\lambda+1)(\lambda^2+2\lambda+2)$

$(\lambda+1)^2(\lambda^2+2\lambda+2)$

For $\lambda^2+2\lambda+2$, the roots are

$$\frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{4i^2}}{2}$$

$$= \frac{-2\pm 2i}{2} = -1\pm i$$

The roots of the char. Eq.

- $\underline{\lambda=-1}$ Double real roots

$y_1=e^{-x}$, $y_2=xe^{-x}$

- $\lambda=-1\pm i$ two complex conjugate roots.

note: $\alpha\pm i\beta$

$e^{\alpha x}\cos(\beta x)$

$e^{\alpha x}\sin(\beta x)$

$y_3=e^{-x}\cos x$, $y_4=e^{-x}\sin x$

The general sol:

$$\boxed{y=C_1e^{-x}+C_2xe^{-x}+C_3e^{-x}\cos x+C_4e^{-x}\sin x}$$

Higher Order Euler-Cauchy Homogeneous ODE's:

Recall The second order Euler-Cauchy

$$x^2y''+axy'+by=0$$

We look for solutions of type $y=x^m$

Definition: Order n homogeneous Euler-Cauchy Equation is an ODE of the form:

$$x^ny^n+a_{n-1}x^{n-1}y^{(n-1)}+\dots+a_2x^2y''+axy'+a_0y=0$$

Like in the case $n=2$, we look for solutions of type $y=x^m$. This will give us the Char. Eq. of the ODE!

Case 1: The Char. Eq. has n distinct real roots $m_1, m_2, \dots, m_n$

In this case $y_1 = x^{m_1}, y_2 = x^{m_2}, \dots, y_n x^{m_n}$

is a basis of solutions and the general solution is

$$\boxed{y = C_1 x^{m_1} + C_2 x^{m_2} + \dots + C_n x^n}$$

Case 2: If m is a root of multiplicity K

$$y_1 = x^m \quad , y_2 = x^m \ln x \quad , y_3 = x^m (\ln x)^2 , \dots , y_n = x^m (\ln x)^{k-1}$$

is the contribution of the root m to the basis of Solutions

Case 3 two complex Conjugate roots

$m_1 = \alpha + i\beta \quad , \quad m_2 = \alpha - i\beta$

The contribution of m_1, m_2 to the basis of solutions is

$$y_1 = x^\alpha \cos(\beta \ln x) \, , \, y_2 = x^\alpha \sin(\beta \ln x)$$

Ex/ Solve the IVP

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 0$$

$$y(1) = 1 , y'(1) = 2 , y''(1) = 0$$

Sol Third order homogeneous euler-cauchy

We look for solutions of the following type

$$y = x^m$$

$$y' = m x^{m-1} \quad , \quad y'' = m(m-1)x^{m-2} \quad , \quad y''' = m(m-1)(m-2)x^{m-3}$$

Back into the ODE:

$$\cancel{x^3}\, m(m-1)(m-2)\,\cancel{x^{m-3}} + \cancel{x^2}\, m(m-1)\,\cancel{x^{m-2}} - 2x\,m\,\cancel{x^{m-1}} + 2\cancel{x^m} = 0$$

Divide by x^m

$$m(m-1)(m-2) + m(m-1) - 2m + 2 = 0$$

$$m\underline{(m-1)}(m-2) + \underline{m(m-1)} - 2\underline{(m-1)} = 0$$

$$(m-1)(m^2 - 2m + m - 2) = 0 \implies (m-1)(m^2 - m - 2) = 0$$

$$(m-1)(m+1)(m-2) = 0$$

$$m_1 = 1 \quad , \quad m_2 = -1 \quad , \quad m_3 = 2$$

$y_1 = x^{-1}$, $y_2 = x$, $y_3 = x^2$ is a basis of solutions

The general solution is $y = C_1 x^{-1} + C_2 x + C_3 x^2$

$$y' = -C_1 x^{-2} + C_2 + 2C_3 x$$

$$y'' = 2C_1 x^{-3} + 2C_3$$

$$y(1) = 1 \implies C_1 + C_2 + C_3 = 1$$

$$y'(1) = 2 \implies -C_1 + C_2 + 2C_3 = 2$$

$$y''(1) = 0 \implies 2C_1 + 2C_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \\ 2 & 0 & 2 & 0 \end{array}\right] \begin{array}{l} R_1 + R_2 \to R_2 \\ \\ -2R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & -2 & 0 & -2 \end{array}\right] -\tfrac{1}{2}R_3$$

$$\begin{bmatrix} 0 & 2 & 3 & | & 3 \\ 0 & -2 & 0 & | & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 2 & 3 & | & 3 \\ 0 & 1 & 0 & | & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 2 & 3 & | & 3 \end{bmatrix} \quad -2R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 3 & | & 1 \end{bmatrix} \quad \frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1/3 \end{bmatrix} \quad -R_3 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 2/3 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1/3 \end{bmatrix} \quad -R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -1/3 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1/3 \end{bmatrix}$$

So $c_1 = -1/3 \quad c_2 = 1 \quad c_3 = 1/3$

The solution to the IVP is

$$\boxed{y = -\frac{1}{3}x^{-1} + x + \frac{1}{3}x^2}$$

EX 2/ Solve the ODE

$$x^3 y''' + 6x^2 y'' + 20 x y' - 20y = 0$$

Sol

Third order homogeneous euler-cauchy

We look for solutions of the following type

$$y = x^m$$

$$y' = m x^{m-1}, \quad y'' = m(m-1) x^{m-2}, \quad y''' = m(m-1)(m-2) x^{m-3}$$

Back to the ODE.

$$\cancel{x^3} m(m-1)(m-2) \cancel{x^{m-3}} + 6\cancel{x^2} m(m-1) \cancel{x^{m-2}} + 20 \cancel{x} m \cancel{x^{m-1}} - 20 \cancel{x^m} = 0$$

$$m(m-1)(m-2) + 6m(m-1) + 20(m-1) = 0$$

$$(m-1)(m^2 - 2m + 6m + 20) = 0$$

$$(m-1)(m^2 + 4m + 20) = 0$$

$$m^2 + 4m + 20 = 0 \Rightarrow m = \frac{-4 \pm \sqrt{16-80}}{2} = \frac{-4 \pm 8i}{2} = -2 \pm 4i$$

The roots of the char. Eq are:

- $\underline{m=1}$ Simple real root $\longrightarrow y_1 = x$
- $\underline{m = -2 \pm 4i}$ Two complex conjugate roots

$$y_2 = x^{-2} \cos(4 \ln x), \quad y_3 = x^{-2} \sin(4 \ln x)$$

The general solution:

$$y = C_1 x + C_2 x^{-2} \cos(4 \ln x) + C_3 x^{-2} \sin(4 \ln x)$$