# higher order homogeneous linear ODE'S

In this chapter, we generelize results From the previous chapter. The main goal is to find solutims to linear and homogeneous ODE'S

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

**Key idea**: The general solution of (H) is of the Form

$$y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$$

where $y_1, y_2, \dots, y_n$ are lin. ind. Solutions to (H) as we did in the previous chapter, we treat cases.

**Case 1** (H) has constant coefficients.

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_2 y'' + a_1 y' + a_0 y = 0$$

Where $a_{n-1}, \dots, a_0 \in \mathbb{R}$

Like before, we look for solutions of type $y = e^{\lambda x}$

This leads to the following Characteristic equation:

$$\boxed{\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2\lambda^2 + a_1\lambda + a_0 = 0}$$

Again, the general solution of (H) depends on the nature of the roots of the (C.E.)

**Case 1** (CE) has $\underline{n}$ distinct real roots $\lambda_1, \lambda_2, \dots, \lambda_n$

$$y_1 = e^{\lambda x}, \quad y_2 = e^{\lambda_2 x}, \dots, \quad y_n = e^{\lambda_n x}$$

is a basis of solutions to (H) and the general solution is:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \dots + C_n e^{\lambda_n x}$$

**Case 2**: If $\lambda$ is a root of multiplicity $m$, then $\lambda$ contributes with

note: $m$ = multiplicity of root i.e. $x^2$ has multiplicity of 2

$$y_1 = e^{\lambda x}, \; y_2 = x e^{\lambda x}, \dots, \; y_n = x^{m-1} e^{\lambda x}$$ to the basis of solutions of (H)

**Case 3**: If $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ are 2 complex conjugate roots of (CE), then they contribute with

$$y_1 = e^{\alpha x}\cos(\beta x) \text{ and } y_2 = e^{\alpha x}\sin(\beta x)$$

**EX/** Solve the IVP:

$$y''' - y'' - y' + y = 0 \;; \quad y(0) = 0, \; y'(0) = -5, \; y''(0) = 2$$

Sol: Third order linear homogeneous with constant coefficients.

The Char. Eq. is: $\lambda^3 - \lambda^2 - \lambda + 1 = 0$

In this course guess $\lambda = 1, -1, 2, -2, 3, -3$ as being a factor.

Note that $\lambda = -1$ is a root of (CE)

So $\lambda+1$ is a Factor. Use long division:

$$\begin{array}{l|l} \lambda^3-\lambda^2-\lambda+1 & \lambda+1 \\ \underline{-\lambda^3-\lambda^2} & \overline{\lambda^2-2\lambda+1} \\ \quad -2\lambda^2-\lambda+1 & \\ \quad \underline{-2\lambda^2+2\lambda} & \\ \qquad \lambda+1 & \\ \qquad \underline{-\lambda-1} & \\ \qquad 0 & \end{array}$$

$$\frac{\lambda^3}{\lambda}=\lambda^2$$

$$\frac{-2\lambda^2}{\lambda}=-2\lambda$$

So $\lambda^3-\lambda^2-\lambda+1=(\lambda+1)\underbrace{(\lambda^2-2\lambda+1)}$

$$=(\lambda+1)(\lambda-1)^2$$

The roots of the Char. Eq. are

$\underline{\lambda_1=-1}$ Simple real root $\longrightarrow y_1=e^{-x}$

$\underline{\lambda_2=1}$ root of multiplicity 2 $\longrightarrow y_2=e^x$ , $y_3=xe^x$

A basis of solutions is $y_1=e^{-x}$, $y_2=e^x$, $y_3=xe^x$

The general solution of (H) is

$$\boxed{y=C_1e^{-x}+C_2e^x+xC_3e^x}$$

$$y'=-C_1e^{-x}+C_2e^x+C_3e^x+C_3xe^x$$

$$y''=C_1e^{-x}+C_2e^x+C_3e^x+C_3e^x+C_3xe^x$$

$$=C_1e^{-x}+C_2e^x+2C_3e^x+C_3xe^x$$

$$y(0)=0=C_1+C_2=0$$

$$y'(0)=-C_1+C_2+C_3=-5$$

$$y''(0)=C_1+C_2+2C_3=2$$

$$\begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -5 \\ 1 & 1 & 2 & 2 \end{array}\right] \end{array} \begin{array}{l} R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 2 & 2 \end{array}\right] \begin{array}{l} \frac{1}{2}R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] -\frac{1}{2}R_3 + R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] -R_2+R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$C_1 = 3$, $C_2 = -3$, $C_3 = 1$

The unique solution to the IVP is

$$y = 3e^{-x} - 3e^{x} + xe^{x}$$

**EX2**/ Find the general solution of

$$y^{(4)} - y = 0$$

Sol Order 4, homogeneous with constant coefficients:

The char. eq. is : $\lambda^4 - 1 = 0$

$$\Rightarrow (\lambda^2-1)(\lambda^2+1) = 0$$
$$(\lambda-1)(\lambda+1)(\lambda^2+1) = 0$$

$\underline{\lambda_1 = 1}$ : Simple real root $\longrightarrow y_1 = e^x$

$\underline{\lambda_2 = -1}$ : // // // $\longrightarrow y_2 = e^{-x}$

$\lambda^2+1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$

2 complex conjugate roots :

$$\longrightarrow y_3 = e^{0x}\cos(x) = \cos x$$
$$y_4 = \sin x$$

So $e^x$, $e^{-x}$, $\cos x$, $\sin x$ For a basis of solutions for (H)

The general solution is

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$