Lec 7 Ch.2 Second Order Linear Homogenous ODE's with Constant Coefficients/ Euler-Cauchy Equation

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"Last time"

Try exponential solution $y = e^{\lambda x}$: this leads to the

Characteristic Equation : $\boxed{\lambda^2 + a\lambda + b = 0}$ (CE)

3 possible cases according to roots of (CE)

Case 1 (CE) has two distinct real roots λ_1, λ_2

$y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$ 2 solutions for (H)

Moreover $\frac{y_1}{y_2} = \frac{e^{\lambda_1 x}}{e^{\lambda_2 x}} = e^{(\lambda_1 - \lambda_2)x} \neq$ constant, so

y_1, y_2 form a basis of solution for (H):

The general solution is $\boxed{y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}}$

Case 2 (CE) has double real roots: $\lambda_1 = \lambda_2 = \lambda$

In this case, $y_1 = e^{\lambda x}$, $y_2 = x e^{\lambda x}$

is a basis for solutions. The general solution is

$$\boxed{y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}}$$

Case 3 (CE) has two complex conjugates roots $\alpha + i\beta$ and $\alpha - i\beta$

In this case, one can show that

$y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$ is a basis of solutions. The general solution is

$$y = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$$

Summary

$$y'' + ay' + by = 0 \qquad (H)$$

$$\Downarrow$$

$$\lambda^2 + a\lambda + b = 0$$

① $\underline{\lambda_1 \neq \lambda_2}$ $\quad y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

② $\underline{\lambda_1 = \lambda_2} = \lambda \in \mathbb{R} \Rightarrow y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$

③ $\underline{\lambda = \alpha + i\beta} \Rightarrow \quad y = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$

Ex 1 Solve the IVP:

$$y'' - 6y' + 10y = 0; \qquad y(0) = 1, \quad y'(0) = 2$$

Sol Kind of ODE?

Second order homogeneous linear with constant coefficients.

The characteristic equation is:

$\lambda^2 - 6\lambda + 10 = 0 \quad \Leftarrow$ assume $e^{\lambda x}$ as a sol.

note: $\sqrt{-1} = i^2$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm \sqrt{-4}}{2}$$

$$= 6 \pm \sqrt{4i^2} \Rightarrow \lambda = \frac{6 \pm 2i}{2} = 3 \pm i$$

2 complex conjugate roots $\lambda_1 = 3+i$, $\lambda_2 = 3-i$

A basis of solutions is $y_1 = e^{3x}\cos x$, $y_1 = e^{3x}\sin x$

The general solution is $y = C_1 e^{3x}\cos x + C_2 e^{3x}\sin x$

$$y' = 3C_1 e^{3x}\cos x - C_1 e^{3x}\sin x + 3C_2 e^{3x}\sin x + C_2 e^{3x}\cos x$$

$$y(0) = 1 \Rightarrow C_1 = 1$$

$$y'(0) = 2 \Rightarrow 3C_1 + C_2 = 2 \Rightarrow C_2 = -1$$

The unique solution to the IVP:

$$y = e^{3x}\cos x - e^{3x}\sin x$$

Ex2/ Solve the IVP:

$$y'' - 8y' + 16y = 0 \; ; \quad y(0) = -1 \; , \; y'(0) = 0$$

$$\downarrow$$

$$\lambda^2 - 8\lambda + 16 = 0 \qquad (CE)$$

One double real root $\lambda_1 = \lambda_2 = 4$

$y_1 = e^{4x}$, $y_2 = x e^{4x}$ is a basis of solutions to (H)

The general solution is :

The general solution is :

$$y = C_1 e^{4x} + C_2 x e^{4x}$$

$$y' = 4C_1 e^{4x} + C_2 e^{4x} + 4C_2 x e^{4x}$$

$$y(0) = -1 \Longrightarrow C_1 = -1$$

$$y'(0) = 0 \Longrightarrow 4C_1 + C_2 = 0$$

$$C_2 = 4$$

The solution For the IVP is $\boxed{y = -e^{4x} + 4x e^{4x}}$

Ex3/ Solve the IVP :

$$y'' - 5y' + 4y = 0 \,, \quad y(0) = 2, \; y'(0) = 1$$

Sol Second order ODE with coefficients constant

The characteristic equation is

$$\lambda^2 - 5\lambda + 4 = 0 \Longrightarrow (\lambda - 1)(\lambda - 4) = 0$$

$$\Longrightarrow \lambda_1 = 1 \,, \; \lambda_2 = 4 : 2 \text{ distict real roots}$$

real roots : $y_1 = e^{x}$, $y_2 = e^{4x}$ is a basis for solutions

The general solution is

$$y = C_1 e^{x} + C_2 e^{4x}$$

$$y' = C_1 e^{x} + 4C_2 e^{4x}$$

$$y(0) = 2 \Longrightarrow C_1 + C_2 = 2 \quad (1)$$

$y'(0)=1 \Longrightarrow c_1+4c_2=1$ (2)

(2) − (1)

$3c_2=-1$

$c_2=-\frac{1}{3}$

(1) $\Rightarrow c_1=2+\frac{1}{3}=\frac{7}{3}$

The unique solution is $\boxed{y=\frac{7}{3}e^{x}-\frac{1}{3}e^{4x}}$

Euler - Cauchy Equation

A second order ODE is called Euler - Cauchy homogeneous if it can be written as

$$\boxed{x^2y''+axy'+by=0}$$

$a, b \in \mathbb{R}$

Homogeneous second order linear ODE

Unlike the case of constant coefficients, we try solutions of type $y=x^m$

$y'=mx^{m-1}$, $y''=m(m-1)x^{m-2}$

Sub back into the ODE:

$$x^2m(m-1)x^{m-2}+axmx^{m-1}+bx^m=0$$

$\Rightarrow m(m-1)x^m+amx^m+bx^m=0$: Divide by x^m

$\Rightarrow m(m-1) + am + b = 0$

$\Rightarrow m^2 - m + am + b = 0$

$\Rightarrow \boxed{m^2 + (a-1)m + b = 0}$ This we call the characteristic equation of Euler-Cauchy

Like before, the general solution depends on the nature of the roots of the char. Eq.

Case 1: 2 distinct real roots of (*) m_1, m_2

$y_1 = x^{m_1}$, $y_2 = x^{m_2}$ is a basis

The general solution is $\boxed{y = C_1 x^{m_1} + C_2 x^{m_2}}$

Case 2: (CE) has double real root

$m_1 = m_2 = m$

$y_1 = x^m$, $y_2 = x^m \ln x$ is a basis

The general solution is

$$\boxed{y = C_1 x^m + C_2 x^m \ln x}$$

Case 3: (CE) has 2 complex roots $\alpha \pm i\beta$

Here one can verify that

$y_1 = x^\alpha \cos(\beta \ln x)$, $y_2 = x^\alpha \sin(\beta \ln x)$

is a basis of solutions.

The general solution is

$$\boxed{y = C_1 x^\alpha \cos(\beta \ln x) + C_2 x^\alpha \sin(\beta \ln x)}$$

Ex/ Solve the IVP:

$$x^2y'' - 4xy' + 4y = 0, \quad x > 0, \quad y(1) = 1, \quad y'(1) = 2$$

Sol <u>Euler - couchy</u> with $a = -4$, $b = 4$

The Char. Eq. is $m^2 + (a-1)m + b = 0$

$$\Longrightarrow m^2 - 5m + 4 = 0 \Longrightarrow (m-1)(m-4) = 0$$

2 distinct real roots

$m_1 = 1$, $m_2 = 4$

The general solution is :

$y_1 = x$, $y_2 = x^4$ is a basis of solutions.

The general solution is

$$y = C_1 x^1 + C_2 x^4$$

$$y' = C_1 + 4C_2 x^3$$

$y(1) = 1 \Longrightarrow C_1 + C_2 = 1$ ①

$y'(1) = 2 \Longrightarrow C_1 + 4C_2 = 2$ ②

② − ① $= 3C_2 = 1 \Longrightarrow C_2 = 1/3$

① $\Longrightarrow C_1 = \frac{2}{3}$

The solution to the IVP is $\boxed{y = \frac{2}{3}x + \frac{1}{3}x^4}$

Ex/

Ex3/ Solve the ODE:

$$x^2y'' - xy' + 2y = 0 \quad , \quad x > 0$$

Sol Euler-cauchy with $a = -1$, $b = 2$

the Char. Eq. is $m^2 + (a-1)m + b = 0 \Rightarrow$

$m^2 - 2m + 2 = 0 \Longrightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2}$

α β

$m_1 = 1 + i$, $m_2 = 1 - 1i$: 2 complex conjugate roots

In this case $y_1 = x^1 \cos(1 \ln x) = x\cos(\ln x)$

$y_2 = x\sin(\ln x)$

is a basis for solutions.

The general solution is

$$\boxed{y = C_1 x\cos(\ln x) + C_2 x\sin(\ln x)}$$