Lec 6 Ch.2 Second Order Linear Homogenous ODE's with Constant Coefficients

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We start by reviewing some concepts from linear algebra needed in this chapter and the coming ones!

Def: Functions are called linearly independent (l.ind for short) on the interval I if

$a_1 y_1(x) + a_2 y_2(x) + \dots a_n y_n(x) = 0$ for all

$x \in I$ implies that $a_1 = a_2 \dots = a_n = 0$

Otherwise, we say that they are linearly dependant (l.d)

Ex/ $y_1 = \sin^2 x$, $y_2 = \cos^2 x$, $y_3 = 1$ on $]-\infty, +\infty[$

We know that $\sin^2 x + \cos^2 x = 1 \Longleftrightarrow$

$y_1(x) + y_2(x) - y_3(x) = 0$ For all x

So y_1, y_2, y_3 are l.d.

Ex/ $y_1 = 2$, $y_2 = 1+x$, $y_3 = 1+x^2$

Prove that they are l.ind.

Sol/

$a y_1 + b y_2 + c y_3 = 0$

$$\left.\begin{aligned} &a y_1 + b y_2 + c y_3 = 0 \\ &2a + b(1+x) + c(1+x^2) = 0 \\ &(2a+b+c) + bx + cx^2 = 0 \end{aligned}\right\}$$ Choose some values for x

$\underline{x=-1}$ $\quad 2a + \cancel{b} + \cancel{c} - \cancel{b} + \cancel{c} = 0 \Rightarrow 2a + 2c = 0 \Rightarrow a + c = 0$ ①

$\underline{x=0}$ $\quad 2a + b + c = 0$ ②

$\underline{x=1}$ $\quad 2a + b + c + b + c = 0 \Rightarrow 2a + 2b + 2c = 0 \Rightarrow a + b + c = 0$ ③

(1) and ③ $\Rightarrow \boxed{b = 0}$

(2) $\Rightarrow 2a + c = 0$ ④ ; $\quad -① + ④ \Rightarrow \boxed{a = 0}$

② $\boxed{c = 0}$

Remark To check clear independence of 2 Functions y_1, y_2:

$\frac{y_1}{y_2}$ or $\frac{y_2}{y_1} \neq$ constant $\Longleftrightarrow y_1, y_2$ are l. ind

ex/ $\frac{y_1}{y_2} = 2$ then $y_1 = 2y_2 \Rightarrow$ l.d

Ex/ $\quad y_1 = e^{3x}, \quad y_2 = e^{-2x}$

Ex/ $y_1 = e^{3x}$, $y_2 = e^{-2x}$

$$\frac{y_1}{y_2} = \frac{e^{3x}}{e^{-2x}} = e^{5x} \neq \text{constant} \implies y_1, y_2 \text{ are l. ind.}$$

Ex/ $y_1 = \sin x$, $y_2 = \cos x$

$$\frac{y_1}{y_2} = \frac{\sin x}{\cos x} = \tan x \neq \text{constant} \implies \{\sin x, \cos x\} \text{ l. ind.}$$

Ex/ $y_1 = e^{3x+2}$, $y_2 = e^{3x}$

$$\frac{y_1}{y_2} = \frac{e^{3x+2}}{e^{3x}} = \frac{\cancel{e^{3x}} \cdot e^{2}}{\cancel{e^{3x}}} = e^{2} = \text{constant} \implies y_1, y_2 \text{ are l.d}$$

Definition A second order linear and homogeneous ODE has the form:

$$\boxed{a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0}$$

We call this <u>homogeneous</u> because the right hand side is "0"

Ex/ $\frac{e^x}{1+x^2}\,y'' + 2y' - (\tan(x))\,y = 0$

Important Facts about (H) :

① If y_1, y_2 are 2 solutions of (H), then $y_1 + y_2$ is also a solution!

② If y is a solution of (H) and $c \in \mathbb{R}$, then cy is also a solution of (H)

So the set of all solutions of (H) is a vector space!

Theorem: The set of all solution to (H) is a vector space of $\boxed{\text{dimension } 2}$

The theorem tells us how to find the general solution of (H)

① Find 2 l. ind solutions y_1 and y_2 of (H)

② The general solution of (H) is

$$\boxed{y = C_1 y_1 + C_2 y_2}$$

EX/ Consider the following ODE:

$$y'' - 5y' + 4y = 0$$

① Show that $y_1 = e^x$, $y_2 = e^{4x}$ are solutions to (H)

② Find the general solution to (H)

Sol ① $y_1 = e^x \Rightarrow y_1' = e^x \Rightarrow y_1'' = e^x$

$y_1'' - 5y_1' + 4y_1 = e^x - 5e^x + 4e^x = 0 \checkmark$ $\quad y_1 = e^x$ is a sol

$y_2 = e^{4x} \Rightarrow y_2' = 4e^{4x} \Rightarrow y_2'' = 16e^{4x}$

$y_2'' - 5y_2' + 4y_1 = 16e^{4x} - 20e^{4x} + 4e^{4x} = 0 \checkmark \Rightarrow y_2$ is a solution!

② $\dfrac{y_2}{y_1} = \dfrac{e^{4x}}{e^x} = e^{3x} \neq$ constant $\Rightarrow y_1, y_2$ are l. ind.

The general sol of (H) is $\boxed{y = C_1 e^x + C_2 e^{4x}}$

We start the discussion with the case of constant coefficients.

① Second-Order linear homogeneous ODE's with constant coefficients

$$\underbrace{a_2}_{} y'' + \underbrace{a_1}_{} y' + \underbrace{a_0}_{} y = 0 \qquad (H)$$

Constants

Dividing by a_2: $y'' + \frac{a_1}{a_2} y' + \frac{a_0}{a_2} y = 0$ or

$$\boxed{y'' + a y' + b y = 0 \quad (H)}$$

The previous example suggests we try solutions of type, $y = e^{\lambda x}$ where $\lambda \in \mathbb{R}$ (constant)

$y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$ back into (H):

$$\lambda^2 e^{x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0 \implies \boxed{e^{\lambda x}(\lambda^2 + a\lambda + b = 0)}$$

This is called the <u>Characteristic Equation</u> of (H)

3 possibilities for the roots of the charc. Eq.

1. Two distinct real roots λ_1, λ_2

2. One Double real root $\lambda_1 = \lambda_2 = \lambda$

3. Two complex conjugat roots
   $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$