Last time linear First order ODE

$$y' + f(x)y = r(x)$$

func of x only

$$y(x) = \frac{\int e^{\int f(x)dx} r(x)dx + c}{e^{\int f(x)dx}}$$

Rewrite the ODE : $y' = \frac{dy}{dx}$

$$\frac{dy}{dx} + f(x)y = r(x) \Longrightarrow dy + f(x)y\,dx = r(x)dx \Longrightarrow$$

$$\underbrace{(f(x)y - r(x))}_{M}dx + \underbrace{1}_{N}dy = 0$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = f(x)\\ \\ \frac{\partial N}{\partial x} = 0\end{array}\right\}\text{ Not Exact!}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = f(x) \Longrightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{f(x)}{1} = f(x)$$

An integrating factor exists, given by $\mu(x) = e^{\int f(x)dx}$

Multiply ODE by $\mu(x)$:

$$\underbrace{(\mu(x)f(x)y - \mu(x)r(x))}_{M^*}dx + \underbrace{\mu(x)}_{N^*}dy = 0$$

Check exactness

$$\frac{\partial M^*}{\partial y} = \mu(x)f(x)$$

derivative of an integral =)

$$\frac{\partial N^*}{\partial x} = \mu'(x)$$

$$\frac{dN^*}{dx} = \mu'(x)$$

derivative of an integral =

But $\mu(x) = e^{\int F(x)dx} \Rightarrow \mu'(x) = e^{\int F(x)dx} F(x)dx = \mu(x) f(x)$

$\therefore \frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$ = Exact!

We look for $F(x,y)$ such that $\frac{\partial F}{\partial x} = M^*$, $\frac{\partial F}{\partial y} = N^*$

$$\frac{\partial F}{\partial y} = N^* = \mu(x) \Rightarrow F(x,y) = \int \mu(x)\,dy = \mu(x)\int dy$$

$$F(x,y) = \mu(x)\,y + \underline{\underline{h(x)}} \Rightarrow \frac{\partial F}{\partial x} = \mu'(x)\,y + h'(x)$$

$$= \mu(x) F(x)\,y + h'(x)$$

But $\frac{\partial F}{\partial x} = M^* \Rightarrow \cancel{\mu(x) f(x)\,y} + h'(x) = \cancel{\mu(x) f(x)\,y} - \mu(x)\,r(x)$

$$h'(x) = -\mu(x)\,r(x) \Rightarrow h(x) = -\int \mu(x)\,r(x)\,dx + K$$

So $F(x,y) = \mu(x)\,y - \int \mu(x)\,r(x)\,dx + K$

The General solution to the ODE is : $F(x,y) = \text{constant}$

$$\mu(x)\,y - \int \mu(x)\,r(x)\,dx = C$$

$$\mu(x)\,y = \int \mu(x)\,r(x)\,dx + C$$

$$y = \frac{\int \mu(x)\,r(x)\,dx + C}{\mu(x)}$$

$$= \frac{\int e^{\int F(x)dx}\,r(x)\,dx + C}{e^{\int F(x)dx}}$$

Attention: Make sure your linear ODE has form

$$y' + F(x)\,y = r(x)$$

(in particular, the coefficient of y' is 1) before applying the Formula for the solution

Ex 1 Solve the IVP:

$$y' + \underbrace{\frac{1}{x}}_{f(x)} y = \underbrace{e^{-2x}}_{r(x)} \; ; \; x > 0 \; ; \; y(1) = 0$$

Sol This is a linear first order with $f(x) = \frac{1}{x}$ and $r(x) = e^{-2x}$

$$y = \frac{\int e^{\int \frac{1}{x} dx} e^{-2x} dx + c}{e^{\int \frac{1}{x} dx}} = \frac{\int e^{\ln x} e^{-2x} dx + c}{e^{\ln x}}$$

$$= \frac{\int x e^{-2x} dx + c}{x}$$

By parts $\quad u = x \quad , \quad v' = e^{-2x}$

$\qquad\qquad u' = 1 \qquad v = \frac{-1}{2} e^{-2x}$

$$\int x e^{-2x} dx = uv - \int u'v dx = \frac{-x}{2} e^{-2x} \underbrace{- \int \frac{-1}{2} e^{-2x} dx}_{+\frac{1}{2} \underbrace{\int e^{-2x} dx}_{-\frac{1}{2} e^{-2x}}}$$

$$= \frac{-x}{2} e^{-2x} - \frac{1}{4} e^{-2x}$$

Back to Formula:

So the general solution to the ODE is

$$y = \frac{\frac{-x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + c}{x} \quad , \quad y(1) = 0$$

$$0 = \frac{\frac{-1}{2} e^{-2} - \frac{1}{4} e^{-2} + c}{1} \Rightarrow c = \frac{3}{4e^2}$$

The solution to the IVP:

$$y = \frac{-\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + \frac{3}{4e^2}}{x}$$

Ex 2 Solve the IVP:

$$xy' + 2y = \sin x \quad ; \quad x > 0 \quad ; \quad y(\pi/2) = -\frac{2}{\pi}$$

Sol Divide the ODE by x:

$$y' + \frac{2}{x}y = \sin x$$

linear with $f(x) = \frac{2}{x}$, $r(x) = \frac{\sin x}{x}$

$$y = \frac{\int e^{\int f(x)dx} r(x)dx + C}{e^{\int f(x)dx}}$$

$$= \frac{\int e^{\int \frac{2}{x}dx} \frac{\sin x}{x} dx + C}{e^{\int \frac{2}{x}dx}}$$

$$= \frac{\int e^{2\ln x} \frac{\sin x}{x} dx + C}{e^{2\ln x}} = \frac{\int e^{\ln x^2} \frac{\sin x}{x} dx + C}{e^{\ln x^2}}$$

$$= \frac{\int \frac{x^2 \sin x}{x} dx + C}{x^2} = \frac{\int x\sin x \, dx + C}{x^2}$$

By parts: $u = x$, $v' = \sin x$

$u' = 1$, $v = -\cos x$

$$\int x\sin x\, dx = -x\cos x + \underbrace{\int \cos x\, dx}_{\sin x}$$

$$= -x\cos x + \sin x + k$$

General solution

$$y = \frac{\int x\sin x\, dx + C}{x^2} = \frac{-x\cos x + \sin x + C}{x^2}$$

$$y=\frac{-x\cos x+\sin x+C)}{x^2}\,;\; y(\tfrac{\pi}{2})=-\frac{2}{\pi}\Rightarrow\frac{1+c}{\frac{\pi^2}{4}}=-\frac{2}{\pi}\Rightarrow 1+c=-\frac{2}{\pi}\frac{\pi^2}{4}=-\frac{\pi}{2}\Rightarrow C=-1-\frac{\pi}{2}$$

$$y=\frac{-x\cos x+\sin x-1-\pi/2}{x^2}$$

6 Bernouilli Equation :

Def : A First order ODE is called Bernouilli type if it can be written under the form :

$$y'+f(x)\,y=r(x)\,y^a$$

where $a\in\mathbb{R}$

Ex / $y'+\underbrace{\frac{2}{x}e^{-2x}}_{f(x)}y=\underbrace{\frac{\sin x}{1+x^2}}_{r(x)}\sqrt{y}$; $a=\frac{1}{2}$

Any Bernouilli Equation can be made linear using the solution $\boxed{u=y^{1-a}}$

Ex 2 / Solve the following IVP :

$y'-3y=2y^2$; $y(0)=1$

Sol Bernouilli type with $f(x)=-3$, $r(x)=2$, $a=2$

Let $u=y^{1-a}=y^{1-2}=y^{-1}$

$$u'=-\frac{1}{y^2}y'=-y^{-2}y'=-y^{-2}\underbrace{(3y+2y^2)}_{\text{From the ODE}}$$

$$u'=-3y^{-1}-2\Rightarrow u'=-3u-2\Rightarrow u'+3u=-2$$

linear first order with $r(x)=3$ and $r(x)=-2$

The general solution is :

$$u=\frac{\int e^{\int 3\,dx}(-2)\,dx+C}{e^{\int 3\,dx}}$$

$$=\frac{-2\int e^{3x}\,dx+C}{e^{3x}}=\frac{-\frac{2}{3}e^{3x}+C}{e^{3x}}$$

$$=-\frac{2}{3}+Ce^{-3x}$$ $y(0)=1$: $x=0$, $y=1$ $\Rightarrow u(0)=\frac{1}{y(0)}=\frac{1}{1}=1$

$$1=\frac{-2}{3}+C\Rightarrow C=\frac{5}{3}$$

$$u=-\frac{2}{3}+\frac{5}{3}e^{-3x}=\frac{-2+5e^{-3x}}{3}\Rightarrow y=\frac{1}{u}=\boxed{\frac{2}{-2+5e^{-3x}}}$$

Ex / Solve the IVP:

$$y'+\frac{2}{x}y=\frac{1}{x^2}y^3\ ;\ y(1)=1$$

Sol Bernoulli type with $F(x)=\frac{2}{x}$, $r(x)=\frac{1}{x^2}$; $a=3$

Let $u=y^{1-a}=y^{1-3}=y^{-2}$

$u=y^{-2}$

$u'=-2y^{-3}y'$

$$u'=-2y^{-3}\underbrace{\left(\frac{1}{x^2}y^3-\frac{2}{x}y\right)}_{\text{Form the ODE}}$$

$$u'=\frac{4}{x}y^{-2}-\frac{2}{x^2}$$

$$=\frac{4}{x}u-\frac{2}{x^2}\Rightarrow u'-\frac{4}{x}u=\frac{-2}{x^2}$$ linear in u

$$u=\frac{\int e^{\int -\frac{4}{x}dx}\left(\frac{-2}{x^2}\right)dx+C}{e^{\int -\frac{4}{x}dx}}$$