# Lec 4 Ch.1 cont. Int. fact./ Linear First Order

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Last time (integrating Factors)

Ex/ $\underbrace{(3x^2y + 2xy + y^3)}_{M}dx + \underbrace{(x^2+y^2)}_{N}dy = 0$

Sol check exactness:

$$\left.\begin{aligned} \frac{dM}{dy} &= 3x^2 + 2x + 3y^2 \\ \frac{dN}{dx} &= 2x \end{aligned}\right\} \quad \frac{dM}{dy} \neq \frac{dN}{dx} \Rightarrow \text{ODE not exact!}$$

$$\frac{dM}{dy} - \frac{dN}{dx} = 3x^2 + 3y^2 = 3(x^2+y^2)$$

$$\frac{\frac{dM}{dy} - \frac{dN}{dx}}{N} = \frac{3(x^2+y^2)}{x^2+y^2} = 3 = f(x)$$

An integrating Factor is given by $\mu(x) = e^{\int f(x)dx} = e^{\int 3dx} = e^{3x}$

Multiplying the ODE by $e^{3x}$

$$\underbrace{(3x^2y\,e^{3x} + 2xy\,e^{3x} + y^3e^{3x})}_{M*}dx + \underbrace{(x^2e^{3x} + y^2e^{3x})}_{N*}dy = 0$$

$$\left.\begin{aligned} \frac{dM*}{dy} &= 3x^2e^{3x} + 2xe^{3x} + 3y^2e^{3x} \\ \frac{dN*}{dx} &= 2xe^{3x} + 3x^2e^{3x} + 3y^2e^{3x} \end{aligned}\right\} \quad \text{Exact!}$$

We look for a Function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M^* \; ; \; \frac{\partial F}{\partial y} = N^*$$

$$\frac{\partial F}{\partial y} = x^2e^{3x} + y^2e^{3x} \Rightarrow F = \int (x^2e^{3x} + y^2e^{3x})\,dy$$

$$F(x,y) = x^2e^{3x}y + \frac{1}{3}y^3e^{3x} + h(x)$$

$$\frac{\partial F}{\partial x} = 2xe^{3x}y + 3x^2e^{3x}y + y^3e^{3x} + h'(x)$$

But $\frac{\partial F}{\partial x} = M^*$

$$\cancel{2xe^{3x}y} + \cancel{3x^2e^{3x}y} + \cancel{y^3e^{3x}} + h'(x) = \cancel{3x^2ye^{3x}} + \cancel{2xye^{3x}} + \cancel{y^3e^{3x}}$$

$\therefore h'(x) = 0$ and $h(x) = K$ (constant)

$$F(x,y) = x^2e^{3x}y + \frac{1}{3}y^3e^{3x} + K$$

The general solution is $F(x,y) = \text{constant}$

$$x^2e^{3x}y + \frac{1}{3}y^3e^{3x} = C \qquad y(0) = -3$$

$$(0)^2e^{30}(-3) + \frac{1}{3}(-3)^3e^{3(0)} = C$$

$$C = -9$$

The unique sol is $\boxed{x^2e^{3x}y + \frac{1}{3}y^3e^{3x} = -9}$

implicit solution!

**Ex 2** Solve the I.V.P:

$$\underbrace{(ye^{\sin x}\cos x - y^3 + 2xy)}\,dx + \underbrace{(2e^{\sin x} - 4xy^2 - 4y^2 + 2x^2)}\,dy = 0 \; ; \quad y(0) = 2$$

$$\underbrace{(ye^{\sin x}\cos x - y^3 + 2xy)}_{M}dx + \underbrace{(2e^{\sin x} - 4xy^2 - 4y^2 + 2x^2)}_{N}dy = 0 \; ; \quad y(0) = 2$$

Sol

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= e^{\sin x}\cos x - 3y^2 + 2x \\ \frac{\partial N}{\partial x} &= 2\cos(x)e^{\sin x} - 4y^2 + 4x \end{aligned}\right\} \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{Not Exact!}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -e^{\sin x}\cos x + y^2 - 2x$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-e^{\sin x}\cos x + y^2 - 2x}{ye^{\sin x}\cos x - y^3 + 2xy}$$

$$= \frac{-e^{\sin x}\cos x + y^2 - 2x}{-y(-e^{\sin x}\cos x + y^2 - 2x)}$$

$$= \frac{-1}{y} = g(y)$$

An integrating factor exists, given by

$$\mu(y) = e^{-\int g(y)\,dy} = e^{-\int -\frac{1}{y}\,dy} = e^{\int \frac{1}{y}\,dy} = e^{\ln y} = |y|$$

Just looking for one integrating factor so use easiest one like y positive

Multiply the Whole ODE by $y$

$$\underbrace{(y^2e^{\sin x}\cos x - y^4 + 2xy^2)}_{M^*}dx + \underbrace{(2ye^{\sin x} - 4xy^3 - 4y^3 + 2x^2y)}_{N^*}dy = 0$$

$$\frac{\partial M^*}{\partial y} = 2ye^{\sin x}\cos x - 4y^3 + 4xy$$

$$\frac{\partial N^*}{\partial x} = 2ye^{\sin x}\cos x - 4y^3 + 4xy$$

$$\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

Exact

Look at function $F(x,y)$ such that: $\frac{\partial F}{\partial x} = M^*$ and $\frac{\partial F}{\partial y} = N^*$

$$\frac{\partial F}{\partial y} = 2ye^{\sin x} - 4xy^3 - 4y^3 + 2x^2y \Rightarrow F(x,y) = y^2e^{\sin x} - xy^4 - y^4 + x^2y^2 + h(x)$$

So $\frac{\partial F}{\partial x} = y^2\cos x\, e^{\sin x} - y^4 + 2xy^2 + h'(x)$

But $\frac{\partial F}{\partial x} = M^* = y^2e^{\sin x}\cos x - y^4 + 2xy^2$

Composing gives $h'(x) = 0 \Rightarrow h(x) = K$

So $F(x,y) = y^2e^{\sin x} - xy^4 - y^4 + x^2y^2 + K$

The general solution is $F(x,y) = \text{constant}$

So $\boxed{y^2e^{\sin x} - xy^4 - y^4 + x^2y^2 = C}$

$y(0) = 2 \Rightarrow 4 - 16 = C \Rightarrow C = -12$

the unique sol is

$$y^2e^{\sin x} - xy^4 - y^4 + x^2y^2 = -12$$

Ex/ $\underbrace{\left(e^{x+y} + ye^{y}\right)}_{M}dx + \underbrace{\left(xe^{y} - 1\right)}_{N}dy = 0 \quad ; \quad y(0) = 0$

Sol

$$\frac{\partial M}{\partial y} = e^{x+y} + e^{y} + ye^{y}$$

$$\frac{\partial N}{\partial x} = e^{y}$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Not Exact!

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{x+y} + ye^{y}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial y}}{M} = \frac{e^{x+y} + ye^{y}}{e^{x+y} + ye^{y}} = 1 = g(y)$$

An integrating Factor exists and is given by $\mu(y) = e^{-\int g(y)dy}$

$$\mu(y) = e^{-\int 1 dy} = e^{-y}$$

Multiplying the ODE by $e^{-y}$:

$$e^{-y}(e^{x+y} + ye^{y})dx + e^{-y}(xe^{y} - 1)dy = 0$$

$$\underbrace{(e^{x} + y)}_{M*}dx + \underbrace{(x - e^{-y})}_{N*}dy = 0$$

$$\frac{\partial M*}{\partial y} = 1$$

$$\frac{\partial N*}{\partial x} = 1$$

$\frac{\partial M*}{\partial y} = \frac{\partial N*}{\partial x}$ Exact!

Look for a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M * \quad ; \quad \frac{\partial F}{\partial y} = N *$$

$$\frac{\partial F}{\partial x} = e^x + y \Rightarrow F(x,y) = e^x + xy + h(y)$$

$$\frac{\partial F}{\partial y} = x + h'(y)$$

But $\frac{\partial F}{\partial y} = N * \Rightarrow x + h'(y) = x - e^{-y} \Rightarrow h'(y) = -e^{-y}$

$$\Rightarrow h(y) = -\int e^{-y}\,dy = e^{-y} + K$$

So $F(x,y) = e^x + xy + e^{-y} + K$

The general solution is $F(x,y) = A$

$$e^x + xy + e^{-y} = C$$

Now $y(0) = 0$

$e^0 + 0 \cdot 0 + e^{-0} = C$

$C = 2$

$$e^x + xy + e^{-y} = 2$$

## (5) Linear First order ODE's

**Def**: A first order ODE is called linear if it can be written under the form:

the form :

$$\boxed{y' + \underbrace{f(x)}_{} \cdot y = \underbrace{r(x)}_{}}$$

function of x only.

Ex/ $(x^2+1)y' - 3e^x y = x + 3\sin x \Longleftrightarrow$ Divide everything by $(x^2+1)$

$$y' - \frac{3e^x}{x^2+1} y = \frac{x+3\sin x}{x^2+1}$$

This is linear with $f(x) = \frac{-3e^x}{x^2+1}$ and $r(x) = \frac{x+3\sin x}{x^2+1}$

Next time, we are going to prove that the general solution (Explicit) of $(*)$ is

$$y = \frac{\int e^{\int f(x)dx} \cdot r(x)\,dx + C}{e^{\int f(x)dx}}$$