Lec 3 Ch1 cont. Exact/ Integrating Factor

September 14, 2018 9:58 AM

IF $F(x,y) = \text{constant} \Rightarrow \text{constant}$

$\Rightarrow \frac{dF}{dx} = \frac{\partial F}{\partial y} = 0 \Rightarrow dF = 0$

The converse is also true (if $F, \frac{\partial F}{\partial x}, \frac{dF}{\partial y}$ are continuous) $dF = 0 \Longleftrightarrow F(x,y) = \text{constant}$

<u>Definition</u> A First order ODE :

$M(x,y)dx + N(x,y)dy = 0$

is called <u>exact</u> if there exists a Func. $F(x,y)$ such that $\frac{dF}{\partial x} = M(x,y)$ and $\frac{dF}{\partial y} = N(x,y)$

if the ODE is exact, then we can write it under the form :

$\frac{dF}{dx}dx + \frac{dF}{\partial y}dy = 0 \Rightarrow dF = 0 \Longleftrightarrow \boxed{F(x,y) = C}$

implicit solution to the ODE.

Given an exact ODE :

$M(x,y)\,dx + N(x,y)\,dy = 0$ Suppos it is exact

then $\frac{dF}{\partial x} = M$ and $\frac{dF}{dy} = N$ for some Func. $F(x,y)$

$$\left.\begin{aligned} &\text{So } \frac{\partial M}{dy} = \frac{d}{dy}\left(\frac{dF}{\partial x}\right) = \frac{d^2F}{\partial y dx} \\ &\text{and } \frac{dN}{dx} = \frac{d}{\partial x}\left(\frac{dF}{dy}\right) = \frac{d^2F}{\partial x dy} \end{aligned}\right\} \text{Same second derivative.}$$

But $\frac{d^2F}{dydx} = \frac{\partial^2 F}{\partial x dy}$ Provided F and its partial derivatives are continuous.

So if the ODE $Mdx + Ndy = 0$ is exact then $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$. The converse is also true.

Theorem : If $M(x,y)$ and $N(x,y)$ are contin. Func.'s and their First order partial derivatives are also continuous on a open rectangle R of $\mathbb{R}^2$, then the ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is <u>exact</u> if and only if $\left(\frac{\partial M}{\ } \quad \frac{\partial N}{\ } \right)$

are also continuous on a open rectangle R of $\mathbb{R}$,
then the ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is
exact if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Steps to solve exact ODEs:

① Verify the ODE is exact: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

② Find a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

③ The general solu. to the ODE is

$$\boxed{F(x,y) = C}$$

④ If initial condition is given we use it to find C.

Ex Solve the IVP

$$\underbrace{(4xy+1)}_{M}\,dx + \underbrace{(2x^2+\cos y)}_{N}\,dy = 0 \qquad y(0) = \pi/2$$

Sol

Check exactness:

$$\left.\begin{array}{l} \frac{\partial M}{\partial y} = 4x \\ \frac{\partial N}{\partial x} = 4x \end{array}\right\} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{Exact ODE}$$

Look for a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

$$\frac{\partial F}{\partial x} = 4xy + 1 \Rightarrow F(x,y) = \int (4xy+1)\,dx$$

$$= 4y\int x\,dx + \int 1\,dx$$

$$= 2x^2y + x + h(y)$$

constant with respect to x. i.e. does not contain a x in it

$$\frac{\partial F}{\partial y} = 2x^2 + h'(y) \qquad \text{But } \frac{\partial F}{\partial y} = N = 2x^2 + \cos y$$

So $\cancel{2x^2} + h'(y) = \cancel{2x^2} + \cos y$

$\Rightarrow h'(y) = \cos y$

$\Rightarrow h(y) = \int \cos y \, dy = \sin y + K$

So $F(x,y) = 2x^2 y + x + \sin y + K$

So the general sol. to the ODE is

$F(x,y) = \text{constant}$

$2x^2 y + x + \sin y + K = A$

$\boxed{2x^2 y + x + \sin y = C}$ implicit.

But $y(0) = \pi/2$

$0 + 0 + \sin(\tfrac{\pi}{2}) = C$

$C = 1$

The unique sol. is

$\boxed{2x^2 y + x + \sin y = 1}$

Ex 2 $(2xy^3 - 2y^2)\,dx + (3x^2y^2 - 4xy)\,dy = 0 \qquad y(1) = 1$

Sol Check Exactness: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\dfrac{\partial M}{\partial y} = 6xy^2 - 4y$

$\dfrac{\partial N}{\partial x} = 6xy^2 - 4y$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Exact ODE

Look for $F(x,y)$ such that $\dfrac{\partial F}{\partial x} = M$ and

$\dfrac{\partial F}{\partial y} = N$

Pushed out of integral

Pushed out

$$N = \frac{dF}{dy} = 3x^2y^2 - 4xy \Rightarrow F(x,y) = \int (3x^2 y^2 - 4xy)\, dy$$

$$= 3x^2 \frac{1}{3} y^3 - 4x \frac{1}{2} y^2 + h(x) = x^2y^3 - 2xy^2 + h(x)$$

So $\frac{dF}{dx} = 2xy^3 - 2y^2 + h'(x)$

But $\frac{dF}{dx} = M$ So

$$\cancel{2xy^3} - \cancel{2y^2} + h'(x) = \cancel{2xy^3} - \cancel{2y^2} \Rightarrow h'(x) = 0$$

$$\Rightarrow h(x) = K$$

So $F(x,y) = x^2y^3 - 2xy^2 + K$

General sol. is $F(x,y) = \text{constant}$

$$x^2y^3 - 2xy^2 = C$$

But $y(1) = 1$

$\therefore 1 - 2 = C \Rightarrow C = -1$

Unique sol. is : $\boxed{x^2y^3 - 2xy^2 = -1}$

(4) First Order ODE's with Integrating Factors

In many cases, our ODE is not exact but can be made Exact by multiplying it with a certain Function

Definition : A Function $\mu(x,y)$ is called an integrating factor of the first-order ODE:

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{iF}$$

$\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$ is Exact.

Finding an integrating Factor is a very hard task in general. However, we have some particular cases where an integrating factor exists:

Case 1: If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$ (a func. of y only)

Then an integrating Factor is given by

$$\mu(y) = e^{-\int g(y)\,dy}$$

Case 2: If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (Func. of x only)

then an integrating Factor is given by $\mu(x) = e^{\int f(x)\,dx}$

ex/ Solve the ODE:

$$\underbrace{(3x^2y + 2xy + y^3)}_{M}\,dx + \underbrace{(x^2+y^2)}_{N}\,dy = 0$$

Sol Check exactness:

$$\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2$$

$$\frac{\partial N}{\partial x} = 2x$$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ Not exact!

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x^2 + \cancel{2x} + 3y^2 - \cancel{2x}$$

$$= 3x^2 + 3y^2$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3(x^2+y^2)}{x^2+y^2} = 3$$

An integrating factor exists by

$$\mu(x) = e^{\int f(x)dx} = e^{\int 3dx} = e^{3x}$$

$$\underbrace{(3x^2y e^{3x} + 2xy e^{3x} + y^3 e^{3x})}_{M*}dx + \underbrace{(x^2 e^{3x} + y^2 e^{3x})}_{N*}dy$$

$$\frac{\partial M*}{\partial y} = 3x^2 e^{3x} + 2x e^{3x} + 3y^2 e^{3x}$$

$$\frac{\partial N*}{\partial x} = 2x e^{3x} + 3x^2 e^{3x} + 3y^2 e^{3x}$$

$\frac{\partial M*}{\partial y} = \frac{\partial N*}{\partial x}$ Exact

and we can solve