Chapter 1 First Order ODE's Lec 1 & 2

September 7, 2018 10:57 AM

In this chapter we look at ODE's of type $y' = F(x,y)$

ex/ $(x^2+1)y' - ye^x \sin y = 3 \Rightarrow y' = \dfrac{ye^x \sin x + 3}{x^2+1} = F(x,y)$

Replacing y' with $\dfrac{dy}{dx}$ in the ODE, we get the standard form of first order ODE.

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

ex/ $x^2 e^x - y' - 3x y' = \sin y \, e^y + 1$

$$y' = \frac{dy}{dx}$$

$$x^2 e^x - \frac{dy}{dx} - 3x\frac{dy}{dx} = \sin y \, e^y + 1$$

Multiply by dx:

$$x^2 e^x dx - dy - 3x\,dy = \sin y \, e^y dx + dx$$

Put dx together, Put dy together

$$\underbrace{\left(x^2 e^x - (\sin y)e^y - 1\right)}_{M(x,y)} dx + \underbrace{(3x-1)}_{N(x,y)} dy = 0$$

① Seperable First Order

Definition: A 1st order ODE is called separable if it can be written under the form $f(x)\,dx = g(y)\,dy$

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(terms in x on one side, terms in y on the other). To solve a separable ODE $f(x)\,dx = g(y)\,dy$

We integrate both sides.

ex/ Solve the following IVP:

$$yy' = xe^x(y^2+1) \quad ; \qquad y(0) = 0.$$

Sol $y' = \dfrac{dy}{dx}$

$$y\frac{dy}{dx} = xe^x(y^2+1) \Rightarrow \frac{y\,dy}{y^2+1} = xe^x\,dx \text{ : Seperable}$$

Integrate both sides :

$$\underbrace{\int \frac{y\,dy}{y^2+1}}_{\text{Subtitution}} = \underbrace{\int xe^x\,dx}_{\text{by Parts } u=x,\ v'=e^x}$$

$u = y^2+1$

$$\frac{1}{2}\ln(y^2+1) \cancel{+C_1} = xe^x - e^x \cancel{+C_2} + C_3$$

But $y(0) = 0 \Rightarrow \frac{1}{2}\ln(0^2+1) = 0e^0 - e^0 + C \Rightarrow$

$$\underbrace{\frac{1}{2}\ln 1}_{0} = -1 + C$$

$C = 1$

$$\frac{1}{2}\ln(y^2+1) = xe^x - e^x + 1 \quad \text{Implicit Solution}$$

$$y = \dots$$

ex/ Solve the following IVP:

$$xy^3 = \sqrt{1+x^2}\, y' \quad ; \quad y(0) = -1$$

Sol

$$y' = \frac{dy}{dx}$$

$$xy^3 = \sqrt{1+x^2}\,\frac{dy}{dx} \Rightarrow xy^3\,dx = \sqrt{1+x^2}\,dy$$

$$\Rightarrow \frac{x}{\sqrt{1+x^2}}\,dx = \frac{dy}{y^3} \quad ; \text{ Separable !} \quad (*)$$

Take integrals on both sides: $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{dy}{y^3}$

For $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$, use the substitution $u = 1+x^2$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{\cancel{x}}{\sqrt{u}}\,\frac{du}{\cancel{2x}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= \sqrt{u} = \sqrt{1+x^2}$$

Now $\displaystyle\int \frac{1}{y^3}\,dy = -\frac{1}{2}y^{-2} = \frac{-1}{2y^2}$

$(*)$:

Back to equation (*):

$$\sqrt{1+x^2} = -\frac{1}{2y^2} + C \qquad \text{note } y(0) = -1$$

$$\sqrt{1+0^2} = -\frac{1}{2(-1)^2} \Rightarrow 1 = -\frac{1}{2} + C \Rightarrow C = \frac{3}{2}$$

So $\sqrt{1+x^2} = -\frac{1}{2y^2} + \frac{3}{2}$ (implicit solution)

$$\sqrt{1+x^2} - \frac{3}{2} = -\frac{1}{2y^2} \Rightarrow -2\sqrt{1+x^2} + 3 = \frac{1}{y^2} \Rightarrow y^2 = \frac{1}{-2\sqrt{1+x^2}+3}$$

$$y = \pm\sqrt{\frac{1}{-2\sqrt{1+x^2}+3}}$$

To satisfy the initial condition of $y(0) = -1$ the $-$ is the explicit solution is $\left[\; y = -\sqrt{\frac{1}{-2\sqrt{1+x^2}+3}}\right.$

② Homogeneous First-order ODE's

Some 1^{st} order ODE's become seperable after a change of the function!

An example of such ODE's are the homogeneous ones.

Def: A function $F(x,y)$ is called homogeneous of degree k if

$$F(\lambda x, \lambda y) = \lambda^k F(x,y)$$

Ex/ $f(x,y) = x^2 + 3xy + 2y^2$

$$F(\lambda x, \lambda y) = (\lambda x)^2 + 3(\lambda x)(\lambda y) + 2(\lambda y)^2$$

$$= \lambda^2 x^2 + 3\lambda^2 xy + \lambda^2 2y^2$$

$$= \lambda^2 x^2 + 3\lambda^2 xy + \lambda^2 2y^2$$

$$= \lambda^2 (x^2 + 3xy + 2y^2) = \lambda^2 F(x,y)$$

$$F(\lambda x, \lambda y) = \lambda^2 F(x,y)$$

$F(x,y)$ is homogeneous of degree 2.

note: in polynomial functions every term must be of degree k in order for it to be homogeneous.

Ex/ $F(x,y) = 3x^2y + x^3 - 3xy^2 - y^3$

homo of degree 3

Ex/ $F(x,y) = 2x - y\cos(y/x)$

$$F(\lambda x, \lambda y) = 2\lambda x - \lambda y \cos\left(\frac{\lambda y}{\lambda x}\right) = 2\lambda x - \lambda y \cos(y/x)$$

$$F(\lambda x, \lambda y) = \lambda \left[2x - y\cos\left(\frac{y}{x}\right)\right] = \lambda F(x,y)$$

$F(x,y)$ = homogeneous of degree 1

Ex/ $F(x,y) = 4x^3y - 2x^2y^2 + y^4 - 2$ — degree 0 not homo.

Def 1^{st} order ODE:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called <u>homogeneous</u> of degree k if both $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree k.

Homogeneous 1^{st} order ODE's can be made seperable by

Homogeneous 1st order ODE's can be made seperable by changing the func. : $u = \frac{y}{x}$ or $U = \frac{x}{y}$

ex/ Solve the IVP: $(3x^2 + 2y^2)\,dx + 4xy\,dy = 0$

$x > 0$, $y(1) = 1$

$$\underbrace{(3x^2 + 2y^2)}_{M}\,dx + \underbrace{4xy}_{N}\,dy = 0$$

Sol: Both are homogeneous of the same degree 2 so our ODE is homo.

Let $u = \frac{y}{x} \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

$\Rightarrow dy = u\,dx + x\,du$

Back to the ODE

$$(3x^2 + 2x^2u^2)\,dx + 4x^2u\,[u\,dx + x\,du] = 0$$

Divide everything by x^2

$$(3 + 2u^2)\,dx + 4u^2\,dx + 4xu\,du = 0$$

$\Rightarrow (3 + 6u^2)\,dx = -4xu\,du \Rightarrow \frac{1}{x}\,dx = \frac{-4u\,du}{3 + 6u^2}$ Seperable

$$\int \frac{1}{x}\,dx = \int \frac{-4u\,du}{3 + 6u^2} \quad *$$

$$\int \frac{-4u\,du}{3 + 6u^2}$$

let $t = 3 + 6u^2$

$dt = 12u\,du \Rightarrow \frac{dt}{du} = 12u$

$$\int\frac{}{3+6u^2}$$

$dt = 12u\,du \Rightarrow \frac{dt}{du} = 12u$

$\Rightarrow du = \frac{dt}{12u}$

So $$\int\frac{-4u\,du}{3+6u^2} = \int\frac{-4\cancel{u}\,\frac{dt}{12\cancel{u}}}{t} = -\frac{1}{3}\int\frac{dt}{t} = -\frac{1}{3}\ln(t)$$

$*$ becomes: $\ln x = -\frac{1}{3}\ln(3+6u^2) + C$

But: when $x=1$, $y=1 \Rightarrow u = \frac{y}{x} = \frac{1}{1} = 1 \Rightarrow u(1) = 1$

$$\ln(1) = -\frac{1}{3}\ln(3+6(1)^2) + C$$

$$0 = -\frac{1}{3}\ln(9) + C$$

$$C = \frac{1}{3}\ln(9)$$

$$\ln x = -\frac{1}{3}\ln(3+6u^2) + \frac{1}{3}\ln(9)$$

$$\ln x = \frac{1}{3}\ln\left(\frac{9}{3+6u^2}\right)$$

$$\ln x = \ln\left(\frac{9}{3+6u^2}\right)^{1/3} = \ln\sqrt[3]{\frac{9}{3+6u^2}}$$

$$x = \sqrt[3]{\frac{9}{3+6u^2}}$$

$$x^3 = \frac{9}{3+6u^2} \Rightarrow 6u^2 = \frac{9}{x^3} - 3$$

$$2u^2 = \frac{3}{x^3} - 1$$

$$u^2 = \frac{3}{2x^3} - \frac{1}{2} = \frac{3-x^3}{2x^3}$$

$$\left(\frac{y}{x}\right)^2 = \frac{3-x^3}{2x^3}$$

$$y^2 = \frac{3-x^3}{2x}$$

Because the IVP is $y(1) = 1$ then

$y = \sqrt{\frac{3-x^3}{}}$ Explicit Solution

$2x$

Ex Solve the IVP:

$$xy' = y + \frac{2x}{\sin\left(\frac{y}{x}\right)} \quad ; \quad x > 0 \quad , \quad y(1) = \pi$$

Sol $y' = \frac{dy}{dx}$

$$x\frac{dy}{dx} = y + \frac{2x}{\sin\left(\frac{y}{x}\right)} \Rightarrow -x\,dy + y\,dx + \frac{2x}{\sin\left(\frac{y}{x}\right)}dx = 0$$

$$\Rightarrow -x\sin\left(\frac{y}{x}\right)dy + y\sin\left(\frac{y}{x}\right)dx + 2x\,dx = 0$$

$$\Rightarrow \underbrace{\left(2x + y\sin\left(\frac{y}{x}\right)\right)dx}_{M(x,y)} \underbrace{- x\sin\left(\frac{y}{x}\right)dy}_{N(x,y)} = 0$$

Both $M(x,y)$ and $N(x,y)$ are homo. of deg. 1

Let $u = \frac{y}{x} \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx} \Rightarrow dy = u\,dx + x\,du$

Back to the ODE.

$$(2x + xu\sin(u))\,dx - x\sin(u)\left[u\,dx + x\,du\right] = 0$$

$$2x\,dx + \cancel{xu\sin u\,dx} - \cancel{xu\sin u\,dx} - x^2\sin u\,du = 0$$

$2x\,dx - x^2\sin u\,du = 0$ Divide by x:

$$2\,dx - x\sin u\,du = 0 \Rightarrow 2\,dx = x\sin u\,du$$

$2\frac{1}{x}\,dx = \sin u\,du$ Separable!

$$2\int \frac{1}{x}\,dx = \int \sin u\,du$$

note $y(1) = \pi \Rightarrow u(1) = \frac{y(1)}{1} = \pi$

$$2\ln x = -\cos u + C$$

$2\ln 1 = -\cos(\pi) + C$

$0 = 1 + C$

$C = -1$

So $2\ln x = -\cos u - 1$

$\cos u = -2\ln x - 1$

$u = \cos^{-1}(-2\ln x - 1)$

$\frac{y}{x} = \cos^{-1}(-2\ln x - 1) \Rightarrow \boxed{y = x\cos^{-1}(-2\ln x - 1)}$

Explicit!

③ Exact First Order

Def: If $F(x,y)$ is a function of 2 variables, then the differential of F, denoted by dF, is defined as,

$$\boxed{dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy}$$

Ex/ $F(x,y) = x^2y + 3xy^2$

$\frac{\partial F}{\partial x} = 2xy + 3y^2$ $\qquad$ $\frac{\partial F}{\partial y} = x^2 + 6xy$

$dF = (2xy + 3y^2)dx + (x^2 + 6xy)dy$