Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Calculus III for Engineers
MAT 2322A - Fall 2017
Final Exam
Professor: Victor G. LeBlanc
Time limit: 3 hours. Closed books.

Name: Solutions ID Number:________________

Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

(Signature): ________________________

Instructions

- This is a 15 pages closed book exam. **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Questions 1 to 10 are multiple choice questions. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
- Questions 11 to 16 are long answer questions and are worth 5 marks each, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages if necessary.

Answers to multiple choice Qs

1	2	3	4	5	6	7	8	9	10
E	F	A	A	A	B	D	B	A	C

Grid below is used for grading
(do not write in this grid)

MCQ	11	12	13	14	15	16	Total
/20	/5	/5	/5	/5	/5	/5	/50

1. A wire has the shape of the parametrized curve $\vec{r}(t) = t^2\,\vec{i} + t\,\vec{j} + 3t\,\vec{k}$, $0 \leq t \leq 2$, and has a mass-density given by $f(x, y, z) = y + z$. The total mass of the wire is

A. $\frac{1}{3}(22\sqrt{22} - 6\sqrt{6})$

B. $\frac{1}{3}(23\sqrt{23} - 7\sqrt{7})$

C. $\frac{1}{3}(24\sqrt{24} - 8\sqrt{8})$

D. $\frac{1}{3}(25\sqrt{25} - 9\sqrt{9})$

E. $\frac{1}{3}(26\sqrt{26} - 10\sqrt{10})$

F. $\frac{1}{3}(27\sqrt{27} - 11\sqrt{11})$

$\vec{r}\,'(t) = 2t\vec{i} + \vec{j} + 3\vec{k}$ $\quad \|\vec{r}\,'(t)\| = \sqrt{4t^2+10}$

$f(\vec{r}(t)) = t + 3t = 4t.$ Mass $= \int_0^2 f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt =$

$\int_0^2 4t\sqrt{4t^2+10}\,dt$ $\quad u = 4t^2+10 \quad du = 8t\,dt$

$\int 4t\sqrt{4t^2+10}\,dt = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}u^{3/2}\cdot\frac{2}{3} + C$

$\Rightarrow$ Mass $= \frac{1}{3}(4t^2+10)^{3/2}\Big|_0^2$

$= \frac{1}{3}\left[26^{3/2} - 10^{3/2}\right]$

2. A thin sheet S has the form of the portion of the cone $z = \sqrt{x^2 + y^2}$ which lies bewteen $z = 1$ and $z = 2$, in the octant $x \geq 0$, $y \geq 0$ and $z \geq 0$, illustrated below. The mass-density of this sheet is given by $f(x, y, z) = x$. The total mass of this sheet is

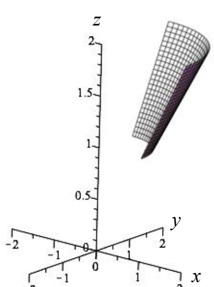
Mass $= \sqrt{2}\int_0^{\pi/2}\int_1^2 a\cos\theta\cdot a\,da\,d\theta$

$\vec{r}(a,\theta) = a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j} + a\,\vec{k}$

$1 \leq a \leq 2, \quad 0 \leq \theta \leq \pi/2$

$\vec{r}_a = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + \vec{k}$

$\vec{r}_\theta = -a\sin\theta\,\vec{i} + a\cos\theta\,\vec{j}$



$\vec{r}_a \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -a\sin\theta & a\cos\theta & 0 \end{vmatrix}$

$= -a\cos\theta\,\vec{i} - a\sin\theta\,\vec{j} + a(\cos^2\theta + \sin^2\theta)\,\vec{k}$

$\|\vec{r}_a \times \vec{r}_\theta\| = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta + a^2} = \sqrt{2}\,a$

A. $\frac{2\sqrt{2}}{3}$

B. $\sqrt{2}$

C. $\frac{4\sqrt{2}}{3}$

D. $\frac{5\sqrt{2}}{3}$

E. $2\sqrt{2}$

F. $\frac{7\sqrt{2}}{3}$

$= \sqrt{2}\int_0^{\pi/2} \frac{a^3}{3}\cos\theta\Big|_1^2\,d\theta$

$= \sqrt{2}\int_0^{\pi/2} \frac{7}{3}\cos\theta\,d\theta$

$= \frac{7\sqrt{2}}{3}\sin\theta\Big|_0^{\pi/2}$

$= 7\sqrt{2}/3$

3. What is the value of the directional derivative of the function $f(x, y) = xy^2$ at the point $(2, 3)$ in the direction of the vector $\vec{u} = \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$?

$f_x = y^2 \quad f_y = 2xy$

$f_x(2,3) = 9 \quad f_y(2,3) = 12$

$D_{\vec{u}} f(2,3) = (9\vec{i} + 12\vec{j}) \cdot \left(\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}\right) = \frac{9\sqrt{3}+12}{2}$

A. $\dfrac{9\sqrt{3} + 12}{2}$

B. $9\vec{i} + 12\vec{j}$

C. $\dfrac{9\sqrt{3}}{2}\vec{i} + 6\vec{j}$

D. 0

E. π

F. 21

$f_x = 2x$, $f_y = -2y \Rightarrow (0,0)$ only critical pt. in D. On boundary $g = x^2 + y^2 = 1$

$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow 2x = 2\lambda x$ & $-2y = 2\lambda y$

$2x(1-\lambda) = 0 \quad 2y(1+\lambda) = 0$

$x = 0$ or $\lambda = 1$

$x = 0 \Rightarrow y = \pm 1$: $(0,1), (0,-1)$

$\lambda = 1 \Rightarrow y = 0 \Rightarrow (x = \pm 1)$: $(1,0), (-1,0)$

(x,y)	$f(x,y)$	
$(0,0)$	2	
$(1,0)$	3	max
$(-1,0)$	3	max
$(0,1)$	1	min
$(0,-1)$	1	min

4. Which of the following statements is true concerning the global extrema of the function $f(x, y) = x^2 - y^2 + 2$ on the set $D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$?

A. f has a global maximum value of 3 at the points $(1, 0)$ and $(-1, 0)$ and a global minimum value of 1 at the points $(0, 1)$ and $(0, -1)$.

B. f has a global maximum value of 3 at the points $(1, 0)$ and $(-1, 0)$ and a global minimum value of 2 at the point $(0, 0)$.

C. f has no global extrema on this set, but has a saddle point at $(0, 0)$.

D. f has a global maximum value of 3 at $(1, 0)$ and no global minimum.

E. f has a global minimum value of 1 at the points $(0, 1)$ and $(0, -1)$, but no global maximum value.

F. f has both a global minimum and a global maximum at $(0, 0)$, i.e. f is constant on the set D.

5. Let C be the portion of the circle $x^2 + y^2 = 1$ which lies in the first quadrant ($x \geq 0$, $y \geq 0$), oriented counterclockwise. Let $\vec{F}(x, y) = xy\,\vec{\imath}$. What is the value of the line integral

$$\int_C \vec{F} \cdot d\vec{r}\ ?$$

$\vec{r}(t) = \cos t\,\vec{\imath} + \sin t\,\vec{\jmath} \quad 0 \le t \le \pi/2$

$\vec{r}\,'(t) = -\sin t\,\vec{\imath} + \cos t\,\vec{\jmath}$

$\vec{F}(\vec{r}(t)) = \cos t \sin t\,\vec{\imath}$

A. $-\frac{1}{3}$

B. $-\frac{4}{3}$

C. $-\frac{7}{3}$

D. $-\frac{10}{3}$

E. $-\frac{13}{3}$

F. $-\frac{16}{3}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{\pi/2} (\cos t \sin t\,\vec{\imath}) \cdot (-\sin t\,\vec{\imath} + \cos t\,\vec{\jmath})\,dt$$

$$= \int_0^{\pi/2} -\cos t \sin^2 t\,dt \qquad u = \sin t \qquad du = \cos t\,dt$$

$$-\int \cos t \sin^2 t\,dt = -\int u^2\,du = -\frac{u^3}{3} \Rightarrow -\frac{\sin^3 t}{3}\Big|_0^{\pi/2}$$

$$= -\frac{1}{3} + 0 = -\frac{1}{3}$$

6. Consider the composite function $g(s, t) = f(x(s, t), y(s, t))$, where all functions are differentiable. Considering the following data:

$$\vec{\nabla} f(1, 3) = \vec{\imath} - \vec{\jmath}, \quad \vec{\nabla} f(2, 5) = 2\vec{\imath} + \vec{\jmath}, \quad x(1, 3) = 2, \quad y(1, 3) = 5, \quad \frac{\partial x}{\partial s}(1, 3) = -1, \quad \frac{\partial x}{\partial t}(1, 3) = 4$$

$$\frac{\partial y}{\partial s}(1, 3) = -2, \quad \frac{\partial y}{\partial t}(1, 3) = 1.$$

What is the value of the partial derivative $\frac{\partial g}{\partial s}(1, 3)$?

A. -5

B. -4

C. -3

D. -2

E. -1

F. 0

$$\frac{\partial g}{\partial s}(1,3) = \frac{\partial f}{\partial x}(x(1,3), y(1,3))\frac{\partial x}{\partial s}(1,3) + \frac{\partial f}{\partial y}(x(1,3), y(1,3))\frac{\partial y}{\partial s}(1,3)$$

$$= \frac{\partial f}{\partial x}(2,5)\frac{\partial x}{\partial s}(1,3) + \frac{\partial f}{\partial y}(2,5)\frac{\partial y}{\partial s}(1,3)$$

$$= 2 \cdot (-1) + 1 \cdot (-2) = -4$$

7. The double integral $\int_0^1 \int_1^{e^x} f(x,y)\,dy\,dx$ describes integration over a region which is described as a type I region. Which of the integrals below describes the same integral, but with the region of integration expressed as a type II region?

A. $\int_0^1 \int_1^{e^x} f(x,y)\,dx\,dy$

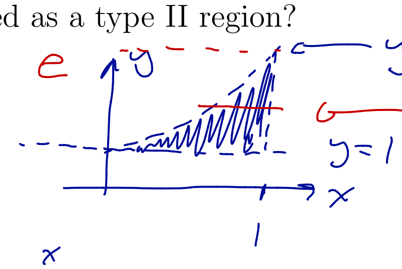
B. $\int_1^{e^x} \int_0^1 f(x,y)\,dx\,dy$

C. $\int_0^1 \int_{\ln y}^1 f(x,y)\,dx\,dy$

D. $\int_1^e \int_{\ln y}^1 f(x,y)\,dx\,dy$

E. $\int_1^e \int_1^{\ln y} f(x,y)\,dx\,dy$

F. None of the above



$y = e^x \Rightarrow x = \ln y$

$\ln y \le x \le 1$

$1 \le y \le e$

$\int_0^1 \int_1^{e^x} f(x,y)\,dy\,dx = \int_1^e \int_{\ln y}^1 f(x,y)\,dx\,dy$

8. Of the vector fields below, only one is such that there exists a scalar function $f(x,y,z)$ with $\vec{F} = \vec{\nabla} f(x,y,z)$. Which one?

A. $\vec{F}(x,y,z) = z\,\vec{i} - x\,\vec{k}$

B. $\vec{F}(x,y,z) = x^2\,\vec{i} + y^3\,\vec{j} + z^4\,\vec{k}$

C. $\vec{F}(x,y,z) = y\,\vec{i} - x\,\vec{j}$

D. $\vec{F}(x,y,z) = z\,\vec{j} - y\,\vec{k}$

E. $\vec{F}(x,y,z) = x\,\vec{i} - z\,\vec{j} + y^2\,\vec{k}$

F. $\vec{F}(x,y,z) = xyz\,\vec{i}$

B. is the only one for which $\vec{\nabla} \times \vec{F} = \vec{0}$. So the answer is B.

9. Consider the function $f(x,y) = y^3x + 2xy^2 - x + y$. Which of the following expressions corresponds to the tangent plane to the graph of $z = f(x,y)$ at the point $(x,y,z) = (2,1,f(2,1))$?

A. $z = 2(x-2) + 15(y-1) + 5$

B. $z = 2\vec{i} + 15\vec{j}$

C. This function is not differentiable at the indicated point, so the tangent plane does not exist.

D. $z = 2x + 15y$

E. $z = (y^3 + 2y^2 - 1)(x-2) + (3y^2x + 4xy + 1)(y-1) + 5$

F. $z = 5$

$f_x = y^3 + 2y^2 - 1$ $\quad f_y = 3y^2x + 4xy + 1$ $\quad f_x(2,1) = 2$ $\quad f_y(2,1) = 15$

$f(2,1) = 5 \Rightarrow$ $z = 2(x-2) + 15(y-1) + 5$

10. Let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = -y\vec{i} + x\vec{j} + z\vec{k}$, and let S be the cylinder $x^2 + y^2 = 4$, between $z = 0$ and $z = 4$, oriented away from the z-axis. Then the value of the flux integral $\iint_S \vec{F} \cdot d\vec{S}$ is

$\hookrightarrow = \int_0^{2\pi}\int_0^4 \vec{F}(\vec{r}(\theta,z)) \cdot (\vec{r}_\theta \times \vec{r}_z)\, dz\, d\theta$

$= \int_0^{2\pi}\int_0^4 0 \, dz\, d\theta = 0$

$\vec{r}(\theta, z) = 2\cos\theta\,\vec{i} + 2\sin\theta\,\vec{j} + z\vec{k}$

$0 \le \theta \le 2\pi \quad 0 \le z \le 4$

$\vec{r}_\theta = -2\sin\theta\,\vec{i} + 2\cos\theta\,\vec{j}$

$\vec{r}_z = \vec{k}$

$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

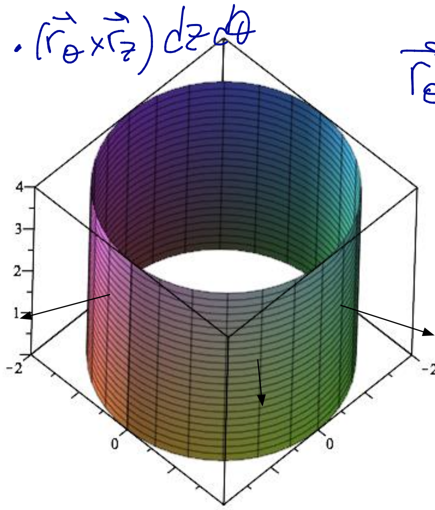
$= 2\cos\theta\,\vec{i} + 2\sin\theta\,\vec{j}$

$\vec{F}(\vec{r}(\theta,z)) = -2\sin\theta\,\vec{i} + 2\cos\theta\,\vec{j} + z\vec{k}$

$\vec{F}(\vec{r}(\theta,z)) \cdot (\vec{r}_\theta \times \vec{r}_z) =$

$-4\cos\theta\sin\theta + 4\cos\theta\sin\theta$

$= 0$



A. 2π

B. π

C. 0

D. $-\pi$

E. -2π

F. -3π

11. Find and classify the critical points of the function $f(x, y) = x^3 + xy + 2y^2$.

$f_x = 3x^2 + y$ $\quad f_y = x + 4y$

Critical points are solutions to

$3x^2 + y = 0$

$x + 4y = 0 \Rightarrow x = -4y$

$3(-4y)^2 + y = 0$

$48y^2 + y = 0$

$y(48y + 1) = 0$

$\Rightarrow y = 0$ or $y = -\frac{1}{48}$

$x = -4y \Rightarrow$ Critical points are

$(0,0)$ and $(\frac{1}{12}, -\frac{1}{48})$

$f_{xx} = 6x$

$f_{xy} = 1$

$f_{yy} = 4$

$D = f_{xx} f_{yy} - f_{xy}^2$

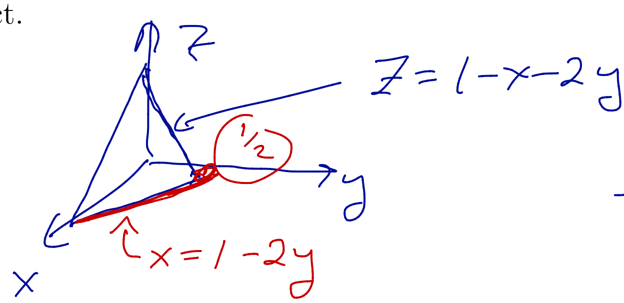
$= 24x - 1$

$(0,0)$: $D = -1 < 0 \Rightarrow$ **$(0,0)$ is a saddle**

$(\frac{1}{12}, -\frac{1}{48})$: $D = 1 > 0$

$f_{xx} = \frac{1}{2} > 0 \Rightarrow$ **$(\frac{1}{12}, -\frac{1}{48})$ local minimum**

12. A solid object is in the first octant, bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $z = 1 - x - 2y$. This object has mass-density given by $f(x, y, z) = 96\,y$. Find the total mass of this object.



The solid is described by the inequalities

$0 \le z \le 1-x-2y$

$0 \le x \le 1-2y$

$0 \le y \le \frac{1}{2}$

$$\Rightarrow \text{Mass} = \int_0^{1/2}\int_0^{1-2y}\int_0^{1-x-2y} 96y \, dz\, dx\, dy$$

$$= \int_0^{1/2}\int_0^{1-2y} 96yz \Big|_0^{1-x-2y} dx\, dy = \int_0^{\frac{1}{2}}\int_0^{1-2y} 96y\,(1-x-2y)\, dx\, dy$$

$$= 96\int_0^{1/2}\int_0^{1-2y} (y - xy - 2y^2)\, dx\, dy = 96\int_0^{1/2} xy - \frac{x^2 y}{2} - 2xy^2 \Big|_0^{1-2y} dy =$$

$$96\int_0^{1/2}\left[y(1-2y) - y\frac{(1-2y)^2}{2} - 2y^2(1-2y)\right] dy = 96\int_0^{1/2} \left(2y^3 - 2y^2 + \frac{1}{2}y\right) dy =$$

$$96\left[\frac{y^4}{2} - \frac{2y^3}{3} + \frac{y^2}{4}\Big|_0^{\frac{1}{2}}\right] = 96\left[\frac{1}{32} - \frac{1}{12} + \frac{1}{16}\right]$$

$$= 96\left[\frac{3 - 8 + 6}{96}\right] = 1$$

13. Consider the vector field $\vec{F}(x,y,z) = (yz+x)\,\vec{i} + (xz^2 - y)\,\vec{j} + z^2\,\vec{k}$.

(a) Compute the divergence and the curl of $\vec{F}$, i.e. compute $\vec{\nabla} \cdot \vec{F}$ and $\vec{\nabla} \times \vec{F}$.

(b) Let E denote the three-dimensional solid brick region defined by $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$, and let S denote the surface which is the boundary of E (i.e. S is the "skin" of E), oriented outwards. Compute the flux integral $\iint_S \vec{F} \cdot d\vec{S}$ using Gauss' Divergence theorem.

a) $$\vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left((yz+x)\vec{i} + (xz^2 - y)\vec{j} + z^2\vec{k}\right) =$$
$$1 - 1 + 2z = 2z$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+x & xz^2 - y & z^2 \end{vmatrix} = \vec{i}(-2xz) - \vec{j}(-y) + \vec{k}(z^2 - z)$$
$$= -2xz\vec{i} + y\vec{j} + (z^2 - z)\vec{k}$$

b) Gauss' Divergence Theorem: $$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \vec{\nabla} \cdot \vec{F}\, dV$$

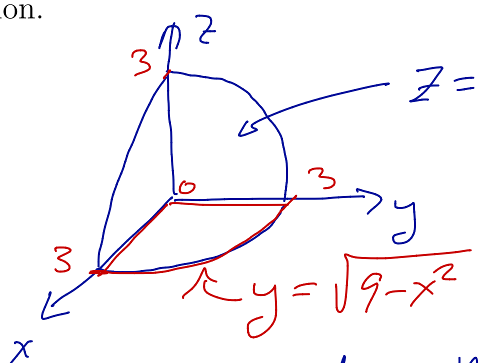
$$= \int_0^3 \int_0^2 \int_0^1 2z\, dx\, dy\, dz = \int_0^3 \int_0^2 2xz \Big|_0^1 dy\, dz = \int_0^3 \int_0^2 2z\, dy\, dz =$$

$$\int_0^3 2yz \Big|_0^2 dz = \int_0^3 4z\, dz = 2z^2 \Big|_0^3 = 18$$

14. Compute the triple integral

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} (x^2+y^2+z^2)^2\, dz\, dy\, dx.$$

Hint: Cartesian coordinates may not be the most appropriate choice of coordinates to do this computation.



(sphere of radius 3)

In spherical coordinates, the region of integration is described by $0 \le \rho \le 3$, $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$.

The integrand is $(x^2+y^2+z^2)^2 = (\rho^2)^2 = \rho^4$, and $dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$. So

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} (x^2+y^2+z^2)^2\, dz\, dy\, dx = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^4 \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta =$$

$$\int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^7}{7} \sin\phi \Big|_0^3 d\phi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/2} \frac{3^7}{7}\sin\phi\, d\phi\, d\theta = \int_0^{\pi/2} -\frac{3^7}{7}\cos\phi\Big|_0^{\pi/2} d\theta$$

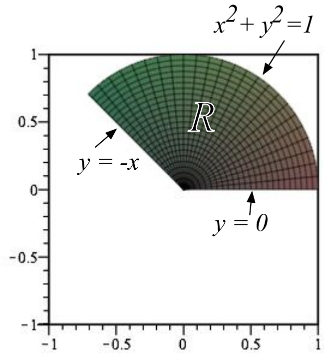
$$= \int_0^{\pi/2} \frac{3^7}{7}\, d\theta = \frac{3^7 \pi}{14} = \frac{2187}{14}\pi$$

15. Compute $\iint_R \sin(x^2+y^2)\,dA$, where R is the two-dimensional region sketched below.

In polar coordinates, R is described as

$0 \le r \le 1$

$0 \le \theta \le 3\pi/4$



$$\Rightarrow \iint_R \sin(x^2+y^2)\,dA = \int_0^{3\pi/4}\int_0^1 \sin(r^2)\cdot r\,dr\,d\theta$$

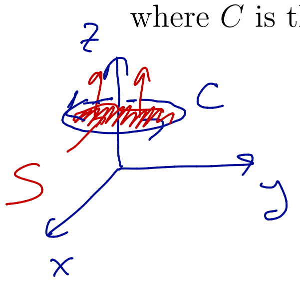
$$= \int_0^{3\pi/4} -\tfrac{1}{2}\cos(r^2)\Big|_0^1\,d\theta = \int_0^{3\pi/4}\left(-\tfrac{1}{2}\cos(1)+\tfrac{1}{2}\right)d\theta$$

$$= \boxed{\frac{3\pi}{8}(1-\cos(1))}$$

16. A vector field $\vec{F}$ is such that its curl is given by $\vec{\nabla} \times \vec{F} = 2z\,\vec{k}$. Using Stokes' theorem, compute the line integral

$$\oint_C \vec{F} \cdot d\vec{r}$$

where C is the oriented closed curve $\vec{r}(t) = 2\cos(t)\,\vec{i} + 2\sin(t)\,\vec{j} + 4\,\vec{k}$, $0 \le t \le 2\pi$.



C is the boundary of the oriented surface S on the diagram on the left:

$$\vec{r}(a,\theta) = a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j} + 4\vec{k} \qquad 0 \le a \le 2,\quad 0 \le \theta \le 2\pi$$

$$\vec{r}_a = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}, \quad \vec{r}_\theta = -a\sin\theta\,\vec{i} + a\cos\theta\,\vec{j}$$

$$\vec{r}_a \times \vec{r}_\theta = a\vec{k}.$$

Stokes' theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{dS} = \int_0^{2\pi}\int_0^2 (2 \cdot 4 \cdot \vec{k}) \cdot (a\vec{k})\, da\, d\theta =$$

$(\vec{\nabla} \times \vec{F}) = 2z\vec{k}$, but $z = 4$ on S.

$$\int_0^{2\pi}\int_0^2 8a\, da\, d\theta = \int_0^{2\pi} 4a^2\Big|_0^2\, d\theta = \int_0^{2\pi} 16\, d\theta = 16 \cdot 2\pi$$

$$= \boxed{32\pi}$$

(Extra page)

(Extra page)

(Extra page)