MAT 1320

Winter 2018

Midterm #1

Short Answer (1 point each):

1. Simplify. Use log rules.

$$e^{5\ln x} = e^{\ln\left(x^5\right)} = x^5$$

2. Simplify. Note that they must give the restriction on *x* because the function "arc sin of sin" is only defined in between those *x* values.

$$\sin^{-1}\left(\sin x\right) - x + 10 = x - x + 10 = 10$$

3. **False**. The function $|x+1|$ is ***not*** differential at *x = -1*, where there is a sharp "corner". Functions are not differentiable at corners.

4. Find the inverse.

$$y = \sin^{-1}\left(10x - 1\right)$$

Swap x and y. Solve for y.

$$x = \sin^{-1}\left(10y - 1\right)$$
$$\sin x = \sin\left[\sin^{-1}\left(10y - 1\right)\right]$$
$$= 10y - 1$$
$$10y = \sin x + 1$$
$$y = \frac{\sin x + 1}{10}$$
$$f^{-1}\left(x\right) = \frac{\sin x + 1}{10}$$

5. For the function to be continuous at $x = 0$, the two functions in the piecewise function must be equal at $x = 0$.

   Let's check!

   First function:
   $$\frac{0+10}{0+5} = \frac{10}{5} = 2$$

   Second function:
   Always equals 2.

   The two functions are equal at $x = 0$.

   Therefore, **yes**, the function is continuous at $x = 0$.

6. We are given position. To find velocity, take the derivative. Then, evaluate the velocity at $t = 2$.

   $$f'(t) = 4t^3 - 4$$
   $$f'(2) = 4(2)^3 - 4 = 28\frac{ft}{\text{sec}}$$

   (required units were given in the problem statement)

Long Answer:

7. (3 points) Definition of the derivative. Sigh.

$$f(x) = \sqrt{x-1}$$
$$f(x+h) = \sqrt{x+h-1}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h-1}-\sqrt{x-1}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h-1}-\sqrt{x-1}}{h}\,\frac{\sqrt{x+h-1}+\sqrt{x-1}}{\sqrt{x+h-1}+\sqrt{x-1}} \\ &= \lim_{h\to 0}\frac{\left(\sqrt{x+h-1}-\sqrt{x-1}\right)\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)} \end{aligned}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h-1}\right)\left(\sqrt{x+h-1}\right)+\left(\sqrt{x+h-1}\right)\left(\sqrt{x-1}\right)-\left(\sqrt{x-1}\right)\left(\sqrt{x+h-1}\right)-\left(\sqrt{x-1}\right)\left(\sqrt{x-1}\right)}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h-1}\right)\left(\sqrt{x+h-1}\right)+\cancel{\left(\sqrt{x+h-1}\right)\left(\sqrt{x-1}\right)}-\cancel{\left(\sqrt{x-1}\right)\left(\sqrt{x+h-1}\right)}-\left(\sqrt{x-1}\right)\left(\sqrt{x-1}\right)}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h-1}\right)\left(\sqrt{x+h-1}\right)-\left(\sqrt{x-1}\right)\left(\sqrt{x-1}\right)}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{(x+h-1)-(x-1)}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{(\cancel{x}+h\cancel{-1})-(\cancel{x}\cancel{-1})}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-1}+\sqrt{x-1}}$$

$$= \frac{1}{\sqrt{x+0-1}+\sqrt{x-1}}$$

$$= \frac{1}{2\sqrt{x-1}}$$

You can always check your answer to “definition of the derivative” questions by taking the derivative using one of the conventional methods. In this case, the chain rule would work. However, if you only evaluate the derivative using the chain rule, you will get 0 marks.

8. Limits!
   a. Rational polynomial.

$$\lim_{x\to\infty}\frac{x\left(\frac{\pi}{x}+3\right)}{x\left(\pi-\frac{10}{x}\right)}=\lim_{x\to\infty}\frac{\cancel{x}\left(\frac{\pi}{x}+3\right)}{\cancel{x}\left(\pi-\frac{10}{x}\right)}$$

$$=\lim_{x\to\infty}\frac{\frac{\pi}{x}+3}{\pi-\frac{10}{x}}$$

$$=\frac{0+3}{\pi-0}$$

$$=\frac{3}{\pi}$$

   b. Factoring.

$$\lim_{x\to-2^-}\frac{x-2}{(x+2)(x-2)}=\lim_{x\to-2^-}\frac{\cancel{x-2}}{(x+2)\cancel{(x-2)}}$$

$$=\lim_{x\to-2^-}\frac{1}{x+2}$$

We are asked to evaluate $\lim\limits_{x\to-2^-}$, which means -2 *from the left*. Yeah… numbers have sides…

To do that, we want to plug in a number that is "to the left" of -2 (i.e. larger in the negative direction) into this limit. Without a calculator, this is a bit difficult. With a calculator, this is pretty simple.

I'll do this without a calculator (assuming the worst case scenario on an exam).

Try -2.1. Plug this into the limit, and we get $\frac{1}{-0.1}$, or 1 divided by a **small negative** number.

Try -2.01. Plug this into the limit, and we get $\frac{1}{-0.01}$, or 1 divided by a **smaller negative** number.

As the values we plug in get closer to -2 from the left, the number on the bottom gets **smaller and smaller**. It is negative. Eventually, it would basically go to "negative zero". (Sounds weird, I know. It's like you're approaching zero from the negative side of numbers.)

Hopefully you have understand from class that the following is true: $\lim_{x\to 0}\frac{1}{x}=\infty$. For us, we have a similar case, except our number is tending towards a small, **negative** value. Therefore, the limit is $-\infty$.

9. The problem statement says **do not simplify**. Sweeeeeet.

a) Product rule.

$$f(x)=5e^x$$
$$g(x)=x^2+1$$
$$f'(x)=5e^x$$
$$g'(x)=2x$$

$$\begin{aligned}h'(x)&=f'(x)g(x)+f(x)g'(x)\\&=(5e^x)(x^2+1)+(5e^x)(2x)\end{aligned}$$

b) Quotient rule.

$$f(x)=x^2+3x+1$$
$$g(x)=x+1$$
$$f'(x)=2x+3$$
$$g'(x)=1$$

$$\begin{aligned}y'&\equiv\frac{dy}{dx}\\&=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}\\&=\frac{(2x+3)(x+1)-(x^2+3x+1)(1)}{(x+1)^2}\end{aligned}$$

MAT 1320

Fall 2017

Midterm #1 Version #1

1. We have the function $g\left(x\right)=\dfrac{xe^{\sqrt{1-2x}}}{x^2-9}$, which contains *both* a square root term and a denominator. Let's analyze both parts separately.

Square root term: The term inside the square root must be greater than or equal to zero, otherwise we would be taking the square root of a negative number, and … yeah, no. Imaginary numbers are only for linear algebra.

So, we require that $1-2x\geq 0$. Then, we can solve for *x*.

$$\begin{aligned} 1-2x &\geq 0 \\ -2x &\geq -1 \\ 2x &\leq 1 \\ x &\leq \frac{1}{2} \end{aligned}$$

If you switch the sign on both sides, the inequality switches direction.

This statement is equivalent to $-\infty < x \leq \frac{1}{2}$. Writing it like this will make it easier to combine with the denominator part.

Denominator: The term in the denominator cannot be equal to zero. If it is, we end up with a "divide by zero" situation. Nope.

So, we require that the denominator is *not* equal to zero, and then solve for *x*.

$$\begin{aligned} x^2-9 &\neq 0 \\ x^2 &\neq 9 \\ x &\neq \pm 3 \end{aligned}$$

Now we need to combine this information into a cohesive expression for the domain. We have

1. $-\infty < x \leq \frac{1}{2}$
2. $x \neq \pm 3$

This basically states that x can be anything between negative infinity and ½ *except* for -3 and +3. Firstly, +3 is not in these bounds, so we can disregard this point. Next, to take care of -3, we need to split the continuous term in #1 into term parts, splitting it at x = -3 where the function cannot exist.

This is the final answer.

$$D:\left\{x \in R \mid -\infty < x < -3 \quad \cup \quad -3 < x \leq \frac{1}{2}\right\}$$

Math version of "and".

2. Ugh. Definition of the derivative. -_-

$$f(x) = \sqrt{x+7}$$
$$f(x+h) = \sqrt{x+h+7}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+7} - \sqrt{x+7}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+7} - \sqrt{x+7}}{h} \frac{\sqrt{x+h+7} + \sqrt{x+7}}{\sqrt{x+h+7} + \sqrt{x+7}} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{x+h+7} - \sqrt{x+7}\right)\left(\sqrt{x+h+7} + \sqrt{x+7}\right)}{h\left(\sqrt{x+h+7} + \sqrt{x+7}\right)} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{x+h+7}\right)\left(\sqrt{x+h+7}\right) + \left(\sqrt{x+h+7}\right)\left(\sqrt{x+7}\right) - \left(\sqrt{x+7}\right)\left(\sqrt{x+h+7}\right) - \left(\sqrt{x+7}\right)\left(\sqrt{x+7}\right)}{h\left(\sqrt{x+h+7} + \sqrt{x+7}\right)} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{x+h+7}\right)\left(\sqrt{x+h+7}\right) + \cancel{\left(\sqrt{x+h+7}\right)\left(\sqrt{x+7}\right)} - \cancel{\left(\sqrt{x+7}\right)\left(\sqrt{x+h+7}\right)} - \left(\sqrt{x+7}\right)\left(\sqrt{x+7}\right)}{h\left(\sqrt{x+h+7} + \sqrt{x+7}\right)} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{x+h+7}\right)\left(\sqrt{x+h+7}\right) - \left(\sqrt{x+7}\right)\left(\sqrt{x+7}\right)}{h\left(\sqrt{x+h+7} + \sqrt{x+7}\right)}
\end{aligned}$$

$$= \lim_{h\to 0} \frac{(x+h+7)-(x+7)}{h\left(\sqrt{x+h+7}+\sqrt{x+7}\right)}$$

$$= \lim_{h\to 0} \frac{(\cancel{x}+h+\cancel{7})-(\cancel{x}+\cancel{7})}{h\left(\sqrt{x+h+7}+\sqrt{x+7}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+7}+\sqrt{x+7}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+7}+\sqrt{x+7}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+7}+\sqrt{x+7}}$$

$$= \frac{1}{\sqrt{x+0+7}+\sqrt{x+7}}$$

$$= \frac{1}{2\sqrt{x+7}}$$

You can always check your answer to "definition of the derivative" questions by taking the derivative using one of the conventional methods. In this case, the chain rule would work. However, if you only evaluate the derivative using the chain rule, you will get 0 marks.

3. Limits!

a) Immediately plugging in $x = 2$ gives 0/0. That's not a thing… Try factoring the denominator.

$$\lim_{x\to 2} \frac{x-2}{2x^2-4x} = \lim_{x\to 2} \frac{x-2}{2x(x-2)}$$

$$= \lim_{x\to 2} \frac{\cancel{x-2}}{2x\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{1}{2x}$$

$$= \frac{1}{2(2)}$$

$$= \frac{1}{4}$$

b) If we "plug in" $x=\infty$ into both polynomials, we get $\frac{\infty}{\infty}$. That's broken… OK… Let's analyze the numerator and denominator separately:

Numerator: $x^3 - x - 7$
As x gets bigger and bigger, x^3 becomes much larger than $-x$ and -7. We can use this to our advantage…

Denominator: $2x^3 + \sqrt{x}$
As x gets bigger and bigger, $2x^3$ becomes much larger than $\sqrt{x}$. Again, we can use this to our advantage…

I will factor out the largest power in both the numerator and the denominator. We normally wouldn't do this in math (it looks pretty dumb, but it ends up helping us solve the limit!) We will actually learn a better way of evaluating limits like this later on in the course, but for now… just… just do it this way…

$$\lim_{x\to\infty}\frac{x^3-x-7}{2x^3+\sqrt{x}} = \lim_{x\to\infty}\frac{x^3\left(1-\frac{x}{x^3}-\frac{7}{x^3}\right)}{x^3\left(2+\frac{\sqrt{x}}{x^3}\right)}$$

$$= \lim_{x\to\infty}\frac{\cancel{x^3}\left(1-\frac{1}{x^2}-\frac{7}{x^3}\right)}{\cancel{x^3}\left(2+\frac{1}{x^{5/2}}\right)}$$

$$= \lim_{x\to\infty}\frac{1-\frac{1}{x^2}-\frac{7}{x^3}}{2+\frac{1}{x^{5/2}}}$$

$$= \frac{1-0-0}{2+0}$$

$$= \frac{1}{2}$$

4. Implicit differentiation.

$$x^3 + xy - e^y = 5$$
$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}(xy) - \frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(5)$$
$$3x^2 + \left(y + x\frac{dy}{dx}\right) - e^y\frac{dy}{dx} = 0$$
$$x\frac{dy}{dx} - e^y\frac{dy}{dx} = -3x^2 - y$$
$$\frac{dy}{dx}\left(x - e^y\right) = -3x^2 - y$$
$$\frac{dy}{dx} = \frac{-3x^2 - y}{x - e^y}$$

5. First we need the derivative of the function.

$$f'(x) = -6x^2 + 1$$

Then, we can evaluate the derivative at the given point, $x = 1$.

$$f'(1) = -6(1)^2 + 1 = -5$$

We also need the y-coordinate of the function at the given point, $x = 1$. To get this, we will plug x into the ***original function***.

$$f(1) = -2(1)^3 + 1 = -1$$

To solve for the equation of the tangent line, there are two methods. I will show both. Either works! Choose your favourite.

Slope-Point Form	*Slope-Intercept Form*
$y - y_0 = m(x - x_0)$ $y - (-1) = -5(x - 1)$ $y + 1 = -5x + 5$ $y = -5x + 4$	$y = mx + b$ $-1 = -5(1) + b$ $b = -1 + 5$ $= 4$ $\therefore y = -5x + 4$

6. Note that the test says to leave the answers unsimplified. Sweeeeet.

a) Product rule and chain rule. (Pain rule?)

$$f(x) = (t^2+\pi)^7$$
$$g(x) = e^t + 1$$
$$f'(x) = \underbrace{7(t^2+\pi)^6 (t^2+\pi)'}_{\text{chain rule}}$$
$$= 7(t^2+\pi)^6 (2t)$$
$$g'(x) = e^t$$

$$\begin{aligned} y' &\equiv \frac{dy}{dx} \\ &= f'(x)g(x) + f(x)g'(x) \\ &= 7(t^2+\pi)^6(2t)(e^t+1) + (t^2+\pi)^7 e^t \end{aligned}$$

b) Chain in a chain rule. Ch-ch-ch-ch chaaaain ruuuule.

$$\begin{aligned} w'(r) &= \underbrace{\frac{1}{\sqrt{1-r^2}}\left(\sqrt{1-r^2}\right)'}_{\text{chain rule once}} \\ &= \frac{1}{\sqrt{1-r^2}}\left[(1-r^2)^{1/2}\right]' \\ &= \underbrace{\frac{1}{\sqrt{1-r^2}}\left(\frac{1}{2}\right)(1-r^2)^{-1/2}(1-r^2)'}_{\text{chain rule twice}} \\ &= \frac{1}{\sqrt{1-r^2}}\left(\frac{1}{2}\right)(1-r^2)^{-1/2}(-2r) \end{aligned}$$

c) Quotient rule.

$$f(x) = \tan x$$
$$g(x) = x^2 + 2^x$$
$$f'(x) = \sec^2 x$$
$$g'(x) = 2x + \ln 2\left(2^x\right)$$

$$\begin{aligned} y' &\equiv \frac{dy}{dx} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2} \\ &= \frac{\left(\sec^2 x\right)\left(x^2 + 2^x\right) - (\tan x)\left[2x + \ln 2\left(2^x\right)\right]}{\left(x^2 + 2^x\right)^2} \end{aligned}$$

7. Logarithmic differentiation.

$$\begin{aligned} \ln\left[f(x)\right] &= \ln\left[\left(x^2 + 2\right)^{\sin x}\right] \\ &= (\sin x)\ln\left(x^2 + 2\right) \end{aligned}$$

$$\frac{d}{dx}\ln\left[f(x)\right] = \frac{d}{dx}\underbrace{(\sin x)\ln\left(x^2 + 2\right)}_{\text{product and chain rule}}$$

$$\frac{1}{f(x)} f'(x) = \cos x \ln\left(x^2 + 2\right) + \sin x\left(\frac{2x}{x^2 + 2}\right)$$

$$\begin{aligned} f'(x) &= f(x)\left[\cos x \ln\left(x^2 + 2\right) + \sin x\left(\frac{2x}{x^2 + 2}\right)\right] \\ &= \left(x^2 + 2\right)^{\sin x}\left[\cos x \ln\left(x^2 + 2\right) + \sin x\left(\frac{2x}{x^2 + 2}\right)\right] \end{aligned}$$

8. Related rates.

We are given the following rates: $\frac{dw}{dt} = 1\frac{cm}{s}$ and $\frac{dA}{dt} = 10\frac{cm^2}{s}$, and asked to determine $\frac{dh}{dt}$.

The equation relating all of these variables is $A = wh$, the area of a rectangle.

Let's take the derivative of this equation with respect to time. To do this, we will use the product rule.

$$\frac{d}{dt}(A) = \frac{d}{dt}(wh)$$
$$\frac{dA}{dt} = h\frac{dw}{dt} + w\frac{dh}{dt}$$

Then, we can rearrange this equation for the unknown rate, $\frac{dh}{dt}$:

$$\frac{dA}{dt} - h\frac{dw}{dt} = w\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{1}{w}\left(\frac{dA}{dt} - h\frac{dw}{dt}\right)$$

At this point, we don't know the width of the rectangle. However, we are told that the rectangle has an area of 100 cm^2 and a height of 20 cm. Using $A = wh$, the width of the rectangle is 5 cm.

Plugging everything in to the related rates equation we developed, we get

$$\frac{dh}{dt} = \frac{1}{5\,cm}\left[10\frac{cm^2}{s} - (20\,cm)\left(1\frac{cm}{s}\right)\right]$$
$$= -2\frac{cm}{s}$$

MAT 1320

Fall 2017

Midterm #1 Version #2

1. We have the function $g(x)=\dfrac{xe^{\sqrt{2-x}}}{x^2-9}$, which contains *both* a square root term and a denominator. Let's analyze both parts separately.

Square root term: The term inside the square root must be greater than or equal to zero, otherwise we would be taking the square root of a negative number, and … yeah, no. Imaginary numbers are only for linear algebra.

So, we require that $2-x\geq 0$. Then, we can solve for *x*.

$$2-x\geq 0$$
$$-x\geq -2$$
$$x\leq 2$$

If you switch the sign on both sides, the inequality switches direction.

This statement is equivalent to $-\infty<x\leq 2$. Writing it like this will make it easier to combine with the denominator part.

Denominator: The term in the denominator cannot be equal to zero. If it is, we end up with a "divide by zero" situation. Nope.

So, we require that the denominator is *not* equal to zero, and then solve for *x*.

$$x^2-9\neq 0$$
$$x^2\neq 9$$
$$x\neq \pm 3$$

Now we need to combine this information into a cohesive expression for the domain. We have

1. $-\infty<x\leq 2$
2. $x\neq \pm 3$

This basically states that *x* can be anything between negative infinity and 2 *except* for -3 and +3. Firstly, +3 is not in these bounds, so we can disregard this point. Next, to take care of -3, we need to split the continuous term in #1 into term parts, splitting it at *x* = -3 where the function cannot exist.

This is the final answer.

$$D:\{x \in R \mid -\infty < x < -3 \quad \cup \quad -3 < x \le 2\}$$

Math version of "and".

2. Ugh. Definition of the derivative. -_-

$$f(x) = \sqrt{2x+1}$$
$$f(x+h) = \sqrt{2(x+h)+1}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}\,\frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}} \\
&= \lim_{h\to 0}\frac{\left(\sqrt{2(x+h)+1}-\sqrt{2x+1}\right)\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)}{h\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)} \\
&= \lim_{h\to 0}\frac{\left(\sqrt{2(x+h)+1}\right)\left(\sqrt{2(x+h)+1}\right)+\left(\sqrt{2(x+h)+1}\right)\left(\sqrt{2x+1}\right)-\left(\sqrt{2x+1}\right)\left(\sqrt{2(x+h)+1}\right)-\left(\sqrt{2x+1}\right)\left(\sqrt{2x+1}\right)}{h\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)} \\
&= \lim_{h\to 0}\frac{\left(\sqrt{2(x+h)+1}\right)\left(\sqrt{2(x+h)+1}\right)+\cancel{\left(\sqrt{2(x+h)+1}\right)\left(\sqrt{2x+1}\right)}-\cancel{\left(\sqrt{2x+1}\right)\left(\sqrt{2(x+h)+1}\right)}-\left(\sqrt{2x+1}\right)\left(\sqrt{2x+1}\right)}{h\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)} \\
&= \lim_{h\to 0}\frac{\left(\sqrt{2(x+h)+1}\right)\left(\sqrt{2(x+h)+1}\right)-\left(\sqrt{2x+1}\right)\left(\sqrt{2x+1}\right)}{h\left(\sqrt{2(x+h)+1}+\sqrt{2x+1}\right)}
\end{aligned}$$

$$= \lim_{h\to 0} \frac{\left[2\left(x+h\right)+1\right]-\left(2x+1\right)}{h\left(\sqrt{2\left(x+h\right)+1}+\sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{\left(2x+2h+1\right)-\left(2x+1\right)}{h\left(\sqrt{2\left(x+h\right)+1}+\sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{\left(\cancel{2x}+2h+\cancel{1}\right)-\left(\cancel{2x}+\cancel{1}\right)}{h\left(\sqrt{2\left(x+h\right)+1}+\sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2\left(x+h\right)+1}+\sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2\left(x+h\right)+1}+\sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2\left(x+h\right)+1}+\sqrt{2x+1}}$$

$$= \frac{2}{\sqrt{2\left(x+0\right)+1}+\sqrt{2x+1}}$$

$$= \frac{2}{2\sqrt{2x+1}}$$

$$= \frac{1}{\sqrt{2x+1}}$$

You can always check your answer to “definition of the derivative” questions by taking the derivative using one of the conventional methods. In this case, the chain rule would work. However, if you only evaluate the derivative using the chain rule, you will get 0 marks.

3. Limits!

a) Immediately plugging in $x = -3$ gives 0/0. That's not a thing… Try factoring the denominator.

$$\lim_{x\to -3}\frac{x+3}{4x^2+12x} = \lim_{x\to -3}\frac{x+3}{4x(x+3)}$$
$$= \lim_{x\to -3}\frac{\cancel{x+3}}{4x\cancel{(x+3)}}$$
$$= \lim_{x\to -3}\frac{1}{4x}$$
$$= \frac{1}{4(-3)}$$
$$= -\frac{1}{12}$$

b) If we "plug in" $x=\infty$ into both polynomials, we get $\frac{\infty}{\infty}$. That's broken… OK… Let's analyze the numerator and denominator separately:

Numerator: $x^2-\sqrt{x}-7$

As x gets bigger and bigger, x^2 becomes much larger than $\sqrt{x}$ and -7. We can use this to our advantage…

Denominator: x^2+x

As x gets bigger and bigger, x^2 becomes much larger than x. Again, we can use this to our advantage…

I will factor out the largest power in both the numerator and the denominator. We normally wouldn't do this in math (it looks pretty dumb, but it ends up helping us solve the limit!) We will actually learn a better way of evaluating limits like this later on in the course, but for now… just… just do it this way…

$$\lim_{x\to\infty}\frac{x^2-\sqrt{x}-7}{x^2+x}=\lim_{x\to\infty}\frac{x^2\left(1-\frac{\sqrt{x}}{x^2}-\frac{7}{x^2}\right)}{x^2\left(1+\frac{x}{x^2}\right)}$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}\left(1-\frac{x^{1/2}}{x^2}-\frac{7}{x^2}\right)}{\cancel{x^2}\left(1+\frac{1}{x}\right)}$$

$$=\lim_{x\to\infty}\frac{1-\frac{1}{x^{3/2}}-\frac{7}{x^2}}{1+\frac{1}{x}}$$

$$=\frac{1-0-0}{1+0}$$

$$=1$$

4. Implicit differentiation.

$$x^4+xy-e^y=-1$$

$$\frac{d}{dx}\left(x^4\right)+\frac{d}{dx}(xy)-\frac{d}{dx}\left(e^y\right)=\frac{d}{dx}(1)$$

$$4x^3+\left(y+x\frac{dy}{dx}\right)-e^y\frac{dy}{dx}=0$$

$$x\frac{dy}{dx}-e^y\frac{dy}{dx}=-4x^3-y$$

$$\frac{dy}{dx}\left(x-e^y\right)=-4x^3-y$$

$$\frac{dy}{dx}=\frac{-4x^3-y}{x-e^y}$$

5. First we need the derivative of the function.

$f'(x) = -12x^3 - 1$

Then, we can evaluate the derivative at the given point, $x = 1$.

$f'(1) = -12(1)^3 - 1 = -13$

We also need the *y*-coordinate of the function at the given point, $x = 1$. To get this, we will plug *x* into the ***original function***.

$f(1) = -3(1)^4 - 1 = -4$

To solve for the equation of the tangent line, there are two methods. I will show both. Either works! Choose your favourite.

Slope-Point Form	*Slope-Intercept Form*
$y - y_0 = m(x - x_0)$ $y - (-4) = -13(x - 1)$ $y + 4 = -13x + 13$ $y = -13x + 9$	$y = mx + b$ $-4 = -13(1) + b$ $b = -4 + 13$ $= 9$ $\therefore y = -13x + 9$

6. Note that the test says to leave the answers unsimplified. Sweeeeet.

a) Quotient rule.

$f(x) = x^2 + 2^x$

$g(x) = \tan x$

$f'(x) = 2x + \ln 2(2^x)$

$g'(x) = \sec^2 x$

$$\begin{aligned} y' &\equiv \frac{dy}{dx} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{\left[2x + \ln 2(2^x)\right](\tan x) - (x^2 + 2^x)(\sec^2 x)}{(\tan x)^2} \end{aligned}$$

b) Product and chain rule.

$$f(x)=\left(t+e^t\right)^7$$
$$g(x)=e^t+2$$
$$f'(x)=\underbrace{7\left(t+e^t\right)^6\left(t+e^t\right)'}_{\text{chain rule}}$$
$$=7\left(t+e^t\right)^6\left(1+e^t\right)$$
$$g'(x)=e^t$$

$$\begin{aligned} y' &\equiv \frac{dy}{dx} \\ &= f'(x)g(x)+f(x)g'(x) \\ &= 7\left(t+e^t\right)^6\left(1+e^t\right)\left(e^t+2\right)+\left(t+e^t\right)^7 e^t \end{aligned}$$

c) Ch-ch-ch-chaaaain rule.

$$\begin{aligned} w'(r) &= \underbrace{\frac{1}{\sqrt{r-r^2}}\left(\sqrt{r-r^2}\right)'}_{\text{chain rule once}} \\ &= \frac{1}{\sqrt{r-r^2}}\left[\left(r-r^2\right)^{1/2}\right]' \\ &= \underbrace{\frac{1}{\sqrt{r-r^2}}\left(\frac{1}{2}\right)\left(r-r^2\right)^{-1/2}\left(r-r^2\right)'}_{\text{chain rule twice}} \\ &= \frac{1}{\sqrt{r-r^2}}\left(\frac{1}{2}\right)\left(r-r^2\right)^{-1/2}\left(1-2r\right) \end{aligned}$$

7. Logarithmic differentiation.

$$\ln\left[f(x)\right] = \ln\left[\left(x^2+2\right)^{\cos x}\right]$$
$$= (\cos x)\ln\left(x^2+2\right)$$

$$\frac{d}{dx}\ln\left[f(x)\right] = \frac{d}{dx}\underbrace{(\cos x)\ln\left(x^2+2\right)}_{\text{product and chain rule}}$$

$$\frac{1}{f(x)}f'(x) = -\sin x\ln\left(x^2+2\right)+\cos x\left(\frac{2x}{x^2+2}\right)$$

$$f'(x) = f(x)\left[-\sin x\ln\left(x^2+2\right)+\cos x\left(\frac{2x}{x^2+2}\right)\right]$$
$$= \left(x^2+2\right)^{\cos x}\left[-\sin x\ln\left(x^2+2\right)+\cos x\left(\frac{2x}{x^2+2}\right)\right]$$

8. Related rates.

We are given the following rates: $\frac{dw}{dt} = 1\frac{cm}{s}$ and $\frac{dA}{dt} = 5\frac{cm^2}{s}$, and asked to determine $\frac{dh}{dt}$.

The equation relating all of these variables is $A = wh$, the area of a rectangle.

Let’s take the derivative of this equation with respect to time. To do this, we will use the product rule.

$$\frac{d}{dt}(A) = \frac{d}{dt}(wh)$$
$$\frac{dA}{dt} = h\frac{dw}{dt} + w\frac{dh}{dt}$$

Then, we can rearrange this equation for the unknown rate, $\frac{dh}{dt}$:

$$\frac{dA}{dt} - h\frac{dw}{dt} = w\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{1}{w}\left(\frac{dA}{dt} - h\frac{dw}{dt}\right)$$

At this point, we don’t know the width of the rectangle. However, we are told that the rectangle has an area of 100 cm^2 and a height of 20 cm. Using $A = wh$, the width of the rectangle is 5 cm.

Plugging everything in to the related rates equation we developed, we get

$$\frac{dh}{dt} = \frac{1}{5\,cm}\left[5\frac{cm^2}{s} - (20\,cm)\left(1\frac{cm}{s}\right)\right]$$
$$= -3\frac{cm}{s}$$

MAT 1320

Fall 2017

Midterm #1 Version #3

1. We have the function $g\left(x\right)=\dfrac{xe^{\sqrt{1-x}}}{x^2-9}$, which contains *both* a square root term and a denominator. Let's analyze both parts separately.

Square root term: The term inside the square root must be greater than or equal to zero, otherwise we would be taking the square root of a negative number, and … yeah, no. Imaginary numbers are only for linear algebra.

So, we require that $1-x\geq 0$. Then, we can solve for *x*.

$$1-x\geq 0$$
$$-x\geq -1$$
$$x\leq 1$$

If you switch the sign on both sides, the inequality switches direction.

This statement is equivalent to $-\infty<x\leq 1$. Writing it like this will make it easier to combine with the denominator part.

Denominator: The term in the denominator cannot be equal to zero. If it is, we end up with a "divide by zero" situation. Nope.

So, we require that the denominator is *not* equal to zero, and then solve for *x*.

$$x^2-9\neq 0$$
$$x^2\neq 9$$
$$x\neq \pm 3$$

Now we need to combine this information into a cohesive expression for the domain. We have

1. $-\infty<x\leq 1$
2. $x\neq \pm 3$

This basically states that *x* can be anything between negative infinity and 1 *except* for -3 and +3. Firstly, +3 is not in these bounds, so we can disregard this point. Next, to take care of -3, we need to split the continuous term in #1 into term parts, splitting it at *x* = -3 where the function cannot exist.

This is the final answer.

$$D:\{x \in R \mid -\infty < x < -3 \quad \cup \quad -3 < x \leq 1\}$$

Math version of "and".

2. Ugh. Definition of the derivative. -_-

$$f(x) = \sqrt{x+5}$$
$$f(x+h) = \sqrt{x+h+5}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h+5} - \sqrt{x+5}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h+5} - \sqrt{x+5}}{h} \frac{\sqrt{x+h+5} + \sqrt{x+5}}{\sqrt{x+h+5} + \sqrt{x+5}} \\
&= \lim_{h \to 0} \frac{\left(\sqrt{x+h+5} - \sqrt{x+5}\right)\left(\sqrt{x+h+5} + \sqrt{x+5}\right)}{h\left(\sqrt{x+h+5} + \sqrt{x+5}\right)} \\
&= \lim_{h \to 0} \frac{\left(\sqrt{x+h+5}\right)\left(\sqrt{x+h+5}\right) + \left(\sqrt{x+h+5}\right)\left(\sqrt{x+5}\right) - \left(\sqrt{x+5}\right)\left(\sqrt{x+h+5}\right) - \left(\sqrt{x+5}\right)\left(\sqrt{x+5}\right)}{h\left(\sqrt{x+h+5} + \sqrt{x+5}\right)} \\
&= \lim_{h \to 0} \frac{\left(\sqrt{x+h+5}\right)\left(\sqrt{x+h+5}\right) + \cancel{\left(\sqrt{x+h+5}\right)\left(\sqrt{x+5}\right)} - \cancel{\left(\sqrt{x+5}\right)\left(\sqrt{x+h+5}\right)} - \left(\sqrt{x+5}\right)\left(\sqrt{x+5}\right)}{h\left(\sqrt{x+h+5} + \sqrt{x+5}\right)} \\
&= \lim_{h \to 0} \frac{\left(\sqrt{x+h+5}\right)\left(\sqrt{x+h+5}\right) - \left(\sqrt{x+5}\right)\left(\sqrt{x+5}\right)}{h\left(\sqrt{x+h+5} + \sqrt{x+5}\right)}
\end{aligned}$$

$$= \lim_{h\to 0} \frac{(x+h+5)-(x+5)}{h\left(\sqrt{x+h+5}+\sqrt{x+5}\right)}$$

$$= \lim_{h\to 0} \frac{(\cancel{x}+h+\cancel{5})-(\cancel{x}+\cancel{5})}{h\left(\sqrt{x+h+5}+\sqrt{x+5}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+5}+\sqrt{x+5}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+5}+\sqrt{x+5}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+5}+\sqrt{x+5}}$$

$$= \frac{1}{\sqrt{x+0+5}+\sqrt{x+5}}$$

$$= \frac{1}{2\sqrt{x+5}}$$

You can always check your answer to “definition of the derivative” questions by taking the derivative using one of the conventional methods. In this case, the chain rule would work. However, if you only evaluate the derivative using the chain rule, you will get 0 marks.

3. Limits!

a) Immediately plugging in $x = 1$ gives 0/0. That’s not a thing… Try factoring the denominator.

$$\lim_{x\to 1} \frac{x-1}{5x^2-5x} = \lim_{x\to 1} \frac{x-1}{5x(x-1)}$$

$$= \lim_{x\to 1} \frac{\cancel{x-1}}{5x\cancel{(x-1)}}$$

$$= \lim_{x\to 1} \frac{1}{5x}$$

$$= \frac{1}{5(1)}$$

$$= \frac{1}{5}$$

b) If we "plug in" $x = \infty$ into both polynomials, we get $\frac{\infty}{\infty}$. That's broken… OK… Let's analyze the numerator and denominator separately:

Numerator: $3x^5 - x^2 - 7x$
As x gets bigger and bigger, $3x^5$ becomes much larger than $-x^2$ and $-7x$. We can use this to our advantage…

Denominator: $4x^5 + \sqrt{x}$
As x gets bigger and bigger, $4x^5$ becomes much larger than $\sqrt{x}$. Again, we can use this to our advantage…

I will factor out the largest power in both the numerator and the denominator. We normally wouldn't do this in math (it looks pretty dumb, but it ends up helping us solve the limit!) We will actually learn a better way of evaluating limits like this later on in the course, but for now… just… just do it this way…

$$\lim_{x\to\infty}\frac{3x^5 - x^2 - 7x}{4x^5 + \sqrt{x}} = \lim_{x\to\infty}\frac{x^5\left(3 - \frac{x^2}{x^5} - \frac{7x}{x^5}\right)}{x^5\left(4 + \frac{\sqrt{x}}{x^5}\right)}$$

$$= \lim_{x\to\infty}\frac{\cancel{x^5}\left(3 - \frac{1}{x^3} - \frac{7}{x^4}\right)}{\cancel{x^5}\left(4 + \frac{x^{1/2}}{x^5}\right)}$$

$$= \lim_{x\to\infty}\frac{3 - \frac{1}{x^3} - \frac{7}{x^4}}{4 + \frac{1}{x^{9/2}}}$$

$$= \frac{3 - 0 - 0}{4 + 0}$$

$$= \frac{3}{4}$$

4. Implicit differentiation.

$$2x^3 + 5xy - e^y = 7$$

$$\frac{d}{dx}\left(2x^3\right) + \frac{d}{dx}(5xy) - \frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(7)$$

$$6x^2 + \left(5y + 5x\frac{dy}{dx}\right) - e^y\frac{dy}{dx} = 0$$

$$5x\frac{dy}{dx} - e^y\frac{dy}{dx} = -6x^2 - 5y$$

$$\frac{dy}{dx}\left(5x - e^y\right) = -6x^2 - 5y$$

$$\frac{dy}{dx} = \frac{-6x^2 - 5y}{5x - e^y}$$

5. First we need the derivative of the function.

$f'(x) = -6x^2 - 1$

Then, we can evaluate the derivative at the given point, $x = 1$.

$f'(1) = -6(1)^2 - 1 = -7$

We also need the y-coordinate of the function at the given point, $x = 1$. To get this, we will plug x into the ***original function***.

$f(1) = -2(1)^3 - 1 = -3$

To solve for the equation of the tangent line, there are two methods. I will show both. Either works! Choose your favourite.

Slope-Point Form	*Slope-Intercept Form*
$y - y_0 = m(x - x_0)$ $y - (-3) = -7(x - 1)$ $y + 3 = -7x + 7$ $y = -7x + 4$	$y = mx + b$ $-3 = -7(1) + b$ $b = -3 + 7$ $= 4$ $\therefore y = -7x + 4$

6. Note that the test says to leave the answers unsimplified. Sweeeeet.

a) Product and chain rule.

$$f(x) = \left(t^3 - 5\right)^7$$
$$g(x) = e^t + 2$$
$$f'(x) = \underbrace{7\left(t^3 - 5\right)^6 \left(t^3 - 5\right)'}_{\text{chain rule}}$$
$$= 7\left(t^3 - 5\right)^6 \left(3t^2\right)$$
$$g'(x) = e^t$$

$$\begin{aligned} y' &\equiv \frac{dy}{dx} \\ &= f'(x)g(x) + f(x)g'(x) \\ &= 7\left(t^3 - 5\right)^6 \left(3t^2\right)\left(e^t + 2\right) + \left(t^3 - 5\right)^7 e^t \end{aligned}$$

b) Quotient rule.

$$f(x) = 2^x$$
$$g(x) = x^2 + \tan x$$
$$f'(x) = \ln 2\left(2^x\right)$$
$$g'(x) = 2x + \sec^2 x$$

$$\begin{aligned} y' &\equiv \frac{dy}{dx} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2} \\ &= \frac{\left[\ln 2\left(2^x\right)\right]\left(x^2 + \tan x\right) - \left(2^x\right)\left(2x + \sec^2 x\right)}{\left(x^2 + \tan x\right)^2} \end{aligned}$$

c) Ch-ch-ch-chaaaain rule.

$$w'(r) = \underbrace{\frac{1}{\sqrt{r^2-1}}\left(\sqrt{r^2-1}\right)'}_{\text{chain rule once}}$$

$$= \frac{1}{\sqrt{r^2-1}}\left[\left(r^2-1\right)^{1/2}\right]'$$

$$= \underbrace{\frac{1}{\sqrt{r^2-1}}\left(\frac{1}{2}\right)\left(r^2-1\right)^{-1/2}\left(r^2-1\right)'}_{\text{chain rule twice}}$$

$$= \frac{1}{\sqrt{r^2-1}}\left(\frac{1}{2}\right)\left(r^2-1\right)^{-1/2}(2r)$$

7. Logarithmic differentiation.

$$\ln\left[f(x)\right] = \ln\left[\left(x^4\right)^{\sin x}\right]$$

$$= (\sin x)\ln\left(x^4\right)$$

$$\frac{d}{dx}\ln\left[f(x)\right] = \frac{d}{dx}\underbrace{(\sin x)\ln\left(x^4\right)}_{\text{product and chain rule}}$$

$$\frac{1}{f(x)}f'(x) = \cos x\ln\left(x^4\right) + \sin x\left(\frac{4x^3}{x^4}\right)$$

$$= \cos x\ln\left(x^4\right) + \sin x\left(\frac{4}{x}\right)$$

$$f'(x) = f(x)\left[\cos x\ln\left(x^4\right) + \sin x\left(\frac{4}{x}\right)\right]$$

$$= \left(x^4\right)^{\sin x}\left[\cos x\ln\left(x^4\right) + \sin x\left(\frac{4}{x}\right)\right]$$

8. Related rates.

We are given the following rates: $\frac{dw}{dt}=1\frac{cm}{s}$ and $\frac{dA}{dt}=15\frac{cm^2}{s}$, and asked to determine $\frac{dh}{dt}$.

The equation relating all of these variables is $A=wh$, the area of a rectangle.

Let's take the derivative of this equation with respect to time. To do this, we will use the product rule.

$$\frac{d}{dt}(A)=\frac{d}{dt}(wh)$$
$$\frac{dA}{dt}=h\frac{dw}{dt}+w\frac{dh}{dt}$$

Then, we can rearrange this equation for the unknown rate, $\frac{dh}{dt}$:

$$\frac{dA}{dt}-h\frac{dw}{dt}=w\frac{dh}{dt}$$
$$\frac{dh}{dt}=\frac{1}{w}\left(\frac{dA}{dt}-h\frac{dw}{dt}\right)$$

At this point, we don't know the width of the rectangle. However, we are told that the rectangle has an area of 100 cm^2 and a height of 20 cm. Using $A=wh$, the width of the rectangle is 5 cm.

Plugging everything in to the related rates equation we developed, we get

$$\frac{dh}{dt}=\frac{1}{5\,cm}\left[15\frac{cm^2}{s}-(20\,cm)\left(1\frac{cm}{s}\right)\right]$$
$$=-1\frac{cm}{s}$$

MAT 1320

Fall 2016

Midterm #1

Multiple Choice (1 point each):

1. We have the function $f\left(x\right)=\sqrt{4-x^2}$, which contains a square root term. We cannot take the square root of a negative number, so the term inside the square root must be greater than or equal to 0. Mathematically,

$$\begin{aligned} 4-x^2 &\geq 0 \\ -x^2 &\geq -4 \\ x^2 &\leq 4 \end{aligned}$$

This is equivalent to $-2 \leq x \leq 2$.

Why?

Firstly… take the square root of 4, and you get $x = -2$ and $x = 2$, which are the bounds on the interval.

Next, we want x^2 to be ***less than or equal to*** 4. If we choose numbers that are smaller than -2, this will not be true. Similarly, if we choose numbers that are larger than 2, this will not be true. So, the only interval that makes sense is the interval as shown.

Answer: B

2. Inverse function.

$$y = \ln(x+6)$$

Swap x and y. Solve for y.

$$x = \ln(y+6)$$
$$e^x = e^{\ln(y+6)}$$
$$= y+6$$
$$y = e^x - 6$$
$$f^{-1}(x) = e^x - 6$$

Answer: C

3. First we need the derivative of the function.

$$f'(x) = 2 + 6x$$

Then, we can evaluate the derivative at the given point, $x = 1$.

$$f'(1) = 2(1) + 6(1) = 8$$

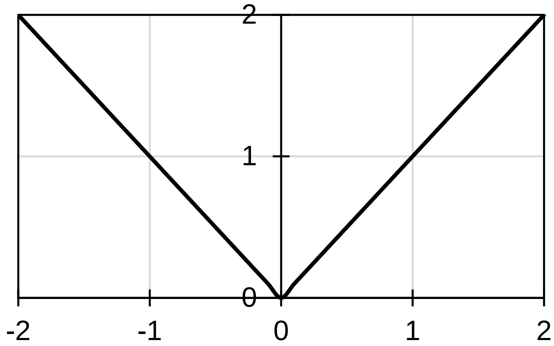
To solve for the equation of the tangent line, there are two methods. I will show both. Either works! Choose your favourite.

Note that we are subbing in the point (1,5), given in the problem statement.

Slope-Point Form	*Slope-Intercept Form*
$y - y_0 = m(x - x_0)$ $y - 5 = 8(x-1)$ $y = 8x - 8 + 5$ $y = 8x - 3$	$y = mx + b$ $5 = 8(1) + b$ $b = 5 - 8$ $= -3$ $\therefore y = 8x - 3$

Answer: C

4. Here's an example where this is *not* true.



This is $y = |x|$. It is continuous (i.e. there are no holes, jumps, etc.), BUT you cannot evaluate the derivative of it at $x = 0$. You can try… but it won't work… there is a "corner" at that point (i.e. the function changes direction too rapidly). Therefore, the derivative is undefined.

My point is that just because a function is continuous doesn't mean that it can have a derivative everywhere. This is one example of that.

Answer: B

5. Rational polynomial (or use L'Hôpitals rule if you know it.)

$$\lim_{x\to\infty}\frac{5x^3+7x^2-8x+9}{11x^2+7x-6} = \lim_{x\to\infty}\frac{x^3\left(5+\frac{7x^2}{x^3}-\frac{8x}{x^3}+\frac{9}{x^3}\right)}{x^2\left(11+\frac{7x}{x^2}-\frac{6}{x^2}\right)}$$

$$= \lim_{x\to\infty}\frac{x\left(5+\frac{7}{x}-\frac{8}{x^2}+\frac{9}{x^3}\right)}{11+\frac{7}{x}-\frac{6}{x^2}}$$

$$= \frac{\infty(5+0-0+0)}{11+0-0}$$

$$= \infty$$

Answer: C

Long Answer:

6. (2 points) Direct evaluation of limit gives 0/0. Let's use the sneaky 1 trick!

$$\lim_{x\to 2}\frac{\sqrt{x^2+12}-4}{x-2}\frac{\sqrt{x^2+12}+4}{\sqrt{x^2+12}+4}=\lim_{x\to 2}\frac{\sqrt{x^2+12}\sqrt{x^2+12}-4\sqrt{x^2+12}+4\sqrt{x^2+12}-16}{(x-2)\left(\sqrt{x^2+12}+4\right)}$$

$$=\lim_{x\to 2}\frac{\sqrt{x^2+12}\sqrt{x^2+12}\cancel{-4\sqrt{x^2+12}}\cancel{+4\sqrt{x^2+12}}-16}{(x-2)\left(\sqrt{x^2+12}+4\right)}$$

$$=\lim_{x\to 2}\frac{\left(x^2+12\right)-16}{(x-2)\left(\sqrt{x^2+12}+4\right)}$$

$$=\lim_{x\to 2}\frac{x^2-4}{(x-2)\left(\sqrt{x^2+12}+4\right)}$$

$$=\lim_{x\to 2}\frac{(x+2)(x-2)}{(x-2)\left(\sqrt{x^2+12}+4\right)}$$

$$=\lim_{x\to 2}\frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}\left(\sqrt{x^2+12}+4\right)}$$

$$=\lim_{x\to 2}\frac{x+2}{\sqrt{x^2+12}+4}$$

$$=\frac{2+2}{\sqrt{2^2+12}+4}$$

$$=\frac{4}{8}$$

$$=\frac{1}{2}$$

7. (3 points) Definition of the derivative. Barf.

$$f(x)=\frac{2x^2}{x+3}$$

$$f(x+h)=\frac{2(x+h)^2}{x+h+3}=\frac{2(x+h)(x+h)}{x+h+3}=\frac{2(x^2+2xh+h^2)}{x+h+3}=\frac{2x^2+4xh+2h^2}{x+h+3}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{2x^2+4xh+2h^2}{x+h+3}-\frac{2x^2}{x+3}}{h}$$

$$=\lim_{h\to 0}\left[\left(\frac{1}{h}\right)\left(\frac{2x^2+4xh+2h^2}{x+h+3}\right)-\left(\frac{1}{h}\right)\left(\frac{2x^2}{x+3}\right)\right]$$

$$=\lim_{h\to 0}\left[\left(\frac{1}{h}\right)\left(\frac{2x^2+4xh+2h^2}{x+h+3}\right)\left(\frac{x+3}{x+3}\right)-\left(\frac{1}{h}\right)\left(\frac{2x^2}{x+3}\right)\left(\frac{x+h+3}{x+h+3}\right)\right]$$

$$=\lim_{h\to 0}\frac{(2x^2+4xh+2h^2)(x+3)-(2x^2)(x+h+3)}{h(x+h+3)(x+3)}$$

$$=\lim_{h\to 0}\frac{(2x^3+6x^2+4x^2h+12xh+2xh^2+6h^2)-(2x^3+2x^2h+6x^2)}{h(x+h+3)(x+3)}$$

$$=\lim_{h\to 0}\frac{(\cancel{2x^3}+\cancel{6x^2}+4x^2h+12xh+2xh^2+6h^2)-(\cancel{2x^3}+2x^2h+\cancel{6x^2})}{h(x+h+3)(x+3)}$$

$$=\lim_{h\to 0}\frac{(4x^2h+12xh+2xh^2+6h^2)-2x^2h}{h(x+h+3)(x+3)}$$

$$=\lim_{h\to 0}\frac{2x^2h+12xh+2xh^2+6h^2}{h(x+h+3)(x+3)}$$

$$=\lim_{h\to 0}\frac{h(2x^2+12x+2xh+6h)}{h(x+h+3)(x+3)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x^2+12x+2xh+6h)}{\cancel{h}(x+h+3)(x+3)}$$

$$=\lim_{h\to 0}\frac{2x^2+12x+2xh+6h}{(x+h+3)(x+3)}$$

$$=\frac{2x^2+12x+2x(0)+6(0)}{(x+0+3)(x+3)}$$

$$=\frac{2x^2+12x}{(x+3)^2}$$

Verify with the Quotient Rule.

$$f(x) = 2x^2$$
$$g(x) = x + 3$$
$$f'(x) = 4x$$
$$g'(x) = 1$$

$$\begin{aligned} y' &\equiv \frac{dy}{dx} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2} \\ &= \frac{(4x)(x+3) - \left(2x^2\right)(1)}{(x+3)^2} \\ &= \frac{4x^2 + 12x - 2x^2}{(x+3)^2} \\ &= \frac{2x^2 + 12x}{(x+3)^2} \end{aligned}$$

Same answer.

8. (5 points) Derivatives!

a) Product rule and chain rule.

$$f(x) = e^{-3x}$$
$$g(x) = \sin\left(4x^2\right)$$

$$f'(x) = e^{-3x}(-3x)'$$
$$= -3e^{-3x}$$

$$g'(x) = \cos\left(4x^2\right)\left(4x^2\right)'$$
$$= 8x\cos\left(4x^2\right)$$

$$y' \equiv \frac{dy}{dx}$$
$$= f'(x)g(x) + f(x)g'(x)$$
$$= \left(-3e^{-3x}\right)\sin\left(4x^2\right) + e^{-3x}\left[8x\cos\left(4x^2\right)\right]$$
$$= e^{-3x}\left[-3\sin\left(4x^2\right) + 8x\cos\left(4x^2\right)\right]$$

b) Chain rule.

$$g'(t) = \frac{1}{2}\left(3t^2 + 7t - 2\right)^{-1/2}\left(3t^2 + 7t - 2\right)'$$
$$= \frac{1}{2\left(3t^2 + 7t - 2\right)^{1/2}}(6t + 7)$$
$$= \frac{6t + 7}{2\sqrt{3t^2 + 7t - 2}}$$

c) Ch(chain) rule. Chain-ception.

$$\varphi'(\theta) = \underbrace{2\tan\left(3e^{\theta}\right)\left[\tan\left(3e^{\theta}\right)\right]'}_{\text{first chain rule}}$$

$$= 2\tan\left(3e^{\theta}\right)\underbrace{\sec^2\left(3e^{\theta}\right)\left(3e^{\theta}\right)'}_{\text{second chain rule}}$$

$$= 2\tan\left(3e^{\theta}\right)\sec^2\left(3e^{\theta}\right)\left(3e^{\theta}\right)$$

$$= 6e^{\theta}\tan\left(3e^{\theta}\right)\sec^2\left(3e^{\theta}\right)$$

d) Chain rule.

$$p'(t) = 5^{\sqrt{t}}(\ln 5)\left(\sqrt{t}\right)'$$

$$= 5^{\sqrt{t}}(\ln 5)\left(t^{1/2}\right)'$$

$$= 5^{\sqrt{t}}(\ln 5)\left(\frac{1}{2}t^{-1/2}\right)$$

$$= 5^{\sqrt{t}}(\ln 5)\left(\frac{1}{2t^{1/2}}\right)$$

$$= 5^{\sqrt{t}}(\ln 5)\left(\frac{1}{2\sqrt{t}}\right)$$

$$= \left(\frac{\ln 5}{2\sqrt{t}}\right)5^{\sqrt{t}}$$

e) Chain and product rule.

$$y' = e^{x\cos(2x)}\underbrace{\left[x\cos(2x)\right]'}_{\text{do product rule}}$$

$$= e^{x\cos(2x)}\left\{(1)\cos(2x) + x\left[-\sin(2x)(2x)'\right]\right\}$$

$$= e^{x\cos(2x)}\left\{(1)\cos(2x) + x\left[-\sin(2x)(2)\right]\right\}$$

$$= e^{x\cos(2x)}\left[\cos(2x) - 2x\sin(2x)\right]$$

MAT 1320

Fall 2015

Midterm #1

Multiple Choice (1 mark each):

1. First, find the inverse function.

$$y=\frac{x-1}{2x-3}$$

Swap variables.

$$x=\frac{y-1}{2y-3}$$

Solve for y.

$$x(2y-3)=y-1$$
$$2xy-3x=y-1$$
$$2xy-y=-1+3x$$
$$y(2x-1)=-1+3x$$
$$y=\frac{3x-1}{2x-1}$$
$$f^{-1}(x)=\frac{3x-1}{2x-1}$$

Sub in point.

$$f^{-1}\left(\frac{2}{5}\right)=\frac{3\left(\frac{2}{5}\right)-1}{2\left(\frac{2}{5}\right)-1}$$
$$=\frac{\frac{6}{5}-1}{\frac{4}{5}-1}$$
$$=\frac{1/5}{-1/5}$$
$$=-1$$

Answer: C

2. Solve for x.

$$\ln(4x-3)=6$$
$$e^{\ln(4x-3)}=e^6$$
$$4x-3=e^6$$
$$4x=e^6+3$$
$$x=\frac{e^6+3}{4}$$

Answer: C

3. Solve for x.

$$2\sin\left(2x-\frac{\pi}{2}\right)=1$$
$$\sin\left(2x-\frac{\pi}{2}\right)=\frac{1}{2}$$
$$\arcsin\left[\sin\left(2x-\frac{\pi}{2}\right)\right]=\underbrace{\arcsin\left(\frac{1}{2}\right)}_{\text{something you're just supposed to "know"}}$$
$$2x-\frac{\pi}{2}=\frac{\pi}{6}$$
$$2x=\frac{\pi}{6}+\frac{\pi}{2}$$
$$2x=\pi\left(\frac{1}{6}+\frac{1}{2}\right)$$
$$=\pi\left(\frac{1}{6}+\frac{3}{6}\right)$$
$$=\frac{4\pi}{6}$$
$$x=\frac{4\pi}{12}$$
$$=\frac{\pi}{3}$$

Answer: B

Long Answer:

4. (4 points)

   a. Limit evaluates to 0/0. That's not good… Let's factor!

$$\lim_{x\to 1}\frac{5x^2-x-4}{2x^2+x-3}=\lim_{x\to 1}\frac{(x-1)(5x+4)}{(x-1)(2x+3)}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}(5x+4)}{\cancel{(x-1)}(2x+3)}$$

$$=\lim_{x\to 1}\frac{5x+4}{2x+3}$$

$$=\frac{5(1)+4}{2(1)+3}$$

$$=\frac{9}{5}$$

   b. Direct evaluation gives $\infty-\infty$ which is ***<u>not</u>*** equal to 0. Let's use the sneaky 1 trick!

$$\lim_{x\to\infty}\left(\sqrt{x^2+4x+5}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+4x+5}-x\right)\frac{\sqrt{x^2+4x+5}+x}{\sqrt{x^2+4x+5}+x}$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4x+5}-x\right)\left(\sqrt{x^2+4x+5}+x\right)}{\sqrt{x^2+4x+5}+x}$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4x+5}\right)\left(\sqrt{x^2+4x+5}\right)+x\left(\sqrt{x^2+4x+5}\right)-x\left(\sqrt{x^2+4x+5}\right)-x^2}{\sqrt{x^2+4x+5}+x}$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4x+5}\right)\left(\sqrt{x^2+4x+5}\right)\cancel{+x\left(\sqrt{x^2+4x+5}\right)}\cancel{-x\left(\sqrt{x^2+4x+5}\right)}-x^2}{\sqrt{x^2+4x+5}+x}$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4x+5}\right)\left(\sqrt{x^2+4x+5}\right)-x^2}{\sqrt{x^2+4x+5}+x}$$

$$=\lim_{x\to\infty}\frac{\left(x^2+4x+5\right)-x^2}{\sqrt{x^2+4x+5}+x}$$

$$=\lim_{x\to\infty}\frac{\left(\cancel{x^2}+4x+5\right)-\cancel{x^2}}{\sqrt{x^2+4x+5}+x}$$

$$=\lim_{x\to\infty}\frac{4x+5}{\sqrt{x^2+4x+5}+x}$$

At this point, we have a rational polynomial limit. So, we will factor the highest polynomial out of the numerator and denominator functions. In the denominator, to deal with the square root, we will factor out x^2 from each term *inside* the square root, and remove the x^2 from inside the square root. It will then become x once removed to the outside of the square root.

$$= \lim_{x\to\infty} \frac{x\left(4+\frac{5}{x}\right)}{\sqrt{x^2\left(1+\frac{4x}{x^2}+\frac{5}{x^2}\right)}+x(1)}$$

$$= \lim_{x\to\infty} \frac{x\left(4+\frac{5}{x}\right)}{x\sqrt{\left(1+\frac{4}{x}+\frac{5}{x^2}\right)}+x(1)}$$

$$= \lim_{x\to\infty} \frac{x\left(4+\frac{5}{x}\right)}{x\left[\sqrt{\left(1+\frac{4}{x}+\frac{5}{x^2}\right)}+1\right]}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\left(4+\frac{5}{x}\right)}{\cancel{x}\left[\sqrt{\left(1+\frac{4}{x}+\frac{5}{x^2}\right)}+1\right]}$$

$$= \lim_{x\to\infty} \frac{4+\frac{5}{x}}{\sqrt{\left(1+\frac{4}{x}+\frac{5}{x^2}\right)}+1}$$

$$= \frac{4+0}{\sqrt{(1+0+0)}+1}$$

$$= \frac{4}{\sqrt{1}+1}$$

$$= 2$$

5. (3 points)

For the function *f(x)* to be continuous, the functions $2x$ and ax^2+b must be equal at the point $x = 1$, and the functions ax^2+b and $4x$ must be equal at the point $x = 2$, where the transitions between functions take place.

Therefore,

$$2x = ax^2 + b \qquad x = 1$$
$$2(1) = a(1)^2 + b$$
$$2 = a + b$$

$$4x = ax^2 + b \qquad x = 2$$
$$4(2) = a(2)^2 + b$$
$$8 = 4a + b$$

We have two equations and two unknowns. There are several ways to solve for *a* and *b*. I will use substitution.

$$b = 2 - a$$

sub into other equation.

$$8 = 4a + (2 - a)$$
$$8 - 2 = 4a - a$$
$$6 = 3a$$
$$a = 2$$
$$b = 2 - a$$
$$= 0$$

6. (4 points)

a. The definition of the derivative is $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

b. Using the definition of the derivative…

$$f(1)=\frac{1+1}{1+2}=\frac{2}{3}$$

$$f(1+h)=\frac{1+h+1}{1+h+2}=\frac{2+h}{3+h}$$

$$\begin{aligned}
f'(1)&=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\
&=\lim_{h\to 0}\frac{\frac{2+h}{3+h}-\frac{2}{3}}{h}\\
&=\lim_{h\to 0}\left[\left(\frac{1}{h}\right)\left(\frac{2+h}{3+h}\right)-\left(\frac{1}{h}\right)\left(\frac{2}{3}\right)\right]\\
&=\lim_{h\to 0}\left[\left(\frac{1}{h}\right)\left(\frac{2+h}{3+h}\right)\left(\frac{3}{3}\right)-\left(\frac{1}{h}\right)\left(\frac{2}{3}\right)\left(\frac{3+h}{3+h}\right)\right]\\
&=\lim_{h\to 0}\frac{3(2+h)-2(3+h)}{3h(3+h)}\\
&=\lim_{h\to 0}\frac{(6+3h)-(6+2h)}{3h(3+h)}\\
&=\lim_{h\to 0}\frac{h}{3h(3+h)}\\
&=\lim_{h\to 0}\frac{\cancel{h}}{3\cancel{h}(3+h)}\\
&=\lim_{h\to 0}\frac{1}{3(3+h)}\\
&=\frac{1}{3(3+0)}\\
&=\frac{1}{9}
\end{aligned}$$

7. (6 points)
   a. Chain rule.

$$f'(x) = e^{3x^2}\left(3x^2\right)' + 2^x(\ln 2)$$
$$= 6xe^{3x^2} + 2^x(\ln 2)$$

   b. Quotient rule.

$$f(x) = \tan x$$
$$g(x) = x^2 + 1$$
$$f'(x) = \sec^2 x$$
$$g'(x) = 2x$$

$$y' \equiv \frac{dy}{dx}$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$
$$= \frac{\left(\sec^2 x\right)\left(x^2+1\right) - (\tan x)(2x)}{\left(x^2+1\right)^2}$$

   c. Chain rule.

$$h'(x) = -\sin\left(x^3 + x + 11\right)\left(x^3 + x + 11\right)'$$
$$= -\left(3x^2 + 1\right)\sin\left(x^3 + x + 11\right)$$