MATH 1005 H Fall 2018

Test One Solutions

1. (6 marks) Solve the initial-value problem

$$2y' = \frac{5x^4 - \cos x}{y}, \quad y(0) = -2.$$

Solution: This is a separable equation. Rearranging, we have

$$2yy' = 5x^4 - \cos x.$$

Integrating with respect to x (and using the substitution rule on the left) yields

$$\int 2y dy = \int (5x^4 - \cos x) dx$$
$$y^2 = x^5 - \sin x + c.$$

Thus $y = \pm\sqrt{x^5 - \sin x + c}$ is the general solution. Since $y(0) = -2$, we have

$$-2 = \pm\sqrt{c}.$$

Thus we must choose the negative sign and set $c = 4$. The solution to the initial value problem is

$$y = -\sqrt{x^5 - \sin x + 4}.$$

2. (6 marks) Find the general solution of the equation

$$xy' = y\big(1 + \ln y - \ln x\big), \quad x > 0, \ y > 0.$$

Hint: Begin by using a log law to combine the two logarithms.

Solution: Rearranging, we have

$$y' = \frac{y}{x}\left(1 + \ln\left(\frac{y}{x}\right)\right).$$

This is a homogeneous equation. We set $u = \frac{y}{x}$, so that $y = ux$ and $y' = u + xu'$. Then

$$u + xu' = u(1 + \ln u).$$

Rearranging, we have

$$\frac{1}{u \ln u} u' = \frac{1}{x}.$$

This is a separable equation. Integrating both sides with respect to x (and using the substitution rule on the left), we have

$$\int \frac{1}{u \ln u} du = \int \frac{1}{x} dx$$
$$\ln|\ln u| = \ln|x| + c_1$$
$$|\ln u| = e^{c_1}|x|$$
$$\ln u = \pm e^{c_1} x$$
$$\ln u = cx$$
$$u = e^{cx}$$

(to evaluate the integral on the left, the substitution $t = \ln u$ may be made). Thus

$$y = xu = xe^{cx}$$

is the general solution.

3. (6 marks) Find the general solution of the equation

$$x^2y' + 2xy = x^4 + x.$$

Solution: This is a linear equation. We divide by x^2 to put it in standard form:

$$y' + \frac{2}{x}y = x^2 + \frac{1}{x}.$$

We choose the integrating factor

$$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2.$$

Multiplying the equation by x^2 yields

$$\begin{aligned} x^2y' + 2xy &= x^4 + x \\ (x^2y)' &= x^4 + x. \end{aligned}$$

We integrate to get

$$\begin{aligned} x^2y &= \int \left(x^4 + x\right) dx \\ &= \frac{1}{5}x^5 + \frac{1}{2}x^2 + c. \end{aligned}$$

Dividing by x^2 yields the general solution

$$y = \frac{1}{5}x^3 + \frac{1}{2} + cx^{-2}.$$

4. (6 marks) Find the general solution of

$$3x^2e^y + (x^3e^y + 3y^2)y' = 0.$$

Solution: We have $P_y = 3x^2e^y$ and $Q_x = 3x^2e^y$. The equation is exact because P_y and Q_x are continuous in the entire plane (which is simply connected), and $P_y = Q_x$. A potential function f exists. We use the fact that $f_x = P$ and integrate with respect to x to get

$$f(x, y) = x^3e^y + g(y).$$

We now use the fact that we must have $f_y = Q$ to get

$$x^3e^y + g'(y) = x^3e^y + 3y^2.$$

Thus $g'(y) = 3y^2$ and so $g(y) = y^3 + c_1$. A potential function is

$$f(x, y) = x^3e^y + y^3 + c_1.$$

To get the solution of the equation, we set $f(x, y) = c_2$. Combining constants, this yields the solution

$$x^3e^y + y^3 = c.$$

5. (6 marks) Find the general solution of the equation

$$y' + y = -2e^{2x}y^2.$$

Solution: This is a Bernoulli equation with $\alpha = 2$. We set $u = y^{1-\alpha} = y^{-1}$. Then $y = u^{-1}$, and so $y^2 = u^{-2}$ and $y' = -u^{-2}u'$. Substituting this in to the equation, we get

$$-u^{-2}u' + u^{-1} = -2e^{2x}u^{-2}.$$

Multiplying by $-u^2$ yields

$$u' - u = 2e^{2x}.$$

This is a linear equation. We choose the integrating factor

$$I(x) = e^{\int -1 dx} = e^{-x}.$$

Multiplying the equation by e^{-x} yields

$$\begin{aligned} e^{-x}u' - e^{-x}u &= 2e^{x} \\ (e^{-x}u)' &= 2e^{x}. \end{aligned}$$

We integrate to get

$$\begin{aligned} e^{-x}u &= 2e^{x} + c \\ u &= 2e^{2x} + ce^{x}. \end{aligned}$$

Thus the general solution is

$$y = u^{-1} = \frac{1}{2e^{2x} + ce^{x}}.$$