CARLETON UNIVERSITY

SOLUTIONS
FINAL EXAMINATION
MATH 2004 A, B, C, D
Fall 2014

DURATION: 3 HOURS

Department Name and Course Number: School of Mathematics and Statistics, MATH 2004 A, B, C, D.
Course Instructor(s): Dr. A.B. Mingarelli (Sect. A), Dr. R. Cova (Sect. B, C, D)

AUTHORIZED MEMORANDA
STUDIO 56 SCIENTIFIC CALCULATOR ONLY AS PER COURSE OUTLINE.

In addition to the examination paper students will require an EXAMINATION BOOKLET, and a SCANTRON SHEET. This exam may be released to the Library.

1. Please verify that you are in possession of a SCANTRON FORM
2. Please fill in your COURSE CODE (e.g., MATH 2004) and COURSE SECTION (e.g., A, B, C, D), as per above list of instructors; YOUR NAME and YOUR STUDENT NUMBER where required on the Scantron form AND on this examination.
3. **The entire examination consists of 5 pages and two parts, A and B, and is marked out of a total of 80**, that is, 40% of your final mark. Part A consists of 12 multiple choice questions each worth 3 marks for a total of 36 marks. **Please fill in only one answer on your Scantron sheets with a pencil** as there is only one answer to any given question. Circling two or more answers to any question invalidates that question (*i.e.,* you get 0 marks for that question). Part B consists of 6 traditional type questions (with explanations required), for a total of 44 marks for that section. **Both part a and b must be submitted along with the scantron sheet, so do not detach nor unstaple this examination. DO NOT SUBMIT THE EXAMINATION BOOKLET CONTAINING ROUGH WORK**.

Print Name :

Student Number:

Section (either A, B, C, or D. See above for your Instructor's name):

PART A

Do All six (12) Questions for a total of 36 marks out of a maximum of 80.
Do not detach nor unstaple this examination. Missing sheets will void any credit for those questions.

- **A1.** Compute the slope of the tangent line to the polar curve $r = 4\cos\theta$ at $\theta = \frac{\pi}{3}$.

  (a) 0 (b) $\frac{\sqrt{3}}{3}$ (c) $\frac{1}{3}$ (d) $\sqrt{3}$ (e) the slope is vertical

  Solution: (b)

- **A2.** Find $\cos\alpha$, where α is the angle between the planes $3x - y + 2z = 2$ and $2x + 3y + z = 4$.

  (a) $\frac{5\sqrt{14}}{14}$ (b) $\frac{1}{\sqrt{14}}$ (c) $\frac{5}{14}$ (d) 69 (e) the planes are parallel

  Solution: (c)

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- **A3.** Find the volume of the parallelepiped defined by the vectors $< 2, 0, 0 >$, $< 0, \sqrt{2}, 0 >$, $< 0, 0, \sqrt{2} >$. (Note: In this examination the notation $< a, b, c >$ denotes the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with components a, b, c.)

  (a) 4 (b) $\sqrt{2}$ (c) 2 (d) $2\sqrt{2}$ (e) $4\sqrt{2}$

  Solution: (a)

- **A4.** Consider the plane Π that contains the point $(3, -2, 4)$ and is perpendicular to the line defined by $\vec{r} =< -1, 2, -4 > +t < -2, 3, 3 >$. Which of the following points belongs to Π ?

  (a) $(-2, 1, 1)$ (b) $(1, 1, 3)$ (c) $(-2, \frac{4}{3}, -1)$ (d) $(3, -2, 0)$ (e) $(0, 0, 0)$

  Solution: (e)

- **A5.** Let $w(x, y) = x^2 + y + 3xy^4$, where $x = \sin 2t$, $y = \cos t$. Use the Chain Rule to evaluate $\dfrac{\partial w}{\partial t}$ at $t = 0$.

  (a) 0 (b) 2 (c) 3 (d) 6 (e) -2

  Solution: (d)

- **A6.** A vector giving the direction in which $f(x, y, z) = \dfrac{x}{y} + \dfrac{y}{z}$ increases most rapidly at the point (1,-1,2) is

  (a) $\hat{i} - \hat{j} - \hat{k}$ (b) $-\hat{i} - \dfrac{1}{2}\hat{j} + \dfrac{1}{4}\hat{k}$ (c) $\hat{i} + \dfrac{1}{2}\hat{j} + \dfrac{1}{4}\hat{k}$ (d) $4\hat{i} + 2\hat{j} - \hat{k}$ (e) cannot be defined

  Solution: (b)

- **A7.** The relative extremum value(s) of $f(x, y) = -x^2 - 3y^2 + 4x - 6y + 8$ are

  (a) Minimum at (4,2) only (b) Minimum at (2,-1), Maximum at (-2,1)
  (c) Maximum at (2,-1) only (d) No extrema; saddle point at (-1,2)
  (e) Minimum at (2,-1), saddle point at (-1,2)

  Solution: (c)

- **A8.** Evaluate the double integral $\int_0^2 \int_0^{x/2} e^{-x^2}\, dy\, dx$.

  (a) $\dfrac{e^4 - 1}{4e^4}$ (b) $\dfrac{e^4 - 1}{4}$ (c) $\dfrac{e^4}{4}$ (d) $\dfrac{4e^2 - 1}{4}$ (e) none of these

  Solution: (a)

- **A9.** Evaluate the triple integral $\iiint\limits_{\mathcal{T}} xyz\, dz\, dy\, dx$ where $\mathcal{T}$ is the region

  $\mathcal{T} = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ \sqrt{x^2 + y^2} \le z \le 2\}$.

  (a) $\dfrac{3}{4}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{3}{2}$ (d) $\dfrac{3}{4}$ (e) $\dfrac{3}{8}$

  Solution: (e)

- **A10**. Under the change of variables $x = 5u,\ y = 2v$, determine which one of the integrals below is equal to $\iint\limits_{\mathcal{R}} xy\ dA$, where $\mathcal{R}$ is the region in the first quadrant bounded by $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$.

  (a) $\iint\limits_{\mathcal{S}} 10\ u^2v^2\ dA$ (b) $\iint\limits_{\mathcal{S}} 100\ u^2v^2\ dA$ (c) $\iint\limits_{\mathcal{S}} 10\ (u^2 + v^2)\ dA$ (d) $\iint\limits_{\mathcal{S}} 100\ uv\ dA$ (e) $\iint\limits_{\mathcal{S}} \dfrac{25}{4}\ uv\ dA$

  where $\mathcal{S} = \{(u, v) \mid u^2 + v^2 \le 1\}$.

  Solution: (d)

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- **A11.** Evaluate the line integral $\int_{\mathcal{C}} [(6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy]$, where $\mathcal{C}$ is the line segment from the point $(1,2)$ to $(3,4)$.

(a) 0 (b) 236 (c) 188 (d) 48 (e) 136

Solution: (b)

- **A12.** Consider the vector field $\vec{F}(x,y,z) =< x + \sin z,\ 2y + \cos x,\ 3z + \tan z >$. Compute the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathcal{S}$ is the unit sphere $x^2 + y^2 + z^2 = 1$ and $\hat{n}$ is an outer unit vector normal to $\mathcal{S}$.

(a) π (b) 16π (c) 8π (d) 4π (e) 8

Solution: (c)

End of Part A

PART B

Do All six (6) Questions for a total of 44 marks out of a maximum of 80.

Do not detach nor unstaple this examination. Missing sheets will void any credit for those questions.

- **B1.** [8 marks] Find and classify all the critical points of the function $f(x,y) = x^2y^2$ subject to the constraint $x + y = 1$.

Solution: $\nabla f = \lambda \nabla g,\ g = x + y - 1$. This implies that $2xy^2 = \lambda = 2x^2y$. Thus, either $x = 0, y = 1$ or $x = 1, y = 0$ or, finally, that $xy \neq 0$ which, in turn, implies that $x = y$. In this case, $x = y = 1/2$ and so $f(1/2, 1/2) = 1/16$. In the first two cases we have local minima at $(0,1), (1,0)$ while in the last case we have a local maximum at $(1/2, 1/2)$.

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- **B2.** [7 marks] Evaluate the double integral $I = \iint_{\mathcal{R}} xy\sqrt{x^2+y^2}\ dx\,dy$, by a suitable change of cordinates where $\mathcal{R} = \{(x,y) : 1 \leq x \leq 4, x \geq 0, y \geq 0\}$

  Solution: $\mathcal{R} = \{(x,y) : 1 \leq x \leq 4, x \geq 0, y \geq 0\}$ is given by $\mathcal{R} = \{(r,\theta) : 1 \leq r \leq 2, 0 \leq \theta \leq \pi/2\}$. Hence $I = \int_0^{\pi/2} \int_1^2 r^4 \sin\theta\cos\theta\ d\theta = \dfrac{31}{10}$.

- **B3.** [7 marks] Evaluate the line integral $I = \int_{\mathcal{C}} z\,dx + x\,dy + y\,dz$ over the arc of a helical curve C given by $x = \cos t$, $y = \sin t$, $z = t$ joining the points $(1,0,0)$ and $(1,0,2\pi)$.

  Solution: We have $x(t) = \cos t$, $y(t) = \sin t$ and $z(t) = t$ so that $x'(t) = -\sin t$, $y'(t) = \cos t$ and $z'(t) = 1$. The point $P(1,0,0)$ corresponds to $t = 0$ while the point $(1,0,2\pi)$ corresponds to $t = 2\pi$. So,

$$\begin{aligned} I &= \int_0^{2\pi} (t(-\sin t) + \cos^2 t + \sin t)\,dt \\ &= \int_0^{2\pi} (-t\sin t + \cos^2 t + \sin t)\,dt \\ &= 3\pi. \end{aligned}$$

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- **B4.** [7 marks] Determine whether there is a vector field $\mathbf{F}(x, y, z)$ whose curl is

$$\text{curl } \mathbf{F}(x, y, z) = 2xy\mathbf{i} - 2yz\mathbf{j} + 2xy\mathbf{k}.$$

If not, show why not. If so, then find it.

Solution: There is NO such vector field since if there were such then $\text{div}(\text{curl}\,\mathbf{F}) = 0$ is necessary for any vector field $\mathbf{F}$. However, in this case, the curl is already given as curl $\mathbf{F}(x, y, z) = 2xy\mathbf{i} - 2yz\mathbf{j} + 2xy\mathbf{k}$, so that $\text{div}(\text{curl}\,\mathbf{F}) = \text{div}(2xy\mathbf{i} - 2yz\mathbf{j} + 2xy\mathbf{k}) = 2y - 2z + 0 = 2y - 2z \neq 0$. The result follows.

- **B5.** [7 marks] Let $\mathbf{F}(x, y, z) = xz^2\,\mathbf{i} + (x^2y - z^3)\,\mathbf{j} + (2xy + y^2z)\,\mathbf{k}$ and $\mathcal{S}$ the surface of the solid hemispherical region $\mathcal{T}$ bounded by $z = \sqrt{1 - x^2 - y^2}$ and $z = 0$. Let $\mathbf{n}$ be an outer unit normal to $\mathcal{S}$.

Evaluate $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \bullet \mathbf{n}\, dS$ either directly or by using some theorem.

Solution: Use the Divergence Theorem, i.e., $I = \displaystyle\iint_{\mathcal{S}} \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_T \text{div } \mathbf{F}\, dV$. In this case, div $\mathbf{F} = z^2 + x^2 + y^2$.

Thus, converting to spherical coordinates it is easy to see that T may be described as

$$T = \{(\rho, \theta, \phi) : 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2\}.$$

Hence

$$\begin{aligned}
\iiint_T \text{div } \mathbf{F}\, dV &= \iiint_T (x^2 + y^2 + z^2)\, dV \\
&= \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 \rho^2 \cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^{\pi/2} \int_0^{2\pi} \sin\phi \left(\int_0^1 \rho^4 d\rho \right) d\theta\, d\phi \\
&= \frac{2\pi}{5}.
\end{aligned}$$

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- **B6.** [8 marks] Let $\mathbf{F}(x,y,z) = x^2\,\mathbf{i} + 2x\,\mathbf{j} + z^2\,\mathbf{k}$ and let $\mathcal{S}$ be the surface interior to the ellipse $4x^2 + y^2 = 4$ on $z = 0$. Evaluate the surface integral $\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\; dS$ using any method. Here $\mathbf{n}$ is an outer normal unit vector to $\mathcal{S}$.

Solution:

a) Use Stokes' Theorem, $\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\; dS = \int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ where $\mathcal{C}$ is the elliptical boundary of $\mathcal{S}$ oriented counterclockwise. We parametrize this curve using elliptic coordinates, i.e., $x = \cos t, y = 2\sin t$ where $0 \le t \le 2\pi$ and $z = 0$. Now,

$$
\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\cos t, 2\sin t, 0) \bullet \mathbf{r}'(t)\; dt \\
&= \int_0^{2\pi} (\cos^2 t, 2\cos t, 0) \bullet (-\sin t, 2\cos t, 0)\; dt \\
&= \int_0^{2\pi} (-\cos^2 t \sin t + 4\cos^2 t) dt \\
&= 4\pi.
\end{aligned}
$$

b) Direct calculation: This gives that curl $\mathbf{F} = (0,0,2)$. In addition, the normal to the plane $z = 0$ is given by $\mathbf{n} = \mathbf{k}$. Hence $\operatorname{curl}\mathbf{F} \cdot \mathbf{n} = 2$. It follows that

$$
\begin{aligned}
\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\; dS &= 2 \iint_{\mathcal{S}} du\, dv \\
&= 2 \cdot \text{Area of } \mathcal{S} \\
&= 2(1 \cdot 2 \cdot \pi) = 4\pi.
\end{aligned}
$$

TOTAL : 80