CONCORDIA UNIVERSITY
Department of Economics

ECON 221 SECTIONS A, BB, C
STATISTICAL METHODS I
Fall 2018 – MIDTERM 1
Sunday, October 21st, 15:00 – 17:00

Student Name: ______________________

Student ID: ______________________

Section Number: ______________________

Question 1 (16 pts.)

The variable X contains the following temperatures in Celsius observed over 5 consecutive days in December 2017: 2, -1, 0, 1, 0.

a) Briefly explain why X is a discrete variable. **(2 pts.)**

X takes on a finite number (N = 5) of values.

b) Briefly explain why X is NOT a random variable **(2 pts.)**

The temperature values are known/observed.

c) Compute the mean of X. **(2 pts)**

$$\bar{X} = \frac{\sum X_i}{N} = \frac{2+(-1)+0+1+0}{5} = 0.4$$

d) Compute the median of X. **(2 pts)**

Rank from lowest to highest

$-1 \quad 0 \quad \underline{0} \quad +1 \quad +2$

Median $(X) = 0$

e) Compute the mode of X. **(2 pts)**

Mode $(X) = 0$

f) Write down the formula for a variance. **(2 pts.)**

$$var(X) = \frac{\sum (X_i - \bar{X})^2}{N}$$

g) Compute the variance of X. **(2 pts.)**

$$\sigma_X^2 = Var(X) = \frac{(-1-0.4)^2 + (0-0.4)^2 + (0-0.4)^2 +}{5}$$

$$+ \frac{(1-0.4)^2 + (2-0.4)^2}{5} = \frac{5.2}{5} = 1.04$$

h) Compute the standard deviation of X. **(2 pts.)**

$$\sigma_X = \sqrt{var(X)} = \sqrt{1.04} = 1.02$$

Question 2 (6 pts.)

For the following function:

$$f(x) = \frac{4-x}{10} \text{ for } x = 3, 5$$

a) Construct the probability distribution of the random variable x. **(2 pts.)**

$$f(x=3) = \frac{4-3}{10} = \frac{1}{10} = 0.1$$

$$f(x=5) = \frac{4-5}{10} = \frac{-1}{10} = -0.1$$

b) Show whether $\Sigma_x f(x) = 1$ holds. **(2 pts.)**

$$\Sigma_x f(x) = 0.1 + (-0.1) = 0 \neq 1$$

$\Rightarrow$ It does not hold.

c) Show whether $f(x) \geq 0$ holds for each x. **(2 pts.)**

$f(x=3) = 0.1 \geq 0$ $\Rightarrow$ It holds.

$f(x=5) = -0.1 < 0$ $\Rightarrow$ It does not hold.

Question 3 (8 pts.)

If $P(A) = 0.37$, $P(B) = 0.25$, and $P(A \cup B) = 0.62$.

a) Show whether the events A and B are mutually exclusive. **(2 pts.)** ~~No~~ Yes, since

$$P(A \cup B) = P(A) + P(B)$$

~~No~~ Yes, since $0.62 = 0.37 + 0.25$

b) Show whether the events A and B are independent. **(2 pts.)** No, they are not.

$\cancel{P(A \cap B) =}$ $P(A)\,P(B) = 0.37(0.25) \approx 0.09$

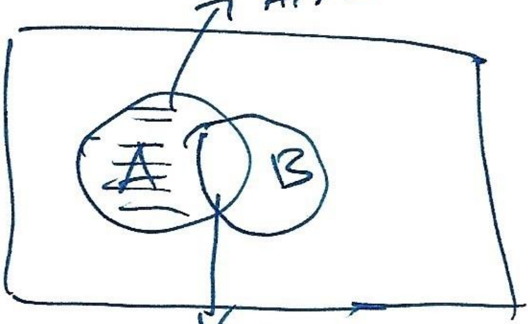
$\cancel{P(A}$ $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= 0.37 + 0.25 - 0.62 = 0$

This implies that $P(A \cap B) \neq P(A)\,P(B)$.

c) Calculate $P(A \cap \bar{B})$. **(2 pts.)**

$A \cap \bar{B}$



$P(A \cap B) = 0$

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$= 0.37 - 0$

$= 0.37$

since A and B are mutually exclusive

d) Calculate the complement of $P(A \cup B)$. **(2 pts.)**

$P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - 0.62 = 0.38$

Question 4 (8 pts.)

Among the 30 applicants for a position at a credit union, they are distinguished based on two characteristics: prior work experience and a possession of a university degree according to the following table:

	University Graduate (u)	Not a university graduate $(\bar{u})$
Some work experience (W)	6	3
No work experience $(\bar{W})$	12	9

a) Calculate the probability that an applicant is a university graduate with some work experience. **(2 pts.)**

$$P(u \cap W) = \frac{6}{30} = 0.2$$

b) Calculate the probability that an applicant has some work experience. **(2 pts.)**

$$P(W) = P(u \cap W) + P(\bar{u} \cap W)$$
$$= \frac{6}{30} + \frac{3}{30} = \frac{9}{30} = 0.3$$

c) Find the odds that an applicant has some work experience. **(2 pts.)**

$$P(W) = 0.3 \Rightarrow P(\bar{W}) = 1 - P(W)$$
$$= 1 - 0.3 = 0.7$$

$$\text{odds} = \frac{P(W)}{1 - P(W)} \qquad \text{Odds} = \frac{0.3}{0.7} = \frac{3}{7}$$

d) Given that an applicant has some work experience, find the probability that they are a university graduate. **(2 pts.)**

$$P(u/W) = \frac{P(u \cap W)}{P(W)} = \frac{0.2}{0.3} = \frac{2}{3}$$

Question 5 (18 pts.)

Consider the grade distribution of Math 333 based on historical data:

Letter Grade	Number of Students
A	40
B	50
C	60
D	30
F	20

In addition, a student must complete Math 333 with a grade of B or higher in order to enrol in Math 444.

a) Calculate the number of trials. **(2 pts.)**

$$n = 40 + 50 + 60 + 30 + 20 = 200$$

b) Calculate the probability of success. **(2 pts.)**

$$p = P(\overset{X=A}{\cancel{A}}) + P(\overset{X=B}{\cancel{B}}) = \frac{40}{200} + \frac{50}{200} = \frac{90}{200} = 0.45$$

c) Calculate the expected value of the random variable *x*, where *x* measures the number of students meeting the prerequisite requirement. **(2 pts.)**

$$E(X) = np = 200(0.45) = 90$$

d) Calculate the variance of the random variable *x*, where *x* measures the number of students meeting the prerequisite requirement. **(2 pts.)**

$$\begin{aligned} var(X) &= np(1-p) = \\ &= 200(0.45)(1-0.45) \\ &= 49.5 \end{aligned}$$

e) State the two conditions required for the normal approximation to the distribution used in part a) may be used. **(2 pts.)**

$np \geq 5$

$n(1-p) \geq 5$

f) Show that each of the two conditions in part e) hold. **(2 pts.)**

$200(0.45) = 90 \geq 5$

$200(1-0.45) = 110 \geq 5$

g) The university administration stipulates that Math 444 be offered if its enrolment is at least 50 students. Use the normal approximation of the distribution you used in part a) to compute the probability of at least 50 students meeting the prerequisite requirement for Math 444:

i. Write down the formula that relates X to a standardized normal variable Z. **(2 pts.)**

$$Z = \frac{X - \mu_X}{\sigma_X} = \frac{X - E(X)}{\sqrt{var(X)}}$$

ii. Calculate the Z value corresponding to $X = 50$. **(2 pts.)**

$$Z = \frac{X - E(X)}{\sqrt{var(X)}} = \frac{X-90}{\sqrt{49.5}} = \frac{X-90}{7.04}$$

iii. Calculate the probability of at least 50 students meeting the prerequisite requirement for Math 444. **(2 pts.)**

$$P(X \geq 50) = P\left(Z \geq \frac{\overset{50}{\cancel{X}} - 90}{7.04}\right) = P(Z \geq 5.68)$$

$$= 1 - P(Z \leq -5.68) = 1 - 0 = 1$$

Question 6 (14 pts.)

Consider the following pairs of observations for the variables X and Y: (-1, 0); (0, 1); (1; 2). The mean (μ_X) and the variance (σ_X^2) of X are 0 and 0.67 respectively. The mean (μ_Y) and the variance (σ_Y^2) of Y are 1 and 0.67 respectively.

a) Write down the formula for the covariance of X and Y (σ_{XY}). **(2 pts.)**

$$\sigma_{XY} = \frac{\sum_i [(X_i - \mu_X)(Y_i - \mu_Y)]}{N}$$

b) Calculate the covariance of X and Y. **(2 pts.)**

$$\sigma_{XY} = \frac{(-1-0)(0-1)+(0-0)(1-1)+(1-0)(2-1)}{3}$$

$$= \frac{1+0+1}{3} = \frac{2}{3}$$

c) Compute $E(5X + 2Y)$ as a function of μ_x and μ_Y. **(2 pts.)**

$$E(5X+2Y) = 5E(X) + 2E(Y)$$

$$= 5\mu_X + 2\mu_Y$$

d) Calculate a numerical value for $E(5X + 2Y)$. **(2 pts.)**

$$E(5X+2Y) = 5\mu_X + 2\mu_Y = 5(0) + 2(1) = 2$$

e) Compute $var(3X - Y)$ as a function of $\sigma_X^2, \sigma_Y^2, \sigma_{XY}$. **(2 pts.)**

$$var(3X - Y) = var(3X) + var(Y) - 2cov(3X, Y)$$
$$= 9var(X) + var(Y) - 6cov(X, Y)$$
$$= 9\sigma_X^2 + \sigma_Y^2 - 6\sigma_{XY}$$

f) Calculate a numerical value for $var(3X - Y)$. **(2 pts.)**

$$var(3X - Y) = 9\sigma_X^2 + \sigma_X^2 - 6\sigma_{XY}$$
$$= 9(0.67) + (0.67) - 6(0.67)$$
$$= 6.7 - 4.0 = 2.7$$

Question 7 (10 pts.)

Among an ambulance service's 16 ambulances, five emit excessive amounts of pollutants. If eight of the ambulances are randomly picked for inspection, determine the probability that this sample will include at least four of the ambulances that emit excessive amounts of pollutants.

a) Write down the formula for the probability density function. **(2 pts.)**

$$P(x) = \frac{C_x^S \, C_{n-x}^{N-S}}{C_n^N} = \frac{\frac{S!}{x!(S-x)!} \, \frac{(N-S)!}{(n-x)!(N-S-n+x)!}}{\frac{N!}{n!(N-n)!}}$$

b) Calculate $P(X = 4)$. **(3 pts.)**

$$P(x=4) = \frac{\frac{5!}{4!\,1!} \, \frac{11!}{4!\,7!}}{\frac{16!}{8!\,8!}} \approx 0.128$$

c) Calculate $P(X = 5)$. **(3 pts.)**

$$P(x=5) = \frac{\frac{5!}{5!\,0!} \, \frac{11!}{3!\,8!}}{\frac{16!}{8!\,8!}} \approx 0.013$$

d) Calculate $P(X \geq 4)$. **(2 pts.)**

$$P(x \geq 4) = P(x=4) + P(x=5)$$
$$= 0.128 + 0.013$$
$$= 0.141$$

Question 8 (12 pts.)

Suppose that a random variable X follows a normal distribution with a mean equal to 82.0 and a standard deviation equal to 4.8.

a) Convert the random variable X to a random variable Z that follows a standard normal distribution. **(2 pts.)**

$$Z = \frac{X - \mu_X}{\sigma_X}$$

b) Show that Z has the mean associated with a standard normal distribution. **(2 pts.)**

$$E(Z) = E\left[\frac{X - \mu_X}{\sigma_X}\right] = \frac{1}{\sigma_X} E[X - \mu_X]$$

$$= \frac{1}{\sigma_X}\left[E(X) - \mu_X\right] = \frac{1}{\sigma_X}\left[\mu_X - \mu_X\right] = 0$$

c) Show that Z has the variance associated with a standard normal distribution. **(2 pts.)**

$$Var(Z) = var\left[\frac{X - \mu_X}{\sigma_X}\right]$$

$$= \frac{1}{\sigma_X^2} var(X - \mu_X)$$

$$= \frac{1}{\sigma_X^2} var(X)$$

$$= \frac{1}{\sigma_X^2} \sigma_X^2 = 1$$

d) To find the probability that X will take on a value between 83.2 and 88.0:

i. Write down the formula for range probabilities for X. **(2 pts.)**

$$P(83.2 \le X \le 88.0)$$

ii. Calculate the Z values corresponding to the range probabilities for X. **(2 pts.)**

$$P\left(\frac{83.2-82}{4.8} \le Z \le \frac{88-82}{4.8}\right)$$

$$P(0.25 \le Z \le 1.25)$$

iii. Calculate the probability that X will take on a value between 83.2 and 88.0. **(2 pts.)**

$$P(0.25 \le Z \le 1.25) =$$

$$= P(Z \le 1.25) - P(Z \le 0.25)$$

$$= 0.8944 - 0.5987$$

$$= 0.2957$$