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DEPARTMENT OF COMPUTER SCIENCE & SOFTWARE ENGINEERING  
COMP232 MATHEMATICS FOR COMPUTER SCIENCE  
FALL 2018

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**Assignment 4.**

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1. Establish the following properties by induction or strong induction.

(a)  $\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$  for all  $n \geq 1$ .

(b)  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$  for all  $n \geq 0$

(c) Every positive integer  $n$  can be represented as a sum of distinct powers of 2, i.e., in the form  $n = 2^{i_1} + \cdots + 2^{i_h}$  with integers  $0 \leq i_1 < \cdots < i_h$ .

(d) Let  $D_n$  denote the number of ways to cover the squares of a 2-by- $n$  board using plain dominos. Then it is easy to see that  $D_1 = 1, D_2 = 2$ , and  $D_3 = 3$ . Compute a few more values of  $D_n$  and guess an expression for the value of  $D_n$  and use induction to prove you are right.

2. Determine whether or not each of the following relations is a partial order and state whether or not each partial order is a total order.

(a)  $(\mathbb{N} \times \mathbb{N}, \leq)$ , where  $(a, b) \leq (c, d)$  if and only if  $a \leq c$ .

(b)  $(\mathbb{N} \times \mathbb{N}, \leq)$ , where  $(a, b) \leq (c, d)$  if and only if  $a \leq c$  and  $b \geq d$ .

3. Which of the following relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(a)  $\{(a, b) | a \text{ and } b \text{ are the same age}\}$

(b)  $\{(a, b) | a \text{ and } b \text{ have the same parents}\}$

(c)  $\{(a, b) | a \text{ and } b \text{ share a common parent}\}$

(d)  $\{(a, b) | a \text{ and } b \text{ have met}\}$

(e)  $\{(a, b) | a \text{ and } b \text{ speak a common language}\}$

4. Consider the following relation  $\simeq$  defined on the set  $\mathbb{N} \times \mathbb{Z}^+$ .

$$(m_1, n_1) \simeq (m_2, n_2) \text{ iff } m_1 n_2 = m_2 n_1.$$

(a) Prove that it is an equivalence and find equivalence classes.

- (b) Provide a concise characterization of the equivalence classes in terms of rational numbers.
5. Consider the following relation  $<$  over reals:  $x < y$  iff  $(x - y) \in \mathbb{Z}$ . Prove that it is an equivalence and characterize its equivalence classes
6. A set  $S$  of jobs can be ordered by writing  $x \leq y$  to mean that either  $x = y$  or  $x$  must be done before  $y$ , for all  $x$  and  $y$  in  $S$ . Given the Hasse diagram represented in Figure 1 for this relation for a particular set  $S$  of jobs, show the following:
- minimal, least, maximal, and greatest elements;
  - a topological sort.

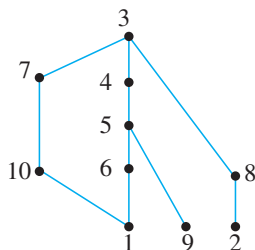
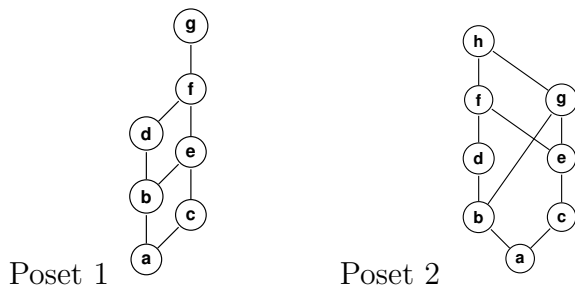


Figure 1: Hasse Diagram

7. Determine whether the posets with these Hasse diagrams are lattices.



8. Determine whether the following posets are lattices:
- $(1, 3, 6, 9, 12, |)$
  - $(1, 5, 25, 125, |)$
  - $(\mathbb{Z}, \geq)$
  - $(P(S), \supseteq)$ , where  $P(S)$  is the power set of a set  $S$ .
9. Let  $R$  be a relation on  $\mathbb{N}$  defined by  $(x, y) \in R$  iff there is a prime  $p$  such that  $y = px$ . Describe in words the reflexive, symmetric and transitive closures of  $R$ , denoted by  $r$ ,  $s$  and  $t$ , respectively.
- Which of the following are true:

$$r(s(R)) = s(r(R))$$

$$r(t(R)) = t(r(R))$$

$$s(t(R)) = t(s(R))$$

You need to justify your answer.

- (b) Which of them hold for all relations on  $\mathbb{N}$ ?
- (c) Using the reflexive, symmetric, and transitive closures, express the smallest equivalence relation containing an arbitrary relation.
- (d) What is the smallest partial order containing  $R$ ?