Université d'Ottawa · University of Ottawa

Faculté des sciences Mathématiques et de statistique — Faculty of Science Mathematics and Statistics

MAT1322A **Midterm Test 2** **Fall 2018**

19 November 2018 **Professor: Z. Montazeri**

Time: 75 minutes **Total: 30 marks**

Student Number:___________ **Family Name:**___________

First Name:___________

You must sign below.

Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur:

You will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. **By signing below, you acknowledge that you have ensured that you are complying with the above statement.**

Your signature:______________

- This is closed book exam.
- Questions 1-5 are **Multiple Choice.** Circle your answer. Not justification is needed.
- Questions 6-9 are **Long Answer.** You must justify your answers with a clear and complete solution.
- You may use the backs of pages and for rough work. **Do not use any of your own scrap paper.**
- Only those calculators explicitly allowed by the Faculty of Sciences (Texas Instruments TI-30, TI-34 et Casio fx-260. fx-300) are authorized. If you are caught with any other make or model of calculator, then it will be confiscated, allegations of academic fraud may be filed, and/or you may receive zero for this test.
- The exam will be marked on a total of 30 points.

MULTIPLE CHOICE QUESTIONS: Circle the correct answer. NO justification is required. Each question has 2.5 marks

1. For all $n \geqslant 1$, let $a_n = \frac{5n^3+2n-3}{2n^3+5n^2}$. Which of the following statements is true?

   A. The sequence $\{a_n\}$ diverges and the series $\sum_{n=1}^{\infty} a_n$ also diverges.
   B. The sequence $\{a_n\}$ diverges and the series $\sum_{n=1}^{\infty} a_n$ converges.
   C. The sequence $\{a_n\}$ converges and the series $\sum_{n=1}^{\infty} a_n$ also converges.
   D. The sequence $\{a_n\}$ converges and the series $\sum_{n=1}^{\infty} a_n$ diverges. (**)

2. Determine the sum of the geometric series $\sum_{n=1}^{\infty}(3^{2n})(2^{-4n})$ if it exists.

   A. $\frac{7}{9}$ **B.** $\frac{9}{7}$ (**) **C.** $\frac{16}{7}$ **D.** $\frac{7}{16}$ **E.** The series is divergent

3. What is the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-4)^n}{\sqrt{n+1}}$

   A. ∞ **B.** 1 **C.** $\frac{1}{4}$ **D.** $\frac{1}{2}$ (**) **E.** None of the above

4. Which of the following series do NOT the converges to finite value?

   A. $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(3n+2)^3}$ **B.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$ (**) **C.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ **D.** $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$
   E. None of the above

5. Use any test to determine the convergence of the following series:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n[\ln(n)]}$$

   A. It converges conditionally (**) **B.** It converges absolutely
   C. It diverges **D.** It converges to the value 1/3 **E.** None of the above

LONG ANSWER QUESTIONS: Give detailed solutions, clearly showing each of your steps.

6. [5] Find the radius and interval of convergence of the power series given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{\sqrt{n+3}}$$

Solution:

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \frac{\sqrt{n+4}}{\sqrt{n+3}} = 1, \rightarrow |x-2| < 1$$

so series converges if $|x-2| < 1 \rightarrow -1 < x-2 < 1 \rightarrow 1 < x < 3$
if $x = 1$, $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$ diverges by using comparison test
If $x = 3$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ is alternating series which $b_n = \frac{1}{\sqrt{n+3}}$ which is decreasing and $\lim b_n = 0$ and series converges
Interval of convergence is $1 < x \leq 3$ or $(1, 3]$

7. [2.5] For the following series, determine whether it converges absolutely, converges conditionally, or diverges. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{(n+1)!}$$

Solution:
$\sum_{n=1}^{\infty} \left| \frac{(-3)^n}{(n+1)!} \right| = \sum_{n=1}^{\infty} \frac{3^n}{(n+1)!}$ converges according to ratio test where

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+2)!} \frac{(n+1)!}{3^n} = \lim_{n\to\infty} \frac{3}{n+2} = 0 < 1$$

so the series converges absolutely.

8. [5] Consider the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

a. Approximate the sum s of the series by s_3 and determine lower and upper bounds for the error R_3.

b. How large must n be taken in order that s_n approximate s with error no greater that 0.001?

Solution:

a) $s_3 = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} = 1.546$

$$\int_4^{\infty} x^{-\frac{3}{2}} dx < R_3 < \int_3^{\infty} x^{-\frac{3}{2}} dx$$

$$\lim_{a\to\infty} -2x^{-\frac{1}{2}}\Big]_4^a < R_3 < -\lim_{b\to\infty} 2x^{-\frac{1}{2}}\Big]_3^b \to 1 < R_3 < \frac{2}{\sqrt{3}}$$

b) $R_n < \int_n^{\infty} x^{-\frac{3}{2}} dx \to \lim_{a\to\infty} -2x^{-\frac{1}{2}}\Big]_n^a \to \frac{2}{\sqrt{n}} < 0.001 \to \sqrt{n} > 200 \to n > 4000000$

9. **a.** [2.5] Find the coefficient of x^4 in the Taylor series of the function $f(x) = \cos(2x)$ about the point $a = 0$.

Solution:
$T(x) = \sum \frac{f^{(n)}(0)}{n!}(x)^n$ so the coefficient of x^4 is $\frac{f^{(4)}(0)}{4!}$
$f'(x) = -2\sin(2x)$, $f''(x) = -4\cos(2x)$, $f^{(3)}(x) = 8\sin(2x)$, $f^{(4)}(x) = 16\cos(2x) \rightarrow \frac{f^{(4)}(0)}{4!} = \frac{16}{4!} = \frac{2}{3}$

b. [2.5] Express $f(x) = \frac{3x^2}{1+x}$ as a power series about 0.

Solution:
$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \quad |t| < 1$ with $t = -x$, then

$$f(x) = \frac{3x^2}{1+x} = 3x^2 \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} 3(-1)^n x^{n+2}, \quad |x| = |t| < 1$$