CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number		Section(s)
Mathematics	209/2		All except EC
Examination	**Date**	**Time**	**Pages**
Final	December 2014	3 Hours	2

Instructors	**Course Examiner**
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Special Instructions:

▷ Answer all questions.

▷ **Only approved calculators are allowed.**

MARKS

[9] **1.** Find the following limits:

(a) $\lim_{x \to 1^-} \frac{|x-1|}{x-1}$ (b) $\lim_{x \to -2} \frac{(x+2)^2}{x^2-4}$ (c) $\lim_{x \to \infty} \frac{x^2+4}{4-25x^2}$

[18] **2.** Find the derivative for each of the following (do not simplify):

(a) $y = 5x^{-7} - 2x^{-4}$

(b) $y = \frac{5}{x^{\frac{1}{5}}} - \frac{8}{x^{\frac{3}{2}}}$

(c) $y = \frac{2x^5 - 4x^3 + 2x}{x^3}$

(d) $y = (1 + e^x)\ln x$

(e) $y = \frac{\log_2 x}{1 + x^2}$

(f) $y = 2\ln(x^2 - 3x + 4)$

[6] **3.** Use implicit differentiation to find $y' = \frac{dy}{dx}$ for $xe^y - y = x^2 - 2$.

[10] **4.** A manufacturer of sunglasses currently sells one type for \$15 a pair. The price p and the demand x for these glasses are related by

$$x = f(p) = 9,500 - 250p$$

(a) Calculate Elasticity E.

(b) Use answer in (a) to find whether revenue increase or decrease.

[12] **5.** Given $f(x) = x^4(x-6)^2$ find:

(a) the critical values of f.

(b) the intervals where $f(x)$ is increasing;

(c) the intervals where $f(x)$ is decreasing;

(d) the local maxima and minima.

[6] **6.** Given $g(x) = \ln(x^2 - 2x + 10)$ find:

(a) the intervals where $g(x)$ is concave upward;

(b) the intervals where $g(x)$ is concave downward;

(c) the inflection point(s);

[6] **7.** Find the absolute extrema of $f(x) = x^4 - 8x^2 + 16$ on the interval $[-3, 4]$.

[9] **8.** Evaluate the following; answers must be accurate to 3 decimals:

(a) $\int_{-5}^{5}(10 - 7x + x^2)dx$

(b) $\int_0^1 xe^{-2x^2}dx$

(c) $\int_0^3 \frac{x}{(1+x^2)^2}dx$

[10] **9.** Compute the antiderivatives:

(a) $\int \frac{x^2e^x - 2x}{x^2}dx$

(b) $\int \frac{x}{\sqrt{x+5}}dx$

(c) $\int x^3(2x^4+5)^5dx$

(d) $\int \frac{e^{-x}}{(e^{-x}+3)}dx$

[10] **10.** Find the area bounded by $y = x^3 + 1$ and $y = x + 1$.

[4] **11.** If the exponential growth law applies to Canada's population growth, at what continuous compound growth rate will the population double over the next 100 years?

Math209 Final Exam December 2014 Solutions (UNEDITED)

1 a) $\lim_{x \to 1^-} \frac{|x-1|}{x-1} = \frac{0}{0}$; $\lim_{x \to 1^-} \frac{-\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \to 1^-} (-1) = -1$

b) $\lim_{x \to -2} \frac{(x+2)^2}{x^2-4} = \frac{0}{0}$; $\lim_{x \to -2} \frac{\cancel{(x+2)}(x+2)}{\cancel{(x+2)}(x-2)} = \frac{0}{-4} = 0$

c) $\lim_{x \to \infty} \frac{x^2+4}{4-25x^2} = \frac{\infty}{-\infty}$; $\lim_{x \to \infty} \frac{\frac{x^2}{x^2} - \frac{4}{x^2}}{\frac{4}{x^2} - \frac{25x^2}{x^2}} = \lim_{x \to \infty} \frac{1 - \frac{4}{x^2}}{\frac{4}{x^2} - 25} = -\frac{1}{25}$

2 a) $y = 5x^{-7} - 2x^{-4}$; $y' = 5(-7)x^{-8} - 2(-4)x^{-5} = -35x^{-8} + 8x^{-5}$

b) $y = \frac{5}{x^{\frac{1}{5}}} - \frac{8}{x^{3/2}}$; $y = 5x^{-\frac{1}{5}} - 8x^{-\frac{3}{2}}$; $y' = 5(-\frac{1}{5})x^{-\frac{6}{5}} - 8(-\frac{3}{2})x^{-\frac{5}{2}}$

$y' = -x^{-\frac{6}{5}} + 12x^{-\frac{5}{2}}$

c) $y = \frac{2x^5 - 4x^3 + 2x}{x^3}$; $y = \frac{2x^5}{x^3} - \frac{4x^3}{x^3} + \frac{2x}{x^3}$; $y = 2x^2 - 4 + 2x^{-2}$

$y' = 4x - 0 - 2(-2)x^{-3}$

$y' = 4x - 4x^{-3}$

d) $y = (1+e^x)\ln x$; $y' = (1+e^x)\frac{1}{x} + \ln x(0 + e^x)$

$y' = \frac{1+e^x}{x} + (\ln x)e^x$

e) $y = \frac{\log_2 x}{1+x^2}$; $y' = \frac{(1+x^2)\frac{1}{x\ln 2} - (\log_2 x)2x}{(1+x^2)^2}$

$y' = \frac{\frac{(1+x^2)}{x}\ln 2 - 2x\log_2 x}{(1+x^2)^2}$

f) $y = 2\ln(x^2-3x+4)$; $y' = 2\,\frac{1}{x^2-3x+4}\,(2x-3)$

$y' = \frac{2(2x-3)}{x^2-3x+4}$

3. $xe^y - y = x^2 - 2$

$x\frac{d}{dx}e^y + e^y\frac{d}{dx}x - \frac{d}{dx}y = \frac{d}{dx}x^2 - \frac{d}{dx}2$

$xe^y\frac{d}{dx}y + e^y - \frac{dy}{dx} = 2x$

$\frac{dy}{dx}[xe^y - 1] = 2x - e^y$

$\frac{dy}{dx} = \frac{2x - e^y}{xe^y - 1}$

4. $E = -\frac{p}{x} * \frac{dx}{dp}$

Step 1

$= -\frac{p}{9500-250p}(-250)$

$E = \frac{250p}{9500-250p}$

$E = \frac{p}{38-p}$

$E|_{p=15} = \frac{15}{38-15} = 0.652$

Step 2 $\frac{dR}{dp} = x(1-E)$

$= \underbrace{x(1-.652)}_{\text{positive}}$

$\frac{dR}{dp} > 0 \Rightarrow$ Rev. Inc

5. $f(x) = x^4(x-6)^2$

$$f'(x) = x^4(2)(x-6) + (x-6)^2(4x^3)$$
$$= 2x^4(x-6) + 4x^3(x-6)^2$$
$$= 2x^3(x-6)[x + 2(x-6)]$$
$$= 2x^3(x-6)[3x-12]$$
$$= 6x^3(x-6)(x-4)$$

$f'(x) = 0$	$f'(x) = \frac{1}{0}$
$6x^3(x-6)(x-4) = 0$	No x here
$x=0 \quad x=6 \quad x=4$	Critical values

x	$-\infty < x < 0$	$x=0$	$0 < x < 4$	$x=4$	$4 < x < 6$	$x=6$	$6 < x < \infty$
$f'(x)$	$f'(-1) = -$	0	$f'(1) = +$	0	$f'(5) = -$	0	$f'(7) = +$
$f(x)$	Dec	Local Min	Inc.	Local Max	Dec	Local Min	Inc

6. $g(x) = \ln(x^2-2x+10)$

$$g'(x) = \frac{1}{x^2-2x+10}(2x-2)$$

$$g'(x) = \frac{2(x-1)}{x^2-2x+10}$$

$$g''(x) = 2 * \frac{[(x^2-2x+10)(1) - (x-1)(2x-2)]}{(x^2-2x+10)^2}$$

$$= 2\frac{[x^2-2x+10 - 2(x^2-2x+1)]}{(x^2-2x+10)^2}$$

$$= \frac{2[x^2-2x+10-2x^2+4x-2]}{(x^2-2x+10)^2}$$

$$g''(x) = \frac{2[-x^2+2x+8]}{(x^2+2x+10)^2} = -\frac{2(x+2)(x-4)}{(x^2+2x+10)^2}$$

$g''(x) = 0$	$g''(x) = \frac{1}{0}$
$2[-x^2+2x+8] = 0$	$(x^2+2x+10)^2 = 0 \Rightarrow x^2+2x+10 = 0$
$x^2-2x-8 = 0$	$b^2-4ac < 0 \Rightarrow$ No sol.
$(x+2)(x-4) = 0$	
$x=-2 \quad x=4$	

x	$-\infty < x < -2$	$x=-2$	$-2 < x < 4$	$x=4$	$4 < x < \infty$
$f''(x)$	$f''(-3) = -$	0	$f''(0) = +$	0	$f''(5) = -$
$f(x)$	Concave down	Inflection pt.	Concave up	Inf. pt	Concave down

7. $f(x) = x^4 - 8x^2 + 16$ on $[-3, 4]$

$f'(x) = 4x^3 - 16x$

$f'(x) = 0$ $\qquad$ $f'(x) = \frac{1}{0}$

$4x^3 - 16x = 0$ $\qquad$ No x

$4x(x^2 - 4) = 0$

$x=0$	$x=2$	$x=-2$

test

$f(-3) =$

$f(4) =$

$f(0) =$

$f(2) =$

$f(-2) =$

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ pick out the MAX & MIN from these values.

8. a) $\int_{-5}^{5} (10 - 7x + x^2)\,dx = \left[10x - \frac{7x^2}{2} + \frac{x^3}{3}\right]_{-5}^{5}$

$= \left(10(5) - \frac{7}{2}(5)^2 + \frac{(5)^3}{3}\right) - \left(10(-5) - \frac{7}{2}(-5)^2 + \frac{(-5)^3}{3}\right)$

$= \frac{550}{3}$

b) $\int_0^1 x e^{-2x^2}\,dx$

let $u = -2x^2$

$\frac{du}{dx} = -4x$

$du = -4x\,dx$

$-\frac{1}{4}du = x\,dx$

$\int e^{-x^2} x\,dx$

$\int e^{u} \left(-\frac{1}{4}du\right)$

$-\frac{1}{4} e^{u} du \quad = \left[-\frac{1}{4} e^{-2x^2}\right]_0^1 = \left[-\frac{1}{4} e^{-2(1)^2} - \left(-\frac{1}{4}e^0\right)\right]$

$= \frac{1}{4}\left[-e^{-2} + 1\right]$

$= \frac{1}{4}\left[1 - \frac{1}{e^2}\right]$

c) $\int_0^3 \frac{x}{(1+x^2)^2}\,dx$

let $u = 1 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int (1+x^2)^{-2} x\,dx$

$\int u^{-2} \left(\frac{1}{2}du\right)$

$\frac{1}{2} \cdot \frac{u^{-2+1}}{-2+1}$

$-\frac{u^{-1}}{2} \quad = -\frac{1}{2} \frac{1}{u} = -\frac{1}{2}\left[\frac{1}{1+x^2}\right]_0^3$

$= -\frac{1}{2}\left[\frac{1}{1+3^2} - \frac{1}{1+0^2}\right]$

$= -\frac{1}{2}\left[\frac{1}{10} - 1\right]$

9a) $\displaystyle\int \frac{x^2e^x - 2x}{x^2}\,dx$

$= \displaystyle\int e^x\,dx - \int \frac{2}{x}\,dx$

$= e^x - 2\ln x + C$

b) $\displaystyle\int (5x^3+1)^{-3}\,15x^2\,dx$

Let $u = 5x^3+1$

$\frac{du}{dx} = 15x^2$

$du = 15x^2\,dx$

$u^{-3}\,du$

$\frac{u^{-2}}{-2} + C$

$-\frac{1}{2u} + C$

$-\frac{1}{2(5x^3+1)} + C$

c) $\displaystyle\int \frac{x}{\sqrt{x+5}}\,dx$

Let $u = x+5 \Rightarrow u-5 = x$

$du = dx$

$\displaystyle\int \frac{u-5}{\sqrt{u}}\,du$

$\displaystyle\int \left(\frac{u}{u^{1/2}} - \frac{5}{u^{1/2}}\right)du$

$\displaystyle\int u^{+\frac{1}{2}}\,du - 5\int u^{-\frac{1}{2}}\,du$

$\frac{u^{3/2}}{\frac{3}{2}} - 5\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$

$\frac{2}{3}(x+5)^{3/2} - 10(x+5)^{\frac{1}{2}} + C$

d) $\displaystyle\int x^3(2x^4+5)^5\,dx$

Let $u = 2x^4+5$

$\frac{du}{dx} = 8x^3$

$du = 8x^3\,dx$

$\frac{1}{8}du = x^3\,dx$

$\displaystyle\int (2x^4+5)^5\,x^3\,dx$

$\displaystyle\int u^5\left(\tfrac{1}{8}\right)du$

$\frac{1}{8}\frac{u^6}{6} + C$

$\frac{1}{48}(2x^4+5)^6 + C$

e) $\displaystyle\int \frac{e^{-x}}{e^{-x}+3}\,dx$

Let $u = e^{-x}+3$

$\frac{du}{dx} = e^{-x}$

$du = e^{-x}\,dx$

$\displaystyle\int \frac{du}{u}$

$\ln|u| + C$

$\ln|e^{-x}+3| + C$

10.

$y = x^3+1$
$y = x+1$

Step 1 Intersection pts:

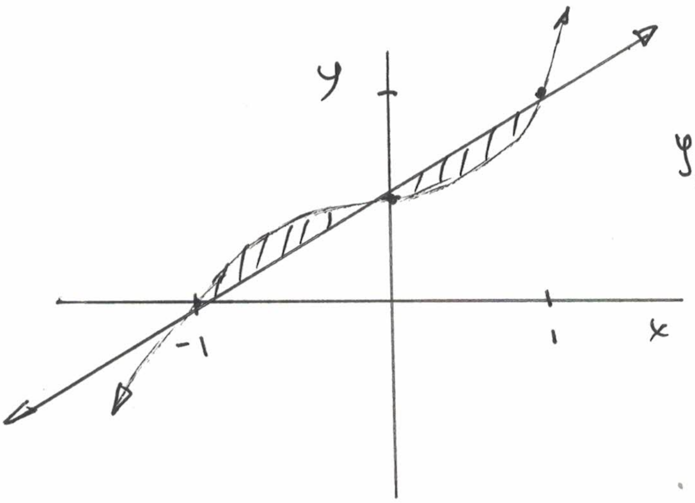
$x^3+1 = x+1$
$x^3 - x = 0$
$x(x^2-1) = 0$

$x=0$ | $x^2-1=0$
$x = \pm 1$

$y = x+1$
$\Rightarrow y = 0+1$
pt $(0,1)$

$x=-1$: $y = x+1$, $y = -1+1$, $y=0$, pt $(-1,0)$

$x=1$: $y = x+1$, $y = 1+1$, $y = 2$, pt $(1,2)$



If $x = -\frac{1}{2}$ $\quad y = x^3+1 \Rightarrow y = (-\frac{1}{2})^3+1 = \frac{7}{8}$
$y = x+1 \Rightarrow y = -\frac{1}{2}+1 = \frac{1}{2}$
y on curve > y on line

If $x = \frac{1}{2}$ $\quad y = x^3+1 \Rightarrow y = (\frac{1}{2})^3+1 = 1\frac{1}{8}$
$y = x+1 \Rightarrow y = \frac{1}{2}+1 = 1\frac{1}{2}$
y on line > y on curve

$$\text{Area} = \int_{-1}^{0}\left[(x^3+1)-(x+1)\right]dx + \int_{0}^{1}\left[(x+1)-(x^3+1)\right]dx$$

$$\text{Area} = \int_{-1}^{0}(x^3-x)\,dx + \int_0^1 (x-x^3)\,dx$$

$$= \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

$$= \left(\frac{0^4}{4}-\frac{0^2}{2}\right)-\left(\frac{[-1]^4}{4}-\frac{[-1]^2}{2}\right) + \left(\frac{1^2}{2}-\frac{1^4}{4}\right)-\left(\frac{0^2}{2}-\frac{0^4}{4}\right)$$

$$= -\frac{1}{4}+\frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{1}{2} \text{ or } 0.5 \text{ SQUARE UNITS}$$

11.

$C = C_0 e^{rt}$

Note: $C = 2C_0$
$t = 10$
$r =$ unknown

$2\cancel{C_0} = \cancel{C_0} e^{r(10)}$

$2 = e^{10r}$

$\log_e 2 = 10r$

$\frac{\ln 2}{10} = r$

$= r$

$\Longrightarrow$ Rate is 0.0693 or 6.93%