ENGR 311: Transform Calculus Summary

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Notes/properties:

- $\sin\left(t + \frac{\pi}{2}\right) = \cos t$
- cos(t)-cos(-t)=0
- Use convolution instead of transform of the integral when you see $\int_0^t f(\alpha)g(t-\alpha)\,d\alpha$
- $\cos^2 3t \Rightarrow \frac{1}{2}(1 + \cos(2(3t)))$ using $\cos a \cos b$ property from formula sheet.
- $\cos t\Big|_{t\to 0=1}^{t\to\infty=1}$, $\sin t\Big|_{t\to 0=0}^{t\to\infty=0}$
- Partial fractions:

Factor in the denominator	Term in partial fraction
$as + b$	$\frac{A}{as+b}$
$(as+b)^k$	$\frac{A}{as+b} + \frac{B}{(as+b)^2} + \cdots + \frac{C}{(as+b)^k}$ $k = 1,2,3\ldots$
$as^2 + bs + c$	$\frac{As+B}{as^2+bs+c}$
$(as^2+bs+c)^k$	$\frac{As+B}{(as^2+bs+c)^1} + \cdots + \frac{Cs+D}{(as^2+bs+c)^k}$

- If you are taking the partial fraction of for example: $\frac{e^{-2s}+e^{-4s}}{(s-1)(s-6)}$ then you do the partial fraction of $\frac{1}{(s-1)(s-6)}$ then multiply that by e^{-2s} and e^{-4s}. You can't take the partial fraction of $e^{(\square)}$
- If you have $\frac{1}{(s-1)^2+25}$ then that + 25 is a sign you shouldnt use partial fractions but instead use sin or cos and e^{at}
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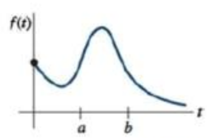
4- If graph of f(t) is given as follow,



match the given graph with one of the given functions

(a) $f(t) - f(t)\,\mathcal{U}(t-a)$
(b) $f(t-b)\,\mathcal{U}(t-b)$
(c) $f(t)\,\mathcal{U}(t-a)$
(d) $f(t) - f(t)\,\mathcal{U}(t-b)$
(e) $f(t)\,\mathcal{U}(t-a) - f(t)\,\mathcal{U}(t-b)$
(f) $f(t-a)\,\mathcal{U}(t-a) - f(t-a)\,\mathcal{U}(t-b)$

θ	0 0	$\frac{\pi}{6}$ 30°	$\frac{\pi}{4}$ 45°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2}$ 90°
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	U
$\csc(\theta)$	U	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec(\theta)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	U
$\cot(\theta)$	U	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Function loses part before a

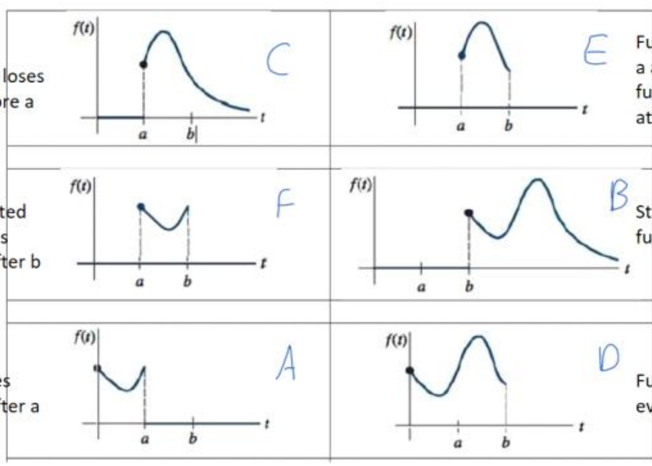
Function loses part before a and then also loses the function that would start at b

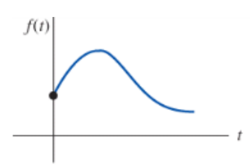
Function shifted by a and loses everything after b

Starting point of function shifted by b

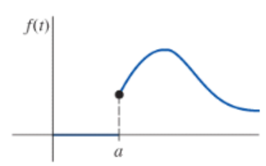
Function loses everything after a

Function loses everything after b





(a) $f(t),\ t \geq 0$



(b) $f(t-a)\,\mathcal{U}(t-a)$

Ch.4: Gamma Function, Transform of Derivatives

Gamma function:

$\mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ Where: $\alpha > (-1)$, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

1) Define each α.
2) Write in the form $\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$
3) Convert $\Gamma(\alpha+1)$ to $\alpha\Gamma(\alpha)$ then split the remaining fraction into a new $\alpha+1$.
4) Repeat until you have $\Gamma\left(\frac{1}{2}\right)$ which equals $\sqrt{\pi}$.

Ex: $\mathcal{L}\left\{6t^{\frac{1}{2}} - 24t^{\frac{5}{2}}\right\}$

1) $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{5}{2}$
2) $= 6\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{3}{2}}} - 24\frac{\Gamma\left(\frac{5}{2}+1\right)}{s^{\frac{7}{2}}}$
3) $= 6*\frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} - 24*\frac{5}{2}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{7}{2}}} = 6*\frac{1}{2}\frac{\sqrt{\pi}}{s^{\frac{3}{2}}} - 24*\frac{5}{2}\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{7}{2}}}$
4) $\Rightarrow 6*\frac{1}{2}\frac{\sqrt{\pi}}{s^{\frac{3}{2}}} - 24*\frac{5}{2}*\frac{3}{2}*\frac{1}{2}\frac{\sqrt{\pi}}{s^{\frac{7}{2}}} = \boldsymbol{\frac{3\sqrt{\pi}}{s^{\frac{3}{2}}} - \frac{45\sqrt{\pi}}{s^{\frac{7}{2}}}}$

Transform of derivatives:

$\mathcal{L}\{f^{n}(t)\} = S^{n}F(s) - S^{n-1}f(0) - S^{n-2}f'(0) - \cdots - f^{n-1}(0)$

1) Take the Laplace of both sides. Find the Laplace of each derivative of y separately and plug in known values for y(0), y'(0) etc.
2) Isolate for Y(s). Use partial fractions to find Y(s)=_________
3) $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ so solve for y(t)

If y(0) or y'(0) is not known, write c in its place and solve for y(t). Then plug in the other given value (ex: y(1)=2) and solve for c then plug it back into y(t).

Ex: $\mathcal{L}\{y'' - 4y' + 4y\} = t^3$, $y(0) = 1$, $y'(0) = 0$,

1) $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$

   $\mathcal{L}\{y'\} = sY - y(0)$

   $\mathcal{L}\{t^3\} = \frac{6}{s^4}$

2) $s^2Y - s - 4sY + 4 + 4Y = \frac{6}{s^4} \Rightarrow Y = \frac{\left(\frac{6}{s^4} + s - 4\right)}{s^2 - 4s + 4} \approx> Y$

   $= \frac{3}{2}*\frac{1}{s^4} + \frac{3}{4}*\frac{1}{s^3} + \frac{9}{8}*\frac{1}{s^2} + \frac{3}{4}*\frac{1}{s} + \frac{\frac{s}{4} - \frac{17}{18}}{(s-2)^2}$

   a. $\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{\left((s-2) - \frac{13}{2}\right)}{(s-2)^2}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{4}*\frac{1}{s-2} - \frac{13}{8}*\frac{1}{(s-2)^2}\right\} \Rightarrow \frac{1}{4}e^{2t} - \frac{13}{8}te^{2t}$

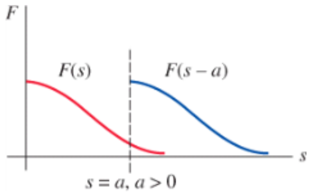
3) $\boldsymbol{y(t) = \frac{1}{4}t^3 + \frac{3}{4}t^2 + \frac{9}{8}t + \frac{3}{4} + \frac{1}{4}e^{2t} - \frac{13}{8}te^{2t}}$

Ch.4: All Translation Theorems

<u>First translation theorem:</u>

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

1) Identify f(t) and a and write s→s-a
2) Solve $\mathcal{L}\{f(t)\} = F(s)$
3) Substitute all s values with s-a



Ex: $\mathcal{L}\{(1-e^{t}+3e^{-4t})\cos 5t\}$

1) $f(t) = \cos 5t\,, a_1 = 1, a_2 = -4 \; so \; s \to s-1 \text{ and } s \to s+4$
2) $\mathcal{L}\{\cos 5t - e^{t}\cos 5t + 3e^{-4t}\cos 5t\} \Rightarrow \dfrac{s}{s^2+25} - \dfrac{s}{s^2+25} + \dfrac{3s}{s^2+25}$
3) $\boldsymbol{F(s) = \dfrac{s}{s^2+25} - \dfrac{(s-1)}{(s-1)^2+25} + \dfrac{3(s+4)}{(s+4)^2+25}}$

<u>Inverse translation theorem:</u>

$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

1) Find (s-a) and identify the "a" value. Make sure that all values of s in the equation are in the form of (s-a).
2) Solve $\mathcal{L}^{-1}\{F(s)\}$ which gives f(t)
3) Multiply by e^{at}

Ex: $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\} \Rightarrow$

1) $\mathcal{L}^{-1}\left\{\dfrac{2(s+3)-1}{(s+3)^2+25}\right\} \Rightarrow a = -3 \; and \; (s+3) \to s$
2) $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s)^2+25} - \dfrac{1}{(s)^2+25}\right\} \Rightarrow 2\cos 5t - \dfrac{1}{5}\sin 5t$
3) $\boldsymbol{f(t) = 2e^{-3t}\cos 5t - \dfrac{1}{5}e^{-3t}\sin 5t}$

<u>Second translation theorem (inverse example):</u>

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

1) Identify e^{-as}, a, and $F(s)$
2) Solve $\mathcal{L}^{-1}\{F(s)\}$ which gives f(t)
3) Substitute t with t-a and multiply f(t) by u(t-a)

Ex: $\mathcal{L}^{-1}\left\{\frac{se^{-\frac{\pi s}{2}}}{s^2+4}\right\}$

1) $e^{-as} = e^{-\frac{\pi s}{2}}, a = \dfrac{\pi}{2}, F(s) = \dfrac{s}{s^2+4}$
2) $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\} = \cos 2t \Rightarrow$
3) $\boldsymbol{\mathrm{f(t)} = \cos(2t-\pi)u(t-\dfrac{\pi}{2})}$

<u>Alternative second translation theorem:</u>

$\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

1) Identify g(t) and a.
2) Write in the form $e^{-as}\mathcal{L}\{g(t+a)\}$ and substitute all t values with t+a.
3) Solve.

Ex: $\mathcal{L}\left\{\sin t\, u\left(t-\frac{\pi}{2}\right)\right\}$

1) $g(t) = \sin t\,, a = \dfrac{\pi}{2}$
2) $e^{-\frac{\pi}{2}s}\mathcal{L}\left\{\sin\left(t+\dfrac{\pi}{2}\right)\right\} \Rightarrow \text{note that } \sin\left(t+\dfrac{\pi}{2}\right) = \cos t \Rightarrow$
3) $\boldsymbol{F(s) = \dfrac{se^{-\frac{\pi}{2}s}}{s^2+1}}$

Ch.4: Derivatives of Transforms, Convolution Theorem

Derivatives of transforms:

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

1) Write $(-1)^n \frac{d^n}{ds^n}$ and then solve $\mathcal{L}\{f(t)\} = F(s)$
2) Derive F(s) n times.

Note: This theorem is usually combined with other theorems. Use this theorem pretty much every time you have $t^{(\square)}$ in front of some function or functions.

Ex: $\mathcal{L}\{te^t \sin t\}$

1) $(-1)^1 \frac{d^1}{ds^1} \mathcal{L}\{e^t sint\} \Rightarrow$ note here you are using translation theorem to solve $\mathcal{L}\{e^t sint\}$
2) $\frac{d}{ds}\left(-\frac{1}{(s-1)^2+1}\right) = \frac{\mathbf{2s-2}}{\left((\mathbf{s-1})^2+\mathbf{1}\right)^2}$

Convolution:

$\mathcal{L}\{f * g\} = \mathcal{L}\{g * f\} = \int_0^t f(\alpha)g(t-\alpha)\,d\alpha = F(s)G(s)$, $*$ is convolution, green is regular multiplicatio.

1) Identify $f(\tau)$ and $g(t-\tau)$ and write in the form f(t) and g(t)
2) Take the Laplace of f(t) and of g(t)
3) Multiply $\mathcal{L}\{f(t)\}$ times $\mathcal{L}\{g(t)\}$

Memorize convolution integral:

$\int_0^t f(\alpha)g(t-\alpha)\,d\alpha$

Ex: $\mathcal{L}\left\{\int_0^t \tau e^{t-\tau}\,d\tau\right\}$

1) $f(\tau) = \tau \Rightarrow f(t) = t,\qquad g(t-\tau) = e^{t-\tau} \Rightarrow g(t) = e^t$
2) $\mathcal{L}\{t\} = \frac{1}{s^2},\qquad \mathcal{L}\{e^t\} = \frac{1}{s-1}$
3) $= \frac{\mathbf{1}}{\mathbf{s^2(s-1)}}$

Inverse form of convolution theorem:

$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g = \int_0^t f(\alpha)g(t-\alpha)\,d\alpha$

1) Separate the equation into F(s) and G(s)
2) Take the inverse Laplace of F(s) and G(s) separately
3) $\mathcal{L}^{-1}\{F(s)G(s)\}$ is equal to f(t)*g(t) which is convolution and is then equal to $\int_0^t f(\alpha)g(t-\alpha)\,d\alpha$ so replace the t from f(t) with α and the t from g(t) with t-α and plug in to the formula.
4) Solve the integral.

Ex: $\mathcal{L}^{-1}\left\{\frac{3s}{(s^2+1)^2}\right\}$

1) $\mathcal{L}^{-1}\left\{\frac{3s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\frac{3}{s^2+1}\right\} = \mathcal{L}^{-1}\{F(s)G(s)\}$
2) $\mathcal{L}^{-1}\{F(s)\} = \cos t,\qquad \mathcal{L}^{-1}\{G(s)\} = 3\sin t$
3) $\int_0^t (\cos\alpha)(3\sin(t-\alpha))\,d\alpha$
4) Knowing that: $\sin a\cos b = \frac{1}{2}(\sin(a+b)+\sin(a-b))$ with: $a=\alpha, b=(t-\alpha)$

$3\int_0^t \frac{1}{2}(\sin t + \sin(2\alpha - t)\,d\alpha \Rightarrow \frac{3}{2}\int_0^t \sin t\,d\alpha + \frac{3}{2}\int_0^t \sin(2\alpha-t)\,d\alpha \Rightarrow \frac{3}{2}\alpha\sin t\,\Big|_0^t + \frac{-3}{4}\cos(2\alpha-t)\Big|_0^t$

$\Rightarrow \frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{t\sin t} + \mathbf{0} \Rightarrow \frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{t\sin t},\qquad$ Note: $-\frac{3}{4}(\cos t - \cos(-t)) = 0$ because $\cos(-x) = -\cos(x)$ property.

Other good examples: problem set 5, #2, #3a

Ch.4: Transform and Inverse Transform of an Integral

Transform and inverse transform of an integral:=

Inverse transform of integral steps:

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

1) Separate until you have some $\frac{F(s)}{s}$ and define F(s)
2) Take the $\mathcal{L}^{-1}\{F(s)\} = f(t)$
3) Plug f(t) into equation $\int_0^t f(\tau)d\tau$ and solve

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

1) $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{(s^2+1)}}{s}\right\}, \qquad F(s) = \frac{1}{(s^2+1)}$
2) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin t = f(t)$
3) $\int_0^t f(\alpha)d\alpha = \int_0^t \sin\alpha\, d\alpha \Rightarrow [-\cos\alpha]_0^t \Rightarrow \mathbf{1 - \cos}\, \boldsymbol{t}$

Other good examples:
problem set 5, #3b

Transform of integral steps:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} = \frac{\mathcal{L}\{f(t)\}}{s}$$

1) Identify $f(\tau)$
2) Plug $f(\tau)$ into $\frac{\mathcal{L}\{f(\tau)\}}{s}$ and solve. Note that when solving $\mathcal{L}\{f(\tau)\}$, you will have to use methods shown previously such as derivatives of transforms etc.

Ex: $\mathcal{L}\left\{\int_0^t \tau\sin\tau\, d\tau\right\}$

1) $f(\tau) = \tau\sin\tau$
2) $\frac{\mathcal{L}\{t\sin t\}}{s} \xrightarrow{using\ derivative\ of\ transform} \frac{\left[\frac{(-1)^n d^n}{ds^n}(\mathcal{L}\{\sin t\})\right]}{s} \Rightarrow \frac{\left(-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right)}{s} = \frac{\mathbf{2}}{\mathbf{(s^2+1)^2}}$

Ch.4: Volterra Equation, Laplace of a Periodic Function

<u>Volterra integral equation:</u>

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)\,d\tau, \qquad \text{where } g(t) \text{ and } h(t) \text{ are known.}$$

1) Take Laplace transform of both sides. Usually you will have to do convolution or transform of an integral on the integral.
2) Isolate F(s)
3) Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$

Other good examples: problem set 5, #3a, #3b

Ex: $f(t) = 4e^{-t} + \sin t - 2\int_0^t f(\tau)\cos(t-\tau)\,d\tau$

1) $F(s) = \dfrac{4}{s-1} + \dfrac{1}{s^2+1} - 2\mathcal{L}\left\{\int_0^t f(\tau)\cos(t-\tau)\,d\tau\right\}$ ← used convolution

$$= \frac{4}{s+1} + \frac{1}{s^2+1} - 2\mathcal{L}\{f(t)\}\mathcal{L}\{\cos t\} \Rightarrow F(s) = \frac{4}{s+1} + \frac{1}{s^2+1} - \frac{2sF(s)}{s^2+1}$$

2) $\left(1 + \dfrac{2s}{s^2+1}\right)F(s) = \dfrac{4}{s+1} + \dfrac{1}{s^2+1} \Rightarrow \left(\dfrac{s^2+2s+1}{s^2+1}\right)F(s) \Rightarrow \left(\dfrac{(s+1)^2}{s^2+1}\right)F(s) \Rightarrow$

$$F(s) = \frac{4(s^2+1)}{(s+1)^3} + \frac{1}{(s+1)^2}$$

3) $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{4(s^2+1)}{(s+1)^3} + \dfrac{1}{(s+1)^2}\right\} \xrightarrow{\text{partial fractions}} F(s) = \dfrac{4}{s+1} - \dfrac{7}{(s+1)^2} + \dfrac{8}{(s+1)^3}$

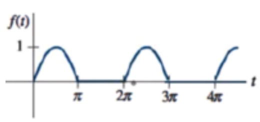
$$\Rightarrow \boldsymbol{f(t) = 4e^{-t} - 7te^{-t} + \frac{8}{2}t^2e^{-t}}$$

<u>Laplace of a periodic function:</u>

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt, \qquad \text{where } f(t) = f(t+T), \qquad \text{where T is the period of } f(t)$$

1) Identify the period (T) of f(t) and write the expression for f(t).
2) Plug in values into $\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$. f(t) could be split into two parts in which case $\int_0^T f(t)dt = \int_0^a f_1(t)dt + \int_a^T f_2(t)dt$
3) Solve.

Ex:



Half-wave rectification of sin *t*

1) $T = 2\pi, \qquad f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi < t \le 2\pi \end{cases}$

2) $\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-s2\pi}}\displaystyle\int_0^{2\pi} e^{-st}f(t)\,dt = \dfrac{1}{1-e^{-s2\pi}}\displaystyle\int_0^{\pi} e^{-st}\sin t\,dt + \int_{\pi}^{2\pi} e^{-st}(0)dt$

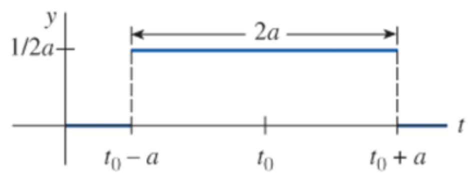
3) $\dfrac{1}{1-e^{-s2\pi}}\displaystyle\int_0^{\pi} e^{-st}\sin t\,dt \xrightarrow{\text{by parts}} \dfrac{1}{1-e^{-s2\pi}}\left[-\dfrac{1}{s^2+1}e^{-st}(\cos t + s\sin t)\right]_0^{\pi} \approx$

$$\approx \frac{1}{(1-e^{-\pi s})(1+e^{-\pi s})}\cdot\frac{1}{s^2+1}(1+e^{-\pi s}) = \boldsymbol{\frac{1}{(1-e^{-\pi s})}\cdot\frac{1}{s^2+1}}$$

Ch.4: Unit Impulse Function, Dirac Delta Function

<u>Unit impulse function:</u>

$$\delta_a(t-t_0)=\begin{cases}0, & 0\le t<t_0-a\\ \dfrac{1}{2a}, & t_0-a\le t<t_0+a\,,\\ 0, & t\ge t_0+a\end{cases}\qquad where\ a>0\ and\ t_0>0$$



And $\int_0^\infty \delta_a(t-t_0)\,dt=1$ which is why it is called the <u>unit</u> impulse function.

<u>Dirac delta function:</u>

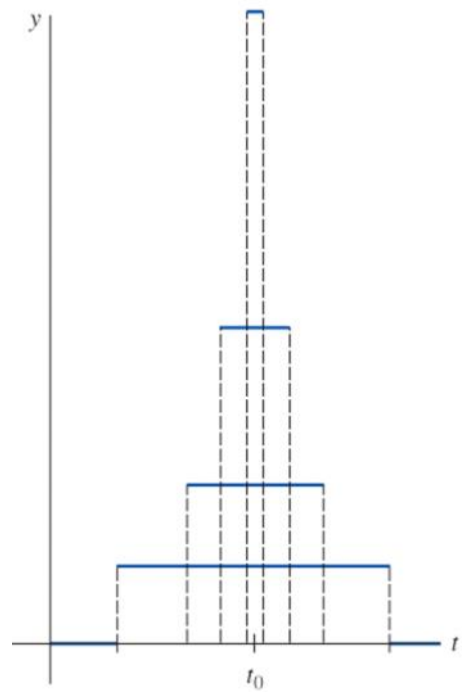
It is the unit impulse function when "a" approaches 0. So literally $\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$

Properties:

$$\delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\ne t_0\end{cases}$$

$$\int_0^\infty \delta(t-t_0)\,dt=1$$

$$\int_0^\infty f(t)\delta(t-t_0)\,dt=f(t_0)$$



(b) Behavior of δ_a as $a\to 0$

Laplace transform of Dirac delta function:

$\mathcal{L}\{\delta(t-t_0)=e^{-st_0}$

And: $\mathcal{L}\{\delta_a(t)\}=\mathcal{L}\{\delta_a(t-0)\}=1\ since\ t_0=0$

1) Find t_0
2) Substitute t_0 into e^{-st_0}

Ex: $\mathcal{L}\left\{\delta\left(t-\frac{\pi}{2}\right)\right\}$

1) $t_0=\dfrac{\pi}{2}$
2) $\Rightarrow \boldsymbol{e^{-\frac{s\pi}{2}}}$

Ch.4: System of Liner Differential Equations

<u>System of linear differential equations:</u>

1) Take the Laplace of both equations
2) Factor out X(s) and Y(s) in both equations
3) Isolate for either X(s) or Y(s) in one of the equations and substitute it back into the other equation. Then solve for X(s) or Y(s) then solve for the other one.
4) Take the Laplace inverse of X(s) and Y(s)

Ex: $\begin{cases} 2\frac{dx}{dt}+\frac{dy}{dt}-2x=1 \\ \frac{dx}{dt}+\frac{dy}{dt}-3x-3y=2 \end{cases} \quad x(0)=0, \;\; y(0)=0$

Use the Laplace transform to solve the given system of differential equations. In other words, find x(t) and y(t).

1) $\begin{cases} \mathcal{L}\left\{2\frac{dx}{dt}\right\}+\mathcal{L}\left\{\frac{dy}{dt}\right\}-\mathcal{L}\{2x\}=\mathcal{L}\{1\} \\ \mathcal{L}\left\{\frac{dx}{dt}\right\}+\mathcal{L}\left\{\frac{dy}{dt}\right\}-\mathcal{L}\{3x\}-\mathcal{L}\{3y\}=\mathcal{L}\{2\} \end{cases} = \begin{cases} 2sX(s)-2X(0)+sY(s)-Y(0)-2X(s)=\frac{1}{s} \\ sX(s)-X(0)+sY(s)-Y(0)-3X(s)-3Y(s)=\frac{2}{s} \end{cases}$

2) $\begin{cases} X(s)(2s-2)+Y(s)(s)=\frac{1}{s} \\ X(s)(s-3)+Y(s)(s-3)=\frac{2}{s} \end{cases}$

3) $a)\; Y(s)=\frac{2}{s(s-3)}-X(s) \Rightarrow X(s)(2s-2)+\left[\frac{2}{s(s-3)}-X(s)\right](s)=\frac{1}{s} \Rightarrow X(s)[(2s-2)-s]$
$=\frac{1}{s}-\frac{2}{s-3} \Rightarrow \boldsymbol{X(s)}=\frac{\mathbf{1}}{\boldsymbol{s(s-2)}}-\frac{\mathbf{2}}{\boldsymbol{(s-3)(s-2)}}$

$b)\; Y(s)=\frac{2}{s(s-3)}-X(s) \Rightarrow \boldsymbol{Y(s)}=\frac{\mathbf{2}}{\boldsymbol{s(s-3)}}-\frac{\mathbf{1}}{\boldsymbol{s(s-2)}}+\frac{\mathbf{2}}{\boldsymbol{(s-3)(s-2)}}$

4) $a)\; \mathcal{L}^{-1}\{X(s)\} \xRightarrow{partial\ fractions} \mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s}+\frac{5}{2}\frac{1}{s-2}-\frac{2}{s-3}\right\} \Rightarrow -\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{5}}{\mathbf{2}}\boldsymbol{e^{2t}}-\mathbf{2}\boldsymbol{e^{3t}}=\boldsymbol{x(t)}$

$b)\; \mathcal{L}^{-1}\{Y(s)\} \xRightarrow{partial\ fractions} \mathcal{L}^{-1}\left\{-\frac{1}{6}\frac{1}{s}+\frac{5}{2}\frac{1}{s-2}-\frac{8}{3}\frac{1}{s-3}\right\} \Rightarrow -\frac{\mathbf{1}}{\mathbf{6}}+\frac{\mathbf{5}}{\mathbf{2}}\boldsymbol{e^{2t}}-\frac{\mathbf{8}}{\mathbf{3}}\boldsymbol{e^{3t}}=\boldsymbol{y(t)}$

Ch.12: Inner Product, Orthogonality, Orthonormality

Inner product:
The inner product of functions $f_1(x)$ and $f_2(x)$ on interval [a,b] is:

$$\left(f_1(x)\,, f_2(x)\right) = \int_a^b f_1(x)\, f_2(x)\ dx$$

Orthogonal functions:
Functions $f_1(x)$ and $f_2(x)$ are orthogonal on [a,b] if the inner product =0:

$$\left(f_1(x)\,, f_1(x)\right) = 0$$

Orthogonal sets:
Multiple cases for proving sets are orthogonal:

1) Given $\{f(x), g(x), h(x)\}$
   a. Find inner product of and show =0:
      i. $\left(f(x), g(x)\right) = 0$
      ii. $\left(f(x), h(x)\right) = 0$
      iii. $\left(g(x), h(x)\right) = 0$
   b. If they are all equal to 0, then the set is orthogonal
2) Given $\{\emptyset_1(x), \emptyset_2(x) \ldots \emptyset_n(x)\}$
   a. Find in form of a single function, $\emptyset_n(x)$:
   b. Find inner product of $\left(\emptyset_n(x), \emptyset_m(x)\right) = 0, \quad$ where $n \neq m$
   c. If it is equal to 0 then the set is orthogonal.
3) If asked to prove something is orthogonal, take inner product and show that it is = 0.

Examples:

1) Given $\{x, \sin x\,, e^x\}, \quad [0,1]$
   a. Find inner product of:
      i. $(x, \sin x) = \int_0^1 x \sin x\, dx = 0$
      ii. $(x, e^x) = \int_0^1 x e^x dx = 0$
      iii. $(\sin x\,, e^x) = \int_0^1 \sin x\, e^x dx = 0$
   b. They all equal 0 so the set of functions is orthogonal.

Normalize/construct orthonormal set of functions:
Divide each function by its norm:

1) $\left\{\frac{f(x)}{\left\|f(x)\right\|}, \frac{g(x)}{\left\|g(x)\right\|}, \frac{h(x)}{\left\|h(x)\right\|}\right\}$

If given a set, write it as a sum so: {___} = $\left\{\frac{1}{2n}\sin nx\right\}$

2) Given $\{\cos x\,, \cos 3x\,, \cos 5x\,, \ldots\}, \quad [0, \frac{\pi}{2}]$
   a. $\{\cos x\,, \cos 3x\,, \cos 5x\,, \ldots\} \Rightarrow \{\cos\left((2n-1)x\right)\}, \qquad n = 1,2,3, \ldots$
   b. $\left(\emptyset_n(x), \emptyset_m(x)\right) \Rightarrow \left(\cos\left((2n-1)x\right), \cos\left((2m-1)x\right)\right) \Rightarrow \int_0^{\frac{\pi}{2}} \cos\left((2n-1)x\right) \cos\left((2m-1)x\right) dx$

Knowing: $\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b)) \rightarrow \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2(n+m-1)x) + \cos(2(n-m)x) \Rightarrow$

$\Rightarrow \frac{1}{2}\left[\left(\frac{\sin 2(n+m-1)x}{2(n+m-1)}\right)_0^{\frac{\pi}{2}} + \left(\frac{\cos(2(n-m)x)}{2(n-m)}\right)_0^{\frac{\pi}{2}}\right]$ and since $\sin(n\pi) = 0$ always, this set $= 0$.

   a. It is equal to 0 so the set of functions is orthogonal.

Orthonormal sets:

1) Are orthogonal so inner product: $\left(\emptyset_n(x), \emptyset_m(x)\right) = 0$
2) Are normal so norm: $\left\|\emptyset_n(x)\right\| = 1, \ \left\|\emptyset_m(x)\right\| = 1$
   - Note: $\left\|f(x)\right\| = \int_a^b f(x)^2 dx = 1$
3) Using both of the above, solve for unknowns if there are unknowns

Ch.12: Fundamental Period, Even and Odd Properties

<u>Fundamental period:</u>
Fundamental period is the lowest common period of all parts of the function. $f(x+T) = f(x)$ where T is the period.

1) Add 2π to the interior of the trig functions and isolate so that you have the form $f\big(x + (\_\_)\big).\ (\_\_) = T$.
2) Repeat until you get common multiples.

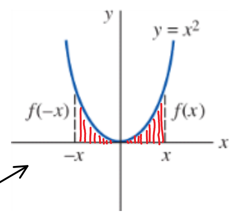
Ex: $f(x) = \sin 3x + \cos 2x$

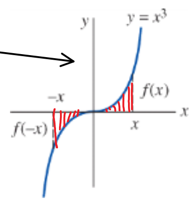
1) $\sin(3x + 2\pi) \to \sin\left(x + \dfrac{2\pi}{3}\right) \to T_1 = \dfrac{2\pi}{3}, \qquad \sin(3x + 4\pi) \to T_2 = \dfrac{4\pi}{3},$

   $\sin(3x + 6\pi) \to T_2 = \dfrac{6\pi}{3} = 2\pi$
2) $\cos(2x + 2\pi) \to T_1 = \pi, \qquad \cos(2x + 4\pi) \to T_1 = 2\pi$
3) Fundamental period is 2π for both $\sin 3x$ and $\cos 2x$

<u>Properties of even and odd functions:</u>

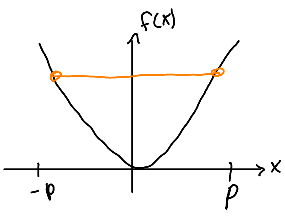
1) even $*$ even $=$ even
2) odd $*$ odd $=$ even
3) even $*$ odd $=$ odd
4) even $\pm$ even $=$ even
5) odd $\pm$ odd $=$ odd
6) If $f(x)$ is **even**, $\int_{-p}^{p} f(x)dx = 2\int_{0}^{p} f(x)dx$
7) If $f(x)$ is **odd**, $\int_{-p}^{p} f(x)dx = 0$





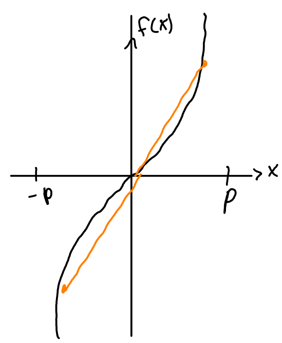
<u>Fourier cosine and Fourier sine series:</u>

$f(-x) = f(x) \to f(x)$ is even

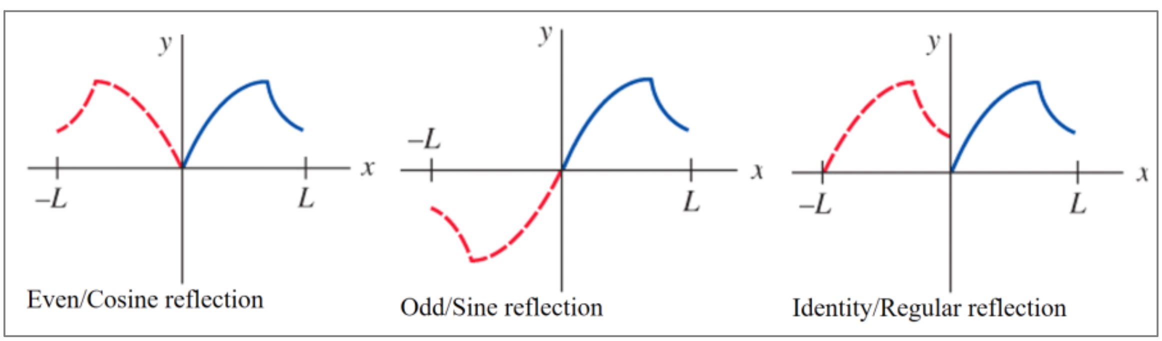


For a symmetrical range [-p,p] function is symmetric with respect to the y axis.

$f(-x) = -f(x) \to f(x)$ is odd



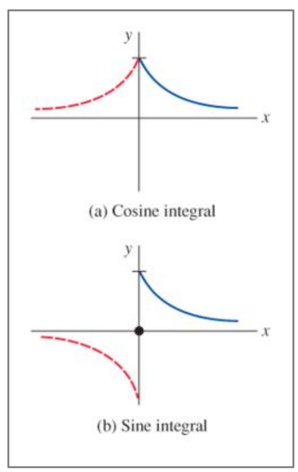
For a symmetrical range [-p,p] function is symmetric with respect to the origin



Even/Cosine reflection

Odd/Sine reflection

Identity/Regular reflection



(a) Cosine integral

(b) Sine integral

Ch.12: Fourier Series, Sine and Cosine Half Range Expansions

Fourier series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right)$$

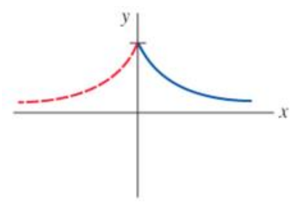
Where:

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,\mathrm{d}x\,, \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)\mathrm{d}x\,, \qquad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right)\mathrm{d}x$$

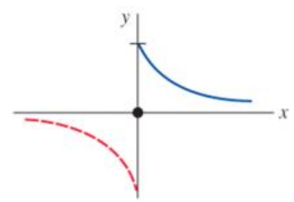
Note: When solving integrals as part of Fourier series, the following trig properties help:

- $\sin(n\pi) = 0$
- $\sin(-n\pi) = 0$
- $\cos(n\pi) = \cos(-n\pi) = (-1)^n$
- $-\cos(n\pi) = (-1)^{n+1}$
- $\sin x \cos nx = \sin\big((1+n)x\big) + \sin\big((1-n)x\big)$ → this formula is on formula sheet.

Half range cosine or sine functions:



(a) Cosine integral



(b) Sine integral

1) Determine if it's odd or even.
   a. $f(-x) = f(x) \rightarrow f(x)$ is even
   b. $f(-x) = -f(x) \rightarrow f(x)$ is odd
2) Find P and T.
3) If even, do cosine half range:
   a. Find a_0 and a_n
      i. Note: $a_0 = \frac{2}{p}\int_0^p f(x)dx\,, \qquad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)dx$
4) If odd, do sine half range:
   a. Find b_n
      i. Note: $b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}x\right)dx$
5) If you get a function such as $\frac{2(-1)^n}{\pi(1-n^2)}$ then n≠1 so solve for $a_1 = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{1\pi}{p}x\right)dx$. The same applies applies for b_1
6) Find $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right)$

Ex: Expand the given function in an appropriate sine cosine function. $f(x)\begin{cases} 1, & -2 < x < -1 \\ 0, & -1 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$

1) Draw it. Even: $\big(f(x) = f(-x)\big)$ so cosine half range
2) T=4, T=2P so P=2
3) $a_0 = \frac{2}{p}\int_{\text{middle (usually 0)}}^{p} f(x)\,\mathrm{d}x = \int_1^2 1dx = 1$
4) $a_n = \frac{2}{p}\int_{\text{middle (usually 0)}}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)\mathrm{d}x = \int_1^2 \cos\left(\frac{n\pi}{p}x\right)\mathrm{d}x = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$
5) $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right) \Rightarrow \frac{1}{2} + \sum_{n=1}^{\infty} -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{p}x\right)$

Because f(x)=0 $-1 < x < 1$, you can ignore that part when integrating.

Ch.12: Fourier Cosine and Sine Half Range Representation

Cosine: reflect/mirror function over y axis and then repeat it on the interval given.
Sine: reflect/mirror function over origin and then repeat it on the interval given.

b- Consider the following function

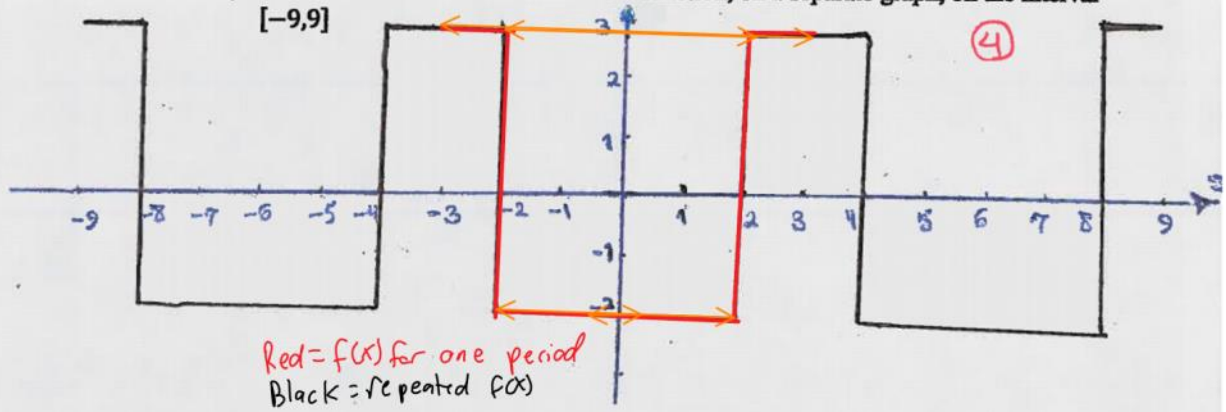
$$f(x) = \begin{cases} -2 & 0 < x \leq 2 \\ 3 & 2 < x \leq 3 \end{cases}$$

<u>WITHOUT FINDING THE FOURIER SERIES</u>

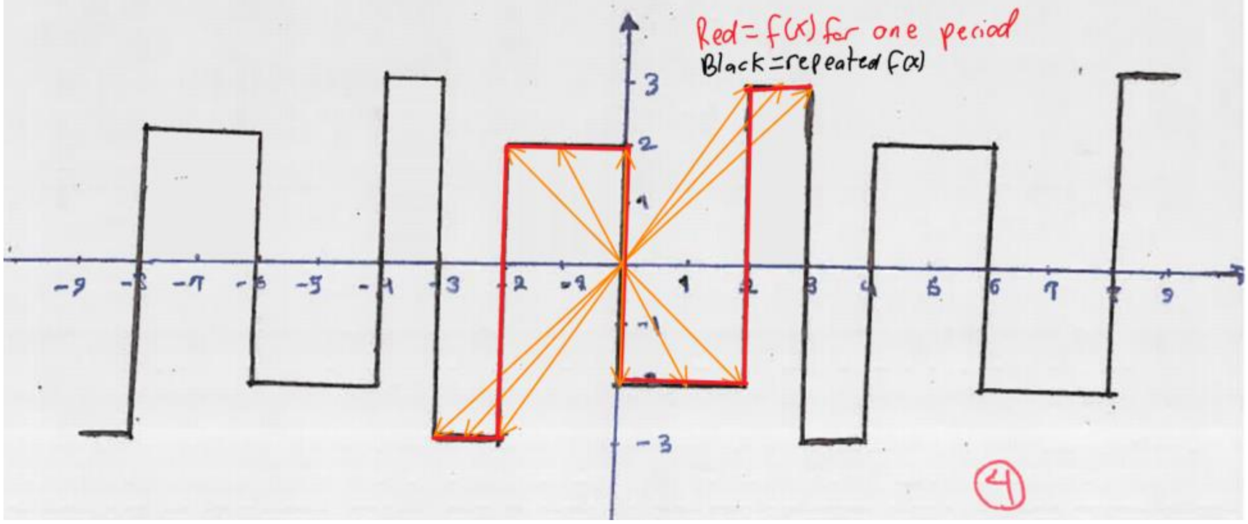
a) Sketch the Fourier Cosine Series of the function on the interval $[-9,9]$

b) To what values will this series converge at $x = 0$, $x = 7$ and $x = -9$?

c) Sketch the Fourier Sine Series of the function, on a separate graph, on the interval $[-9,9]$



b) $g(0) = -2$ ② $g(7) = -2$ ② $g(-9) = 3$ ②



Ch.13: Classifying PDE's, Separation of Variables, BVP

Classification of second order liner PDE:

- $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$
-

$B^2 - 4AC > 0$	Equation is hyperbolic
$B^2 - 4AC = 0$	Equation is parabolic
$B^2 - 4AC < 0$	Equation is elliptic

Remember $x^2 \to xy \to y^2$ as order for A→B→C

-

If G(x,y) =0	The equation is homogeneous
If G(x,y)≠0	The equation is non-homogeneous

Using separation of variables to solve PDE's:

1) $u(x,y) = X(x)Y(y)$ so plug that in and derive.
   a. Note that $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(X(x)Y(y)\right) = X'Y$ since here Y(y) is treated as a constant.
2) Separate both variable and make it equal to $-\lambda$
3) Solve either eigenvalues of the following:
   a. $\lambda = 0, \lambda \neq 0$
   b. $\lambda = 0, \lambda = -\alpha^2, \lambda = \alpha^2$
4) Look at formula sheet to see what X(x) and Y(y) equal to depending on the above conditions.
   $Y' + \alpha Y = 0 \to Y(y) = c_1 e^{-\alpha x}$
   $Y'' = 0 \to Y(y) = my + b$
   $Y'' + \alpha^2 Y = 0 \to Y(y) = c_1 \cos(\alpha y) + c_2 \sin(\alpha y)$
   $Y'' - \alpha^2 Y = 0 \to Y(y) = c_1 \cosh(\alpha y) + c_2 \sinh(\alpha y)$
5) Solution is $u = X(x)Y(y)$

Ex: Use separation of variables to find, if possible, product solutions for the given partial differential equation.
$u_x = u_y + u$

1) $\frac{\partial}{\partial x}(XY) = \frac{\partial}{\partial y}(XY) + XY \to X'Y = XY' + XY$
2) $\frac{X'}{X} = \frac{Y' + Y}{Y} = -\lambda \to \begin{cases} X' + \lambda X = 0 \\ Y' + (1+\lambda)Y = 0 \end{cases}$
3) $\lambda \neq 0$: $\begin{cases} X(x) = c_1 e^{-\lambda x} \\ Y(y) = c_2 e^{-(1+\lambda)y} \end{cases}$
4) $u(x,y) = X(x)Y(y) = c_1 e^{-\lambda x} c_2 e^{-(1+\lambda)y}$

Note:
- λ is an eigen value
- X(x) is an eigen function

Boundary Value Problems:

BVPs have boundary conditions and initial conditions.

There are 3 types of boundary conditions for heat, wave and Laplace equations:

1) Dirichlet $u(\ )$	Ex: $u(x,0) = 0, u(x,L) = 0, u(0,t) = u_0(t),\ etc$ Note: $u(0,t)$ means $X(0) = 0$. Use that as the condition.
2) Neuman $\left.\frac{\partial u}{\partial n}\right\|_{(\ )=(\ )} = (\ )$	Ex: $\left.\frac{\partial u}{\partial x}\right\|_{x=0} = 0,\ \left.\frac{\partial u}{\partial x}\right\|_{x=L} = 0, etc$ Note: If given $\left.\frac{\partial u}{\partial x}\right\|_{x=0} = 0,\quad X(x)$ must be derived in order to use the BC. Ex: $X(x) = Acos(\alpha x) + Bsin(\alpha x) \to X'(x) = -Asin(\alpha x) + Bcos(\alpha x)$ $BC: X'(0) = 0 \to= 0 + B(1) \to B = 0$
3) Robin $\left.\frac{\partial u}{\partial n}\right\|_{(\ )=(\ )} + h * u(\ )$ h is a constant	Ex: $\left.\frac{\partial u}{\partial x}\right\|_{x=L} = -h * \left(u(L,t) - u_m\right)$

Ch.13: BVP Steps

To solve BVP:

1) Use separation of variables to get two ODE's:
   a. Note that $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(X(x)Y(y)\right) = X'Y$ since here Y(y) is treated as a constant.
   b. Attach all constants to terms that aren't ( )″. This term should also be the 2 variable you solve for in step 4.
2) Rewrite BC's in useable terms.
   a. If given $u(0,t) = 0,\quad t > 0$ then you know $u(0,t) = X(0)T(t)$ gives $X(0) = 0$
      i. $X(0) = 0$ is used as a boundary condition for $X(x)$
3) Pick a variable and solve for $X(x)$ or T(t) etc using: $\lambda = 0,\ \lambda = -\alpha^2,\ \lambda = \alpha^2$ and the formula sheet.

   $Y' + \alpha Y = 0 \rightarrow Y(y) = c_1 e^{-\alpha x}$

   $Y'' = 0 \rightarrow Y(y) = my + b$

   $Y'' + \alpha^2 Y = 0 \rightarrow Y(y) = c_1 \cos(\alpha y) + c_2 \sin(\alpha y)$

   $Y'' - \alpha^2 Y = 0 \rightarrow Y(y) = c_1 \cosh(\alpha y) + c_2 \sinh(\alpha y)$

      i. Note: if given $Y' + 4\alpha^2 Y = 0$, then $\alpha^2 = 2\alpha$ so plug 2α into the above equatons.
      ii. Note: if $\lambda = 0$ satisfies the BC but not IC, you will get a A_0 value.

      $Ex\!: \lambda = 0\!:\ X''(x) = 0 \rightarrow X(x) = b_1 x + b_2,$ applying BCs make $b_1 = 0,$ $X(x) = b_2$

      $\lambda = 0\!:\ T' + ht = 0 \rightarrow T(t) = b_3 e^{-ht},$ does not satisfy IC

      Then: $u(x,t) = X(x)T(t) = b_2 * b_3 e^{-ht} = A_0 e^{-ht}$

      Now when doing superposition principle (step 7), after finding that $u(x,t)$ from $\lambda = \alpha^2$ also doesn't satisfy IC, you will have:$u(x,t) = A_0 e^{-ht} + \sum_{n=1}^{\infty} u_n(x,t)$

   a. Write down applicable BC's. Plug in BC's to check if solution is valid.
      i. Remember if BC is $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0,\quad X(x)$ must be derived first in order to use it.
4) Repeat step three with second variable but now you know what λ is equal to.
5) $u(x,t) = X(x)T(t)$, remember to combine constant coefficients.
   a. If you have something like: $\mathrm{u(x,t)} = \cos\left(\frac{\pi x}{(n+1)a}\right) + \sin\left(\frac{\pi x}{(n-1)a}\right)$ then $\mathrm{n} \neq 1$ because that makes one of the terms $= 0$ o solve for A_1 and add that to the A_0 that comes in front of the summation when you do the superposition principle.
6) If there is an IC such that $u(x,0) = f(x)$, show that it will not be satisfied for almost every $u(x,t)$.
7) Use superposition principle: $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ and check/plug in values from IC.
   a. If given $u(x,t) = f(x)$ where f(x) is **not** some cos or sin function:
      i. Solve using Fourier series or half range sine and cosine functions and combine a_0, a_n, b_n to find solution.

      Ex: problem set 9, #1

   Or

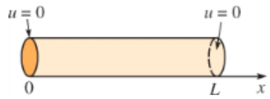
   b. If given $u(x,t) = f(x)$ where f(x) **is** some cos or sin function:
      i. Solve by solving $n = 1,2,3 \ldots$ and making that equal to f(x) from the IC. Use method of coefficients to find which values of the summation stay.

      Ex: if you have $6\sin\left(\frac{\pi x}{2}\right) = \sum A_n sin(\frac{\pi x}{n})$ then $A_n = 6,\quad n = 2$. n now equals only 2 and no other value.

Ch.13: Heat, Wave and Laplace Equations

<u>Heat equation BVP:</u>

General form: $k\frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_x) = \frac{\partial u}{\partial t}$



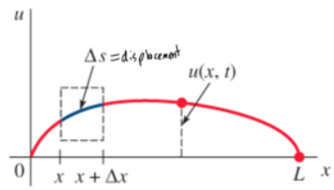
Where:

- $k = \frac{K}{\rho\delta}$ = thermal diffusion
  - $\rho = density\ of\ rod,\quad K = thermal\ conductivity,\quad \delta = specific\ heat$
- $u(x,t)$ = Temperature function
- $F(x,t,u,u_x)$ is effect of surrounding medium on the rod

If we neglect the effect of surrounding medium, $F(x,t,u,u_x) = 0$ gives:

$$\boldsymbol{k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}}$$

Note: When solving for T(t) in heat equation, use $Y' + \alpha^2 Y = 0 \rightarrow Y(y) = c_1 e^{-\alpha^2 x}$ because you have $\boldsymbol{k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}}$ which gives only T' so you can't use the other equations.



(a) Segment of string

<u>Wave equation:</u>

General form: $\boldsymbol{a^2 u_{xx} = u_{tt} \Rightarrow a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}}$,

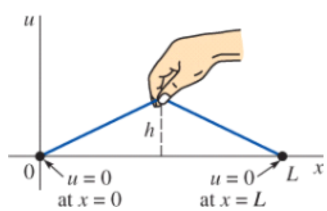
With: $a^2 = \frac{T}{\rho}$, where: T=tension, ρ =density of the string

$BC\colon\ u(0,t) = 0,\quad u(L,t) = 0,\quad IC\colon\ u(x,0) = f(x),\quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) = velocity$

So if given $4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \rightarrow a = 2$

Notes:

- You will have an "a" value when doing separation of variables: $u(x,t) = X(x)T(t) \rightarrow \begin{cases} X'' + \lambda X = 0 \\ T'' + a^2\lambda T = 0 \end{cases}$
- If you have $T'' + a^2\alpha^2 T = 0 \rightarrow T(t) = B_1\cos(a\alpha t) + B_2\sin(a\alpha t)$
- Make $a = 1$ for simplicity when asked to solve a wave function with no given "a".



<u>Laplace equation:</u>

$$\boldsymbol{\nabla u^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0},\quad 0 < x < a,\quad 0 < y < b,$$

$BC\colon \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0,\quad IC\colon\ u(x,0) = 0,\quad u(x,b) = f(x)$

Notes:

- This will give two equations: $\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$
- $\lambda = 0$ may saisfy 2 BCs and 1 IC but if it doesn't satusfy all of them, move on to $\lambda = -\alpha^2$ (lecture 39, 11:46)
  - $u(x,0) = x \rightarrow A_0(-b) \neq x$ that is <u>**never**</u> equal so move on.
- When solving BCs for Y, if you can't eliminate a constant (as shown below), isolate one constant and substitute it back in so that there will only be one constant.

$Y(b) = 0 \rightarrow c_9\cosh\alpha b + c_{10}\sinh\alpha b = 0$

$c_9 = -\frac{\sinh\alpha b}{\cosh\alpha b}c_{10}$

$Y(y) = -\frac{\sinh\alpha b}{\cosh\alpha b}c_{10}\cosh\alpha y + c_{10}\sinh\alpha y$

Heat Equation Example

Given: $\boldsymbol{k}\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}}$, Separate function.

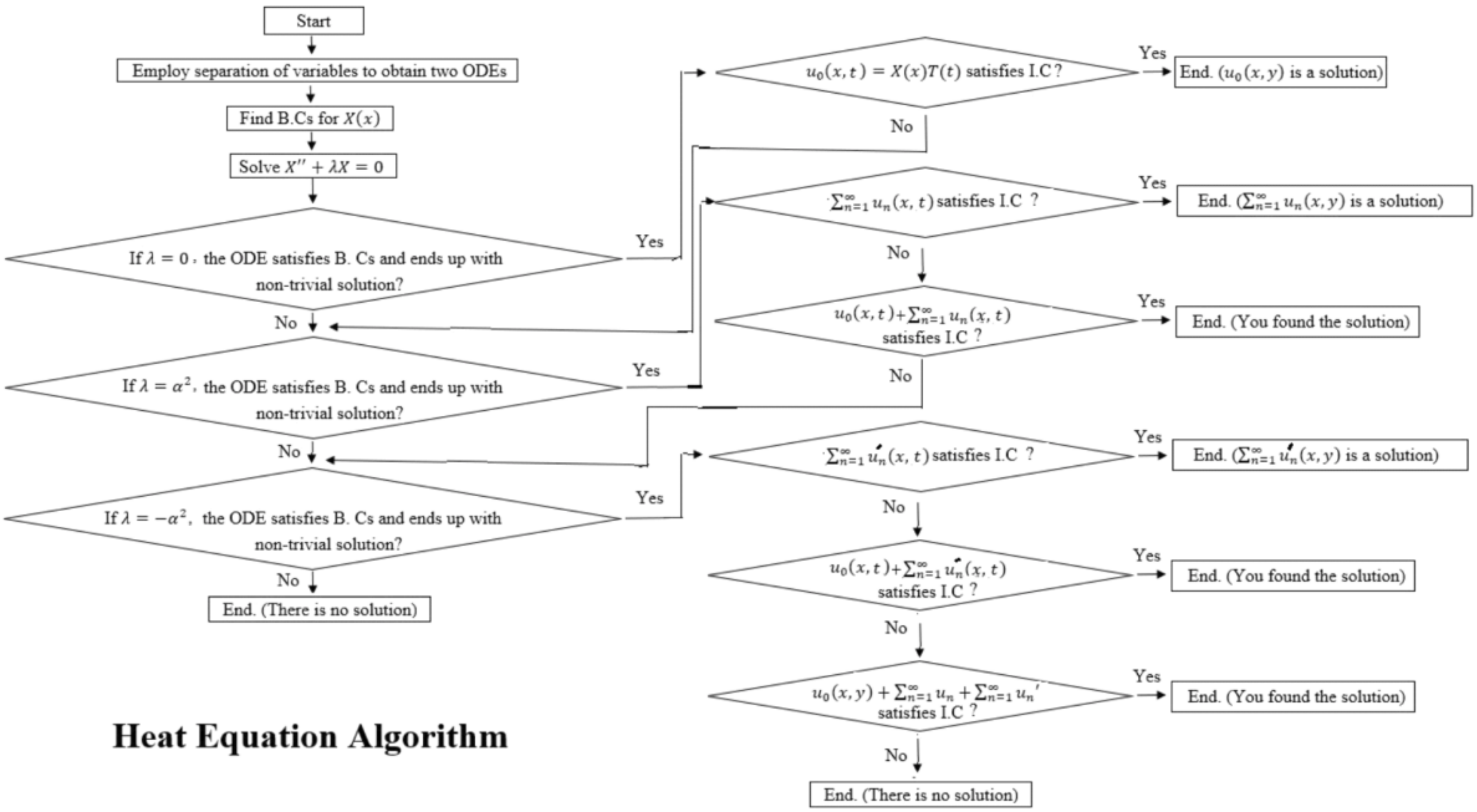
1) Rewrite BC's.
2) Pick a variable and solve for $X(x)$ or T(t) etc using: $\lambda = 0,\ \lambda = -\alpha^2,\ \lambda = \alpha^2$
   a. Plug in BC's for one of the two variables.
3) Repeat step three with second variable but now you know what λ is equal to.
4) $u(x,t) = X(x)T(t)$, remember to combine constant coefficients.
5) If there is an IC such that $u(x,0) = f(x)$, show that it will not be satisfied for almost every $u(x,t)$.
6) Use superposition principle: $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$
   a. Plug in values from IC.
   b. Solve using Fourier series or half range sine and cosine functions **or** method of coefficients.
7) Combine a_0, a_n, b_n to find solution.

Ex: Solve $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < L$, $t > 0$, (B.C) given: $u(0,t) = 0$, $u(L,t) = 0$, $t > 0$, (I.C) given: $u(x,0) = 2\sin\frac{\pi}{L}x$

1) Separate: $u(x,t) = X(x)T(t) \rightarrow \begin{cases} X'' + \lambda X = 0 \\ T' + k\lambda T = 0 \end{cases}$
2) Rewrite BC's:
   a. $u(0,t) = X(0)T(t)$ and $t > 0 \rightarrow X(0) = 0$
   b. $u(L,t) = X(L)T(t)$ and $t > 0 \rightarrow X(L) = 0$
3) Knowing $X'' + \lambda X = 0$ and $X(0) = 0,\ \ X(L) = 0$.
   a. $\lambda = 0 \rightarrow X(x) = Ax + B$
      i. $X(0) = 0 \rightarrow A(0) + B = 0 \rightarrow B = 0$
      ii. $X(L) = 0 \rightarrow A(L) = 0 \rightarrow A = 0$
      This gives $X(x) = 0$ but $X(x) \neq 0$ because then $u(x,t) = X(x)T(t)$ would $= 0$. So $\lambda \neq 0$
   b. $\lambda = -\alpha^2 \rightarrow X'' - \alpha^2 X = 0 \rightarrow$ from formula sheet $\rightarrow \mathrm{Acosh}(\alpha x) + B\sinh(\alpha x)$
      i. $X(0) = 0 \rightarrow A\cosh(\alpha(0)) + B\sinh(\alpha(0)) = 0 \rightarrow A(1) + B(0) = 0 \rightarrow A = 0$
      ii. $X(L) =) \rightarrow B\sinh(\alpha L) = 0 \rightarrow \sinh(\alpha L) = 0$ only at $\mathrm{x} = 0$, never at any $\mathrm{L} \rightarrow \mathrm{B} = 0$
      This gives $X(x) = 0$ again so $\lambda \neq -\alpha^2$
   c. $\lambda = \alpha^2 \rightarrow X'' + \alpha^2 X = 0 \rightarrow$ from formula sheet $\rightarrow X(x) = A\cos(\alpha x) + B\sin(\alpha x)$
      i. $X(0) = 0 \rightarrow A\cos(\alpha(0)) + B\sin(\alpha(0)) = 0 \rightarrow A(1) + B(0) = 0 \rightarrow A = 0$
      ii. $X(L) =) \rightarrow B\sin(\alpha L) = 0 \rightarrow if \begin{cases} B = 0 \rightarrow \text{if B} = 0\text{, then } X(x) = 0 \text{ again so disregard this} \\ \sin(\alpha L) = 0 \rightarrow \alpha L = n\pi \rightarrow \alpha = \frac{n\pi}{L}, \text{since } \lambda = \alpha^2 \rightarrow \frac{n^2\pi^2}{L^2} = \alpha^2 \end{cases}$
   d. $X(x) = B\sin\left(\frac{n\pi}{L}x\right)$
4) Now for $T(t)$: $T' + K\lambda T = 0 \rightarrow$ where $\lambda = \alpha^2$ from previous steps $\rightarrow\ T' + k\alpha^2 T = 0$
   a. From formula sheet: $T' + k\alpha^2 T = 0 \rightarrow T(t) = ce^{-k\alpha^2 t} \rightarrow T(t) = ce^{-k\frac{n^2\pi^2}{L^2}t}$
5) $u(x,t) = X(x)T(t) \rightarrow b\sin\left(\frac{n\pi}{L}x\right)ce^{-k\frac{n^2\pi^2}{L^2}t} \rightarrow D_n\sin\left(\frac{n\pi}{L}x\right)e^{-k\frac{n^2\pi^2}{L^2}t}$
6) Initial conditions: $u(x,0) = 2\sin\frac{\pi}{L}x$ and $\mathrm{u}(x,0) = D_n\sin\left(\frac{n\pi}{L}x\right)e^{-k\frac{n^2\pi^2}{L^2}t} \rightarrow$ is not satisfied because
   $2\sin\frac{\pi}{L}x \neq D_n\sin\left(\frac{n\pi}{L}x\right)e^{-k\frac{n^2\pi^2}{L^2}t}$
7) So use superposition principle: $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$
   a. Now: IC: $u(x,0) = \sum_{n=1}^{\infty} D_n\sin\left(\frac{n\pi}{L}x\right)e^{-k\frac{n^2\pi^2}{L^2}(0)} \rightarrow \sum_{n=1}^{\infty} D_n\sin\left(\frac{n\pi}{L}x\right) = 2\sin\frac{\pi}{L}x$
      i. This is half range sine series so find $D_n \Rightarrow \frac{2}{L}\int_0^L 2\sin\frac{\pi}{L}x * \sin\left(\frac{n\pi}{L}x\right)dx \rightarrow$ solve this
      Or:
      ii. $\sum_{n=1}^{\infty} D_n\sin\left(\frac{n\pi}{L}x\right) = D_1\sin\left(\frac{1\pi}{L}x\right) + D_2\sin\left(\frac{2\pi}{L}x\right) + \cdots = 2\sin\frac{\pi}{L}x$ so $\mathrm{D}_1 = 2, \mathrm{n} = 1$
8) Solution:
      i. $u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L 2\sin\frac{\pi}{L}x * \sin\left(\frac{n\pi}{L}x\right)dx\right) * \sin\left(\frac{n\pi}{L}x\right)e^{-k\frac{n^2\pi^2}{L^2}(t)}$
      ii. Or: $u(x,t) = \sum_{n=1}^{\infty} 2\sin\left(\frac{n\pi}{L}x\right)e^{-k\frac{n^2\pi^2}{L^2}(t)}$

Algorithm for Heat Equation

Algorithm of Solving Heat Equation



Heat Equation Algorithm

Ch.13: BVP Steps

To solve BVPs:

1) Use separation of variables to get two ODE's:
   a. Note that $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(X(x)Y(y)\right) = X'Y$ since here Y(y) is treated as a constant.
   b. Attach all constants to terms that aren't ( )″. This term should also be the 2 variable you solve for in step 4.
2) Rewrite BC's in useable terms.
   a. If given $u(0,t) = 0,\ \ t > 0$ then you know $u(0,t) = X(0)T(t)$ gives $X(0) = 0$
      i. $X(0) = 0$ can be used as a boundary condition for $X(x)$
3) Pick a variable and solve for $X(x)$ or T(t) etc using: $\lambda = 0,\ \lambda = -\alpha^2,\ \lambda = \alpha^2$ and the formula sheet.

   $Y' + \alpha Y = 0 \rightarrow Y(y) = c_1 e^{-\alpha x}$

   $Y'' = 0 \rightarrow Y(y) = my + b$

   $Y'' + \alpha^2 Y = 0 \rightarrow Y(y) = c_1 \cos(\alpha y) + c_2 \sin(\alpha y)$

   $Y'' - \alpha^2 Y = 0 \rightarrow Y(y) = c_1 \cosh(\alpha y) + c_2 \sinh(\alpha y)$

      i. Note: if given $Y' + 4\alpha^2 Y = 0$, then $\alpha^2 = 2\alpha$ so plug 2α into the above equatons.
      ii. Note: if $\lambda = 0$ satisfies the BC but not IC, you will get a A_0 value.

         Ex: $\lambda = 0$: $X''(x) = 0 \rightarrow X(x) = b_1 x + b_2$, applying BCs make $b_1 = 0$, $X(x) = b_2$

         $\lambda = 0$: $T' + ht = 0 \rightarrow T(t) = b_3 e^{-ht}$, does not satisfy IC

         Then:$u(x,t) = X(x)T(t) = b_2 * b_3 e^{-ht} = A_0 e^{-ht}$

         Now when doing superposition principle after finding $u(x,t)$ from $\lambda = \alpha^2$ also doesn't satisfy IC, you will have:$u(x,t) = A_0 e^{-ht} + \sum_{n=1}^{\infty} u_n(x,t)$

   a. Write down applicable BC's. Plug in BC's to check if solution is valid.
      i. Remember if BC is $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$, $X(x)$ must be derived first in order to use it.
4) Repeat step three with second variable but now you know what λ is equal to.
5) $u(x,t) = X(x)T(t)$, remember to combine constant coefficients.
   a. If you have something like: $\mathrm{u(x,t)} = \cos\left(\frac{\pi x}{(n+1)a}\right) + \sin\left(\frac{\pi x}{(n-1)a}\right)$ then $\mathrm{n} \neq 1$ because that makes one of the terms $= 0$ o solve for A_1 and add that to the A_0
6) If there is an IC such that $u(x,0) = f(x)$, show that it will not be satisfied for almost every $u(x,t)$.
7) Use superposition principle: $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ and check/plug in values from IC.
   a. If given $u(x,t) = f(x)$ where f(x) is **not** some cos or sin function:
      i. Solve using Fourier series or half range sine and cosine functions and combine a_0, a_n, b_n to find solution.

         Ex: problem set 9, #1

   Or

   b. If given $u(x,t) = f(x)$ where f(x) **is** some cos or sin function:
      i. Solve by solving $n = 1,2,3$ … and making that equal to f(x) from the IC. Use method of coefficients to find which values of the summation stay.

         Ex: if you have $6\sin\left(\frac{\pi x}{2}\right) = \sum A_n sin(\frac{\pi x}{n})$ then $A_n = 6,\ \ n = 2$. n now equals only 2 and no other value.

Ch.13: Superposition Principle, Dirichlet Problem

Superposition principle in Laplace:

$u(x,y) = u_1(x,y) + u_2(x,y)$

Where:

$u_1(x,y)$ is the solution found for $u(x,y)$ when $\lambda = 0$ that didn't satisfy one of the ICs.

$u_2(x,y)$ is the solution found $u(x,y)$ when $\lambda = \alpha^2$

This will give you usually something like:

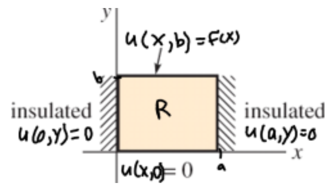
$$u(x,y) = A_0(y) + \sum_{n=1}^{\infty} A_n(\quad y)(\quad x)$$

Then apply last IC/substitute according to last IC and find half range cosine series.

Dirichlet:

Requirements:

1) Has to be elliptic PDE (Laplace is elliptic).
2) u takes prescribed values on the entire boundary of the region R.



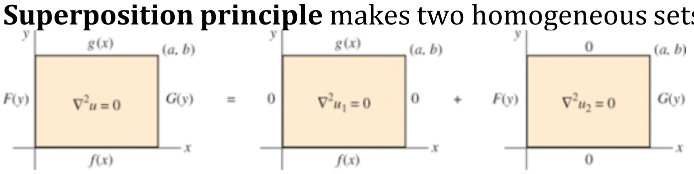
Maximum principle:

A solution u to a Laplace's equation within R takes its max and min on boundaries.

He could ask "Where does the solution to the Laplace equation take its max or min?"

Non homogeneous BVP:

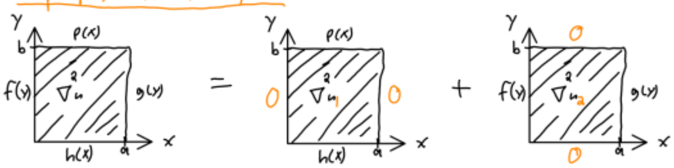
Superposition principle makes two homogeneous sets that can be solved individually



Ex: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b$ with BCs:

$u(0,y) = f(y), \quad u(a,y) = g(y)$

$u(x,0) = h(x), \quad u(x,b) = p(x)$



So superposition gives: $u(x,y) = u_1(x,y) + u_2(x,y)$

Where:

$u_1(x,y) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$ $u_1(0,y) = 0,\ u_1(a,y) = 0,$ $u_1(x,0) = h(x),\ u_1(x,b) = p(x)$. Find $u_1(x,y)$.

$u_2(x,y) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ u_2(0,y) = f(y),\ u_2(a,y) = g(y),$ $u_2(x,0) = 0,\ u_2(x,b) = 0$. Find $u_2(x,y)$.

The highlighted parts are homogeneous so you can solve using separation of variables.

Problem 1

$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \quad 0 < x < a,\ 0 < y < b$

$u_1(0,y) = 0, \quad u_1(a,y) = 0, \ 0 < y < b$

$u_1(x,0) = f(x), \quad u_1(x,b) = g(x), \ 0 < x < a$

Problem 2

$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0, \quad 0 < x < a,\ 0 < y < b$

$u_2(0,y) = F(y), \quad u_2(a,y) = G(y), \ 0 < y < b$

$u_2(x,0) = 0, \quad u_2(x,b) = 0, \ 0 < x < a$

Ch.13: Non-Homogeneous BVP (Time Independent), Integrating Factor

<u>Non homogeneous BVP:</u>
Boundary conditions $\neq 0$ or there is any function that is not being derived.
Time Independent:
Constants and functions of x only are time independent. So below, constants u_1, u_2 and $g(x)$ are time independent.

Given: $k\frac{\partial^2 u}{\partial x^2} + F(x) = \frac{\partial u}{\partial t}$, $\quad u(0,t) = u_0,\ u(1,t) = u_1,\ u(x,0) = g(x)$

1) Solve for $u(x,t) = v(x,t) + \varphi(x)$
   a. Take $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \varphi''(x)$ and $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + 0 \rightarrow$ plug both of those into equation: $k\frac{\partial^2 u}{\partial x^2} + F(x) = \frac{\partial u}{\partial t}$
      i. This gives us: $\boldsymbol{k\frac{\partial^2 v}{\partial x^2} + k\varphi''(x) + F(x) = \frac{\partial v}{\partial t}}$
2) **Assume:** $\varphi''(x) + F(x) = 0$ in order to solve. This gives $k\frac{\partial^2 v}{\partial x^2} + k * (0) = \frac{\partial v}{\partial t}$ which is a homogeneous PDE.
3) BCs and ICs:
   a. BC_1: $u(0,t) = u_0 \rightarrow v(0,t) + \varphi(0) = u_0$
      i. **Assume** $v(0,t) = 0$, $\varphi(0) = u_0$ so that you can get one homogeneous PDE
   b. BC_2: $u(1,t) = u_1 \rightarrow v(1,t) + \varphi(1) = u_1$
      i. **Assume** $v(1,t) = 0$, $\varphi(1) = u_1$ so that you can get one homogeneous PDE
   c. IC: $u(x,0) = g(x) \xrightarrow[knowing]{} v(x,0) = u(x,0) - \varphi(x) \Rightarrow v(x,0) = g(x) - \varphi(x)$

This gives:

	ODE with 2 BC	Homogeneous PDE with 2 BC, 1 IC, can be solved by separation of variables
Equation:	(1) $k\varphi''(x) + F(x) = 0$	(2) $k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$
BCs:	$\varphi(0) = u_0$ $\varphi(1) = u_1$	$v(0,t) = 0$ $v(1,t) = 0$
IC		$v(x,0) = u(x,0) - \varphi(x) = g(x) - \varphi(x)$

4) Solve for (1) and (2).
   a. Solve ODE (1) $\varphi''(x) = -\frac{F(x)}{k} \xrightarrow[integrate\ twice\ to\ get]{} \varphi(x) =$ ______
      i. Apply BCs to find constants c_1, c_2
      ii. Plug new constant values into $\varphi(x) =$ ______ equation
   b. Solve homogeneous PDE (2)$= k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$ with BCs: $v(0,t) = 0,\ v(1,t) = 0$, IC:$v(x,0) = g(x) - \varphi(x)$ but plug your newly found $\varphi(x)$ into that.
5) Then (1)+(2)= $u(x,t) = \varphi(x) + v(x,t)$

<u>Integrating factor:</u>
When you arrive at $v_n'(t) + P(t)v_n(t) = g(t)$, gives integrating factor of $e^{\int P(t)dt}$, then rewrite equation as: $\frac{d}{dt}\left(v_n * e^{\int P(t)dt}\right) = \text{g(t)} * e^{\int P(t)dt}$
Then isolating for $v_n(t)$ gives: $v_n(t) = \frac{1}{e^{\int P(t)dt}}\int g(t) * e^{\int P(t)dt}$

Ch.13: Non-Homogeneous BVP (Time Dependent)

Non-Homogeneous BVP:
Boundary conditions $\neq 0$ or there is any function that is not being derived.
Time Dependent:

Constants and functions of x only are time independent. So below, constants u_1, u_2 and $g(x)$ are time independent.

Given: $k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t}$, $\quad u(0,t) = u_0(t)$, $u(1,t) = u_1(t)$, $u(x,0) = f(x)$

1) Solve $u(x,t) = v(x,t) + \varphi(x,t)$, Note: from step 4; $\varphi(x,t) = u_0(t) + \frac{x}{L}(u_1(t) - u_0(t))$
   a. Take $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 \varphi}{\partial x}$, and $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial \varphi}{\partial t} \rightarrow$ plug both of those into the equation: $k\frac{\partial^2 u}{\partial x^2} + F(x) = \frac{\partial u}{\partial t}$
      i. This gives us: $\boldsymbol{k\frac{\partial^2 v}{\partial x^2} + k\frac{\partial^2 \varphi}{\partial x} + F(x,t) = \frac{\partial v}{\partial t} + \frac{\partial \varphi}{\partial t}}$
2) **Assume:** $\frac{\partial^2 \varphi}{\partial x} = 0$ in order to solve. This leaves us with $k\frac{\partial^2 v}{\partial x^2} + F(x,t) = \frac{\partial v}{\partial t} + \frac{\partial \varphi}{\partial t}$
   a. **Assuming**: $F(x,t) - \frac{\partial \varphi}{\partial t} = G(x,t) \xrightarrow[gives]{} k\frac{\partial^2 v}{\partial x^2} + G(x,t) = \frac{\partial v}{\partial t}$
3) BCs and ICs:
   a. BC_1: $u(0,t) = u_0(t) = v(0,t) + \varphi(0,t) \rightarrow \begin{cases} \varphi(0,t) = u_0(t) \\ v(0,t) = 0 \rightarrow \text{to get homogeneous PDE} \end{cases}$
   b. BC_2: $u(L,t) = u_1(t) = v(L,t) + \varphi(L,t) \rightarrow \begin{cases} \varphi(L,t) = u_1(t) \\ v(L,t) = 0 \rightarrow \text{to get homogeneous PDE} \end{cases}$
   c. IC: $u(x,0) = f(x) = v(x,0) + \varphi(x,0) \rightarrow v(x,0) = f(x,0) - \varphi(x,0)$

This gives:

	ODE with 2 BC	PDE with 2 BC, 1 IC
Equation:	(1) $\varphi(x,t) = \frac{\partial^2 \varphi}{\partial x} = 0$	(2) $v(x,t) = k\frac{\partial^2 v}{\partial x^2} + G(x,t) = \frac{\partial v}{\partial t}$
BCs:	$\varphi(0,t) = u_0(t)$ $\varphi(L,t) = u_1(t)$	$v(0,t) = 0$ $v(L,t) = 0$
IC		$v(x,0) = f(x,0) - \varphi(x,0)$

4) Solve for (1) and (2).
   a. Solve PDE (1) $\frac{\partial^2 \varphi}{\partial x} = 0 \xrightarrow[\text{integrate twice to get}]{} \varphi(x,t) = xc_1 + c_2$
      i. BCs: $\varphi(0,t) = u_0(t) \rightarrow c_2 = u_0(t)$, $\varphi(L,t) = u_1(t) \rightarrow c_1 = \frac{(u_1(t) - u_0(t))}{L}$
      ii. Plug back into $\varphi(x,t) = xc_1 + c_2$ to find $\varphi(x,t) = u_0(t) + \frac{x}{L}(u_1(t) - u_0(t))$
         1) The pink equation can be *used in all scenarios for **time dependent** BVPs*. Simply plug in u_0, u_1 and you skip having to solve for $\varphi(x,t)$.
   b. Solve still non-homogeneous PDE (2) $k\frac{\partial^2 v}{\partial x^2} + G(x,t) = \frac{\partial v}{\partial t}$ but you do have homogenous BCs.
      i. **Assume**: $v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin\frac{n\pi x}{L}$, $\quad G(x,t) = \sum_{n=1}^{\infty} G_n(t) \sin\frac{n\pi x}{L}$
         1) Find $G_n(x,t)$:
            a) $G(x,t)\left[\text{from F(x,t)} - \frac{\partial \varphi}{\partial t}\right] = \sum_{n=1}^{\infty} G_n(t) \sin\frac{n\pi x}{L}$, use half ranges to solve.
         2) Find $v(x,t) = k\frac{\partial^2 v}{\partial x^2} + G(x,t) = \frac{\partial v}{\partial t}$
            a) Find $\frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} v_n(t) \sin\frac{n\pi x}{L}\right), \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} v_n(t) \sin\frac{n\pi x}{L}\right)$
            b) Plug in $G(x,t)$, $\frac{\partial^2 v}{\partial x^2}$, and $\frac{\partial v}{\partial t}$ into $v(x,t) = k\frac{\partial^2 v}{\partial x^2} + G(x,t) = \frac{\partial v}{\partial t}$
            c) Isolate $v_n'(t) + P(t)v_n(t) = g(t)$ out of the summation.
            d) Solve for $v_n(t)$ using integrating factor. You will be left with a C_n after integration.
            e) Superposition principle: $v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin\frac{n\pi x}{L} \rightarrow$ and solve for C_n by using half range identities **or** by using method of cofficients.
5) Plug everything back into $u(x,t) = \varphi(x,t) + v(x,t)$

Ch.13:Non-Homogeneous BVP(Time Dependent) Example

Ex: Find $u(x,t)$ — Purple is step # corresponding to steps on previous page

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad 0 < x < t$$

(3) BC1: $u(0,t) = \cos t$ ← non homogeneous, time dependent

BC2: $u(1,t) = 0$

IC: $u(x,0) = 0$

(1) $u(x,t) = v(x,t) + \Psi(x,t) = v(x,t) + (1-x)\cos t$ — $\Psi(x,t)$

$$\Psi(x,t) = u_0(t) + \frac{x}{L}\left[u_1(t) - u_0(t)\right]$$

$$= \cos t + x(0 - \cos t) = (1-x)\cos t$$

(2) $\frac{\partial^2 \Psi}{\partial x^2} = 0$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial \Psi}{\partial t} \longrightarrow \frac{\partial \Psi}{\partial t} = -(1-x)\sin t$$

1b) $\frac{\partial^2 v}{\partial x^2} + \underbrace{(1-x)\sin t}_{G(x,t)} = \frac{\partial v}{\partial t}$ *

4b) $v(0,t) = 0 \qquad v(1,t) = 0$

$$v(x,0) = 0 - \Psi(x,0) = x - 1$$

$$v(x,t) = \sum_{n=1}^{\infty} V_n(t)\sin(n\pi x)$$

4b i1) $G(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin(n\pi x)$

$$(1-x)\sin t = \sum_{n=1}^{\infty} \underbrace{G_n(t)\sin(n\pi x)}_{\text{Half-range sine}}$$

$$G_n(t) = 2\int_0^1 (1-x)\sin t \sin n\pi x \, dx = \frac{2}{n\pi}\sin t$$

$$(1-x)\sin t = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin t \sin(n\pi x) \quad **$$

4.2) $v(x,t) = \sum_{n=1}^{\infty} V_n(t)\sin(n\pi x)$

4.2a) $\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} -V_n(t)(n\pi)^2 \sin(n\pi x)$

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} V_n'(t)\sin(n\pi x)$$

4.2b) * $\sum_{n=1}^{\infty} -V_n(t)(n\pi)^2 \sin(n\pi x)$

** $+ \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin t \sin(n\pi x)$

$$= \sum_{n=1}^{\infty} V_n'(t)\sin(n\pi x)$$

4.2c) $\sum_{n=1}^{\infty}\left[V_n'(t) + V_n(t)(n^2\pi^2)\right]\sin(n\pi x)$

$$= \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin t \sin(n\pi x)$$

4.2d) $V_n'(t) + V_n(t)(n^2\pi^2) = \frac{2}{n\pi}\sin t$

The ODE can be solved by integrating factor → $e^{n^2\pi^2 t}$

$$\frac{d}{dt}\left(e^{n^2\pi^2 t} V_n(t)\right) = \frac{2}{n\pi} e^{n^2\pi^2 t}\sin t$$

$$V_n(t) = \frac{2}{n\pi}\frac{n^2\pi^2 \sin t - \cos t}{(n^4\pi^4 + 1)} + C_n e^{-n^2\pi^2 t}$$

4.2e) $V(x,t) = \sum_{n=1}^{\infty} V_n(t)\sin(n\pi x)$

$$= \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\frac{n^2\pi^2\sin t - \cos t}{(n^4\pi^4+1)} + C_n e^{-n^2\pi^2 t}\right]\sin(n\pi x)$$

$$v(x,0) = x - 1 = \sum_{n=1}^{\infty}\underbrace{\left[\frac{-2}{n\pi(n^4\pi^4+1)} + C_n\right]\sin(n\pi x)}_{\text{half-range sine}}$$

$$\frac{-2}{n\pi(n^4\pi^4+1)} + C_n = 2\int_0^1 (x-1)\sin(n\pi x)\,dx = -\frac{2}{n\pi}$$

$$\longrightarrow C_n = \frac{2}{n\pi(n^4\pi^4+1)} - \frac{2}{n\pi}$$

(5) $u(x,t) = (1-x)\cos t +$

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{n^2\pi^2\sin t - \cos t + e^{-n^2\pi^2 t}}{n(n^4\pi^4+1)} - \frac{e^{-n^2\pi^2 t}}{n}\right](\sin(n\pi x))$$

Ch.15.3: Fourier Integral

Used to represent certain kind of non-periodic functions that are defined on $(-\infty, \infty)$ or $(0, \infty)$.
The Fourier integral of a function on $(-\infty, \infty)$ is given by:

$$f(x) = \frac{1}{\pi}\int_0^\infty (A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))d\alpha$$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx \rightarrow \text{cosine representation}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin(\alpha x)\,dx \rightarrow \text{sine representation}$$

1) Find $A(\alpha)$
2) Find $B(\alpha)$
3) Plug in $A(\alpha), B(\alpha)$ into $f(x) = \frac{1}{\pi}\int_0^\infty (A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))d\alpha$ and solve. This integral is going to be hella complicated.

<u>Cosine and Sine integrals:</u>
The Fourier integral of an **even** function on $(-\infty, \infty)$ is a cosine integral

$$A(\alpha) = 2\int_0^\infty f(x)\cos(\alpha x)\,dx$$

$$B(\alpha) = 0$$

so: $f(x) = \frac{1}{\pi}\int_0^\infty (A(\alpha)\cos(\alpha x))d\alpha \rightarrow$ its already $\int_0^\infty$ so you dont have to multiply the whole integral by 2 like we did in previous chapters for half range functions.

The Fourier integral of an **odd** function on $(-\infty, \infty)$ is a sine integral.

$$A(\alpha) = 0$$

$$B(\alpha) = 2\int_0^\infty f(x)\sin(\alpha x)\,dx$$

So: $f(x) = \frac{1}{\pi}\int_0^\infty (B(\alpha)\sin(\alpha x))d\alpha$

<u>Convergence of Fourier integral:</u>
Let f and f' be piecewise continuous on every finite interval, and f is absolutely integrable on $(-\infty, \infty)$ then the Fourier integral converges to:

1) $f(x)$ at a point of continuity
2) $\frac{f(x)^+ + f(x)^-}{2}$ at a point of discontinuity.

   Here $f(x)^+$ and $f(x)^-$ are the right and left limits at point x respectively.

Formula sheet midterm 1

Function	Laplace Transform
1	1/s
t^n	$n!/s^{n+1}$
e^{at}	$1/(s-a)$
sin kt	$k/(s^2+k^2)$
cos kt	$s/(s^2+k^2)$
sinh kt	$k/(s^2-k^2)$
cosh kt	$s/(s^2-k^2)$
t^α $\alpha > -1$ Gamma function	$\Gamma(\alpha+1)/s^{\alpha+1}$ $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
$\Gamma(\frac{1}{2})$ $\Gamma(1) = 1$	$\sqrt{\pi}$
$f^{(n)}(t)$ Transform derivatives	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$
$e^{at} f(t)$ Translation theorem	$F(s-a)$
$f(t-a)\mathcal{U}(t-a)$ Second translation theorem	$e^{-as} F(s)$
$g(t)\mathcal{U}(t-a)$ Alternative second translation theorem	$e^{-as} \mathcal{L}\{g(t+a)\}$
$t^n f(t)$ Derivatives of transform	$(-1)^n d^n F(s)/ds^n$
$f*g$ Convolution	$F(s).G(s)$
$\int_0^t f(\tau)d\tau$ Transform of an integral	$F(s)/s$
$f(t)$ where $f(t+T)=f(t)$ Laplace of periodic function	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$
$\delta(t-t_0)$ Dirac Delta function	e^{-st_0}

$\mathcal{U}(t-a)$ is the unit step function

missing Volterra integral func.

$$\int_0^t f(\alpha)g(t-\alpha)\,d\alpha$$

Trigonometric Identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sin A \cdot \sin B = \frac{1}{2}[-\cos(A+B) + \cos(A-B)]$$

$$\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A \cdot \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

Formula sheet midterm 2

Fourier Series for function period (-p to p):

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)dx,\ a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}xdx\quad b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}xdx$$

Fourier sine and cosine series:

Half Range

$$f_{cosine}(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{p}\right)\quad where\ \ a_0=\frac{2}{p}\int_0^p f(x)dx\qquad a_n=\frac{2}{p}\int_0^p f(x)cos\frac{n\pi x}{p}dx$$

Half Range

$$f_{sine}(x)=\sum_{n=1}^{\infty}\left(b_n sin\frac{n\pi x}{p}\right)\qquad where\ \ b_n=\frac{2}{p}\int_0^p f(x)sin\frac{n\pi x}{p}dx$$

Substitute in at any chance?

$\sin(n\pi)=0$
$\sin(-n\pi)=0$
$\cos(n\pi)=(-1)^n$
$\cos(-n\pi)=\cos(n\pi)$
$-\cos(n\pi)=(-1)^{n+1}$

Solutions of Linear Ordinary Differential Equations:

Linear Equation	General Solution
$y'+\alpha y=0$	$y(x)=c_1e^{-\alpha x}$
$y''+\alpha^2y=0,\alpha>0$	$y(x)=c_1\cos\alpha x+c_2\sin\alpha x$
Linear Equation	**General Solution**
$y''-\alpha^2y=0,\alpha>0$	$y(x)=c_1e^{-\alpha x}+c_2e^{\alpha x}$, or $y(x)=c_1\cosh\alpha x+c_2\sinh\alpha x$
$y'-\alpha y=0$	$y(x)=c_1e^{\alpha x}$

Orthogonal Sets:

On the interval [a,b], the set of $\{\varphi_0(x),\ \varphi_1(x),\ \varphi_2(x),\ \ldots\}$ is orthogonal if,

$$(\varphi_m,\varphi_n)=\int_a^b\varphi_m(x).\varphi_n(x).dx=0\qquad;\ m\neq n$$

Trigonometric Identities:

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$
$$\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$$
$$\sin A\cdot\sin B=\frac{1}{2}[-\cos(A+B)+\cos(A-B)]$$
$$\cos A\cdot\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$$
$$\sin A\cdot\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

- $\cosh(0)=1$
- $\sinh(0)=0$
- $\sinh(z)=0$ ONLY IF z = 0
- $\cos(\alpha L)=0\rightarrow\alpha L=(2n-1)\dfrac{\pi}{2}$
- $\sin(n\pi)=0$
- $\sin(-n\pi)=0$
- $\cos(n\pi)=\cos(-n\pi)=(-1)^n$
- $\cos(\alpha)=0\rightarrow\alpha=(2n-1)\dfrac{\pi}{2}$
- $-\cos(n\pi)=(-1)^{n+1}$
- $\sin x\cos nx=\sin\big((1+n)x\big)+\sin((1-n)x)$
  $\rightarrow$ this formula is on formula sheet.
- $\dfrac{\mathrm{d}}{\mathrm{dx}}\cosh(x)=\sinh(x)$
  - No negative sign
- $\dfrac{d}{dx}\sinh(x)=\cosh(x)$
  - No negative sign
- $\cosh(x)=\dfrac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}$

For BCs, remember:

$\sin(\alpha)=0\rightarrow\alpha=n\pi,\qquad\cos(\alpha)=0\rightarrow\alpha=(2n-1)\dfrac{\pi}{2},$

$\cos(0)=1,\qquad\sin(0)=0,\qquad\sinh(0)=0,\qquad\cosh(0)=1$

- $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$