Complete Solutions

CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	209	All except EC

Examination	Date	Pages
Final	December 2017	2

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Special Instructions

▷ **Only approved calculators allowed.**

[MARKS]

[7] **1.** (a) Find $\lim_{x\to\infty} \dfrac{5x - x^2}{x^2 + 7}$.

(b) Give an example of a function f defined for all real numbers which has the property that $\lim_{x\to+2}$ and $\lim_{x\to-3}$ are both equal to $-\infty$.

[6] **2.** Find the derivatives for each of the following functions: (YOU DO NOT NEED TO SIMPLIFY):

(a) $g(x) = (e - \ln(x))\ (7\sqrt{x} - \dfrac{6}{x})$

(b) $h(x) = e^{(-x^3+5)} - x^3 \ln(x)$

[7] **3.** Sketch the graph of the equation $x^2 + y^2 - 97 = 0$ and find y' by implicit differentiation. Then find the slope of the tangent line at $x = -4$ and $y = 9$.

[6] **4.** Find dh if $h = e^{2.5x}$, $x = 2$ and the change in x is 0.3.

[12] **5.** A point is moving on the graph of $y = 45/x$. When the point is at $(5, 9)$, its y coordinate is increasing by 5 units per second. How fast is the x coordinate changing at that moment?

[11] **6.** Use the price-demand equation $0.03x + p = 70$ to find the value of p **(i)** for which the demand is elastic and **(ii)** for which the demand is inelastic.

[5] **7.** For $f(x) = 12x - x^3$ find the absolute maximum and minimum, if either exists, on the interval [-3,3].

[8] **8.** Sales of a company are described by the function $N(x) = 3x^3 - 0.25x^4 + 200,\ 0 \leq x \leq 9$. When is the rate of change of sales increasing and when is it decreasing?

[13] **9.** Find the area bounded by $f(x) = 5 - x^2$ and $g(x) = 2 - 2x$.

[13] **10.** Evaluate the integrals:

(a) $\int \left(\frac{1}{x^5} - \frac{1}{\sqrt{x}}\right) dx$

(b) $\int \sqrt{1 - 5x}\, dx$

(c) $\int_1^5 \left(\frac{2}{x+2}\right) dx$

[12] **11.** (a) Define the notion: the Gini index of a country.

(b) If the Gini index of a country is equal to p, then the number $1 - p$ cannot be the Gini index of any other country. Explain why this is true or false.

(c) It is possible for a country A to have a Gini index equal to three times the Gini index of another country B. Explain why this is true or why it is false.

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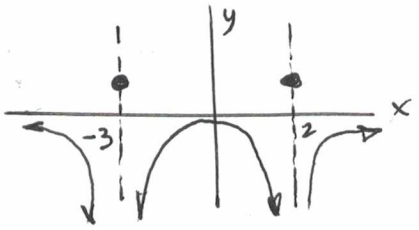
1 a) (3) $\lim_{x\to\infty} \frac{5x-x^2}{x^2+7} = \frac{\infty}{\infty} \Rightarrow \lim_{x\to\infty} \frac{\frac{5x}{x^2}-\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{7}{x^2}} = \lim_{x\to\infty} \frac{\frac{5}{x}-1}{1+\frac{7}{x^2}} = -1$

b) (4)



2 a) (3) $g(x) = (e-\ln x)(7x^{\frac{1}{2}} - 6x^{-1})$

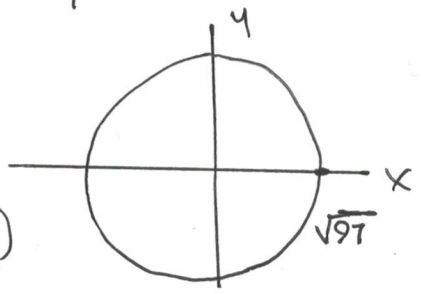
$g'(x) = -\frac{1}{x}(7x^{\frac{1}{2}} - 6x^{-1}) + (7*\frac{1}{2}x^{-\frac{1}{2}} + 6x^{-2})(e-\ln x)$

b) (3) $h(x) = e^{-x^3+5} - x^3 \ln x$

$h'(x) = e^{-x^2+5}(-3x^2) - \left[3x^2(\ln x) + \frac{1}{x}(x^3)\right]$

3. $x^2+y^2-97=0 \Rightarrow x^2+y^2=97 \Rightarrow$ circle

(1)



$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 - \frac{d}{dx}97 = \frac{d}{dx}0$

$2x\frac{d}{dx}x + 2y\frac{d}{dx}y - 0 = 0$

$y\frac{dy}{dx} = -x$

(3) $\frac{dy}{dx} = \frac{-x}{y}$

when $x=-4$

$(-4)^2+y^2=97$

$y^2 = 97-16$

$y = \pm\sqrt{81}$

$y = \pm 9$

when $y=-9$

it is discarded

because of given

(3) $\Rightarrow \left.\frac{dy}{dx}\right|_{\substack{x=-4\\y=9}} = -\frac{(-4)}{9} = \frac{4}{9}$

4. $h = e^{2.5x}$

$\frac{dh}{dx} = e^{2.5x}(2.5)$

$dh = 2.5e^{2.5x}dx$

(6) $\left.dh\right|_{\substack{x=2\\dx=.3}} = 2.5e^{2.5(2)}(.3) = 111.31$

5. Related Rates: Step 1 $\frac{dy}{dt} = \text{5 units/sec}$, $\frac{dx}{dt} = ?$ units/sec

Step 2 $y = \frac{45}{x}$

Step 3 $\frac{d}{dt}y = \frac{d}{dt}\ 45x^{-1}$

$\frac{dy}{dt} = 45(-1)x^{-2}\frac{d}{dt}x$

$\frac{dy}{dt} = -\frac{45}{x^2}\frac{dx}{dt}$

Step 4 $\frac{\frac{dy}{dt}}{-\frac{45}{x^2}} = \frac{dx}{dt}$

$\Rightarrow \left.\frac{dx}{dt}\right|_{\substack{x=5\\y=9}} = \frac{5}{-\frac{45}{5^2}} = -\frac{25}{9}$ units/sec

6. $E = -\frac{p\,f'(p)}{f(p)}$ or $E = -\frac{p\frac{dx}{dp}}{x}$

$E = \frac{-p(-33.33)}{-33.33p + 2333.33}$

$E = \frac{33.33p}{-33.33p + 2333.33}$

$E = \frac{p}{-p + 70}$

(i) Elastic $\Rightarrow E > 1$

$\Rightarrow \frac{p}{-p+70} > 1$

$p > -p + 70$

$2p > 70$

$p > 35.00\$$

(1) $.03x + p = 70$

or $x = -\frac{p}{.03} + \frac{70}{.03}$

$x = -33.33p + 2333.33$

(2) $\frac{dx}{dp} = -33.33$

Divide by 33.33

(ii) Inelastic $E < 1$

$0 < p < 35.00\$$

7. Step 1 $f'(x) = 12 - 3x^2$

Step 2

$f'(x) = 0$	$f'(x) = \frac{1}{0}$
$12 - 3x^2 = 0$	NO x
$3x^2 = 12$	
$x^2 = 4$	
$x = \pm 2$	

Step 3 use Closed Interval test

$f(-3) = 12(-3) - (-3)^3 = -9$

$f(+3) = 12(3) - (3)^3 = 9$

$f(-2) = 12(-2) - (-2)^3 = -16$

$f(+2) = 12(2) - (2)^3 = 16$

$\Rightarrow$ ABS MAX = 16, when $x = 2$

ABS MIN = -16 " " $x = -2$

8. Step 1 $N'(x) = 9x^2 - x^3$ $\quad N''(x) = 18x - 3x^2$

Step 2

$N'(x) = 0$	$N'(x) = \frac{1}{0}$
$9x^2 - x^3 = 0$	NOK
$x^2(9-x) = 0$	
$x=0 \mid x=9$	

$N''(x) = 0$	$N''(x) = \frac{-1}{0}$
$18x - 3x^2 = 0$	NOK
$3x(6-x) = 0$	
$x=0 \mid x=6$	

Step 3

x	0	$0<x<6$	6	$6<x<9$	9
$N''(x)$	0	$N''(1) = +$	0	$N''(7) = -$	$-$
$N'(x)$	LOC MIN	Inc	LocMAX	Dec	Dec

NOTE:

To get INFO. on when SALES increase dec use $\frac{d}{dx}N = N'(x)$

" " " " " $N'(x)$ (Rate of Change of SALES) " " $\frac{d}{dx}N' = N''(x)$

Answer Rate of Change of SALES $N'(x)$ is $\begin{cases} \text{Inc for } 0<x<6 \\ \text{dec } " \ 6<x<9 \end{cases}$

9. Step 1 Intersection pt.

$2 - 2x = 5 - x^2$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$x+1=0$	$x-3=0$
$x=-1$	$x=3$

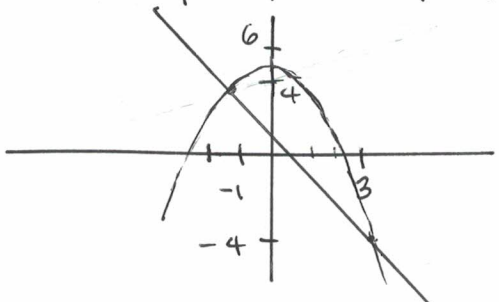
$y = 2-2x$ $\quad y = 2-2x$

$y = 2-2(-1)$ $\quad y = 2-2(3)$

$y = 4$ $\quad y = -4$

pt $(-1, 4)$ $\quad$ pt $(3, -4)$

Step 2 Sketch



Step 3 $\text{Area} = \int_{-1}^{3} [(5-x^2) - (2-2x)]\,dx$

$= \int_{-1}^{3} (-x^2 + 2x + 3)\,dx$

$= \left[-\frac{x^3}{3} + \frac{2x^2}{2} + 3x\right]_{-1}^{3}$

$= \left(-\frac{3^3}{3} + (3)^2 + 3(3)\right) - \left(-\frac{[-1]^3}{3} + [-1]^2 + 3[-1]\right)$

$= -\frac{27}{3} + 9 + 9 - \frac{1}{3} - 1 + 3$

$= 11 - \frac{1}{3} = \frac{33}{3} - \frac{1}{3} = \frac{32}{3}$ SQ. UNITS

10 a) $\int \left(x^{5} - x^{-\frac{1}{2}}\right) dx$

$\int x^{-5} dx - \int x^{-\frac{1}{2}} dx$

$\frac{x^{-4}}{-4} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$

or $-\frac{1}{4x^{4}} - 2x^{\frac{1}{2}} + C$

b) $\int \sqrt{1-5x}\, dx$

$\int \sqrt{u}\left(-\frac{1}{5} du\right)$

$-\frac{1}{5}\int u^{\frac{1}{2}} du$

$-\frac{1}{5}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$

$-\frac{2}{15}u^{\frac{3}{2}} + C$

$-\frac{2}{15}(1-5x)^{\frac{3}{2}} + C$

let $1-5x = u$

$-5 = \frac{du}{dx}$

$-5x\, dx = du$

$dx = -\frac{1}{5} du$

c) $\int_{1}^{5} \frac{2\, dx}{x+2}$

let $u = x+2$

$\frac{du}{dx} = 1$

$du = dx$

$= 2\int \frac{du}{u}$

$= 2\ln|u|$

$= 2\ln|x+2|_{1}^{5}$

$= 2\ln|5+2| - 2\ln|1+2|$

$= 2\ln 7 - 2\ln 3$

$= 2(\ln 7 - \ln 3)$

$= 2\ln\frac{7}{3}$

11 a) Gini Index $= 2\int_{0}^{1}(x - f(x))dx$ where $f(x)$ is the Lorenz curve for a country.

(It is a number between 0 and 1 that is used to compare income distribution between countries).

b) Statement is FALSE: $0 \le p \le 1$ (limits for a Gini Index Number)

$\Rightarrow 0 \ge -p \ge -1$

$\Rightarrow 1+0 \ge 1-p \ge 1-1$

$1 \ge 1-p \ge 0$

or $0 \le 1-p \le 1$

$\Rightarrow$ $1-p$ could be a Gini Index No. for another country because $0 \le 1-p \le 1$ (between 0 and 1)

c) This statement is true.

If Gini Index of Country A : $0 \le p_A \le \frac{1}{3}$

then " " " B : $3(0) \le 3p_A \le 3(\frac{1}{3})$

$0 \le 3p_A \le 1$