

Final sample---solutions

Monday, November 26, 2018 4:06 PM



Fall2017FinalExamSample

MAT1330 A : Instructor Monica Nevins Monday, December 11, 2017 : Final Exam

Duration: 3 hours

Family name: _____

First name: _____

Student number : _____

Seat number : _____

Please read the following instructions carefully.

- You have 3 hours to complete this exam.
- This is a closed book exam. Except for Faculty-approved calculators (models: Texas Instruments TI-30* and TI-34*, Casio FX-260* and Casio FX-300*), no notes, cell phones, smartwatches or related devices of any kind are permitted. All such devices, including cell phones, **must be stored in your bag at the front of the room for the duration of the exam.**
- Read each question carefully — you will save yourself time and grief later on.
- There are 9 questions, each with point values as indicated. **You must show your work, your work must be legible, and when applicable you must record your answers in the boxes provided.**
- Where it is possible to check your work, do so.
- Good luck!

Very important:

Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

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Marker's use only:

Question	Marks
1 (/5)	
2 (/8)	
3 (/6)	
4 (/4)	
5 (/8)	
6 (/4)	
7 (/9)	
8 (/4)	
9 (/3)	
Total (/51)	



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Question 1. (5 points) When a person coughs, their trachea contracts, which affects the speed with which the air flows through the trachea. A mathematical model for the speed \mathcal{A} (in cm^3/s) of the air flowing through the trachea as a function of the radius r (in cm) of the trachea is

$$\mathcal{A}(r) = 5(2 - r)r^2,$$

for $0.5 \leq r \leq 2$.

Find the maximum rate of air flow through the trachea. Justify, with calculations and sentences, why your answer is correct and gives the maximum.

$$\begin{aligned} \mathcal{A}'(r) &= 5[-1 \cdot r^2 + (2-r) \cdot (2r)] \\ &= 5(-r^2 + 4r - 2r^2) \\ &= 5r(4 - 3r) \end{aligned}$$

$$\mathcal{A}'(r) = 0 \Rightarrow \underbrace{r=0}_{\substack{\uparrow \\ \text{not in considered domain.}}} \quad r = \frac{4}{3}$$

$$r : 0.5 \leq r \leq \frac{4}{3} \quad \frac{4}{3} < r \leq 2$$

$$\mathcal{A}'(r) : \quad > 0 \quad < 0$$

$$\mathcal{A}(r) : \quad \nearrow \quad \searrow$$

So $\mathcal{A}(r)$ has a global max when $r = \frac{4}{3}$ cm

the max value is $5 \cdot (2 - \frac{4}{3}) \cdot (\frac{4}{3})^2 \text{ cm}^3/\text{s}$

[Justifying by comparing $\mathcal{A}(\frac{4}{3})$, $\mathcal{A}(0.5)$, $\mathcal{A}(2)$ is also acceptable]

Maximum value of $\mathcal{A} =$ $\frac{160}{27}$ $\text{cm}^3/\text{s}.$

$$\frac{5 \cdot 2 - 16}{27}$$

Question 2. (8 points) Many animals hunt in packs to bring down large prey, for example hyena hunting wildebeest. In consequence, the dynamics of their population demonstrate the so-called Allee effect.

From one year to the next, the density x_t of hyenas in one ecosystem changes according to the discrete time dynamical system (DTDS)

$$x_{t+1} = \frac{6x_t^2}{7 + x_t^2}.$$

(a) Give the updating function for this DTDS:

$$f(x) = \frac{6x^2}{7 + x^2}$$

(b) Find the three fixed points of the DTDS algebraically; show your work.

$$\begin{aligned} x &= \frac{6x^2}{7 + x^2} \\ \Rightarrow 7x + x^3 &= 6x^2 \\ \Rightarrow x(x^2 - 6x + 7) &= 0 \\ \Rightarrow x(x-7)(x+1) &= 0 \\ \Rightarrow x = 0, 7 \text{ or } -1 \end{aligned}$$

Mistake: The factorization is wrong, you should use the Quadratic formula for the solution: $X = 0$ or $3 \pm \sqrt{2}$

$$x_1^* = -1 < x_2^* = 0 < x_3^* = 7$$

This question is continued on the next page.

Question 2 continued. If you were unable to answer part (b), then you may use part (d) to estimate x_1^* , x_2^* and x_3^* to 1 decimal place and use these estimates for part (c).

(c) For this DTDS, we have computed the derivative of the updating function for you:

$$f'(x) = \frac{84x}{(7+x^2)^2}$$

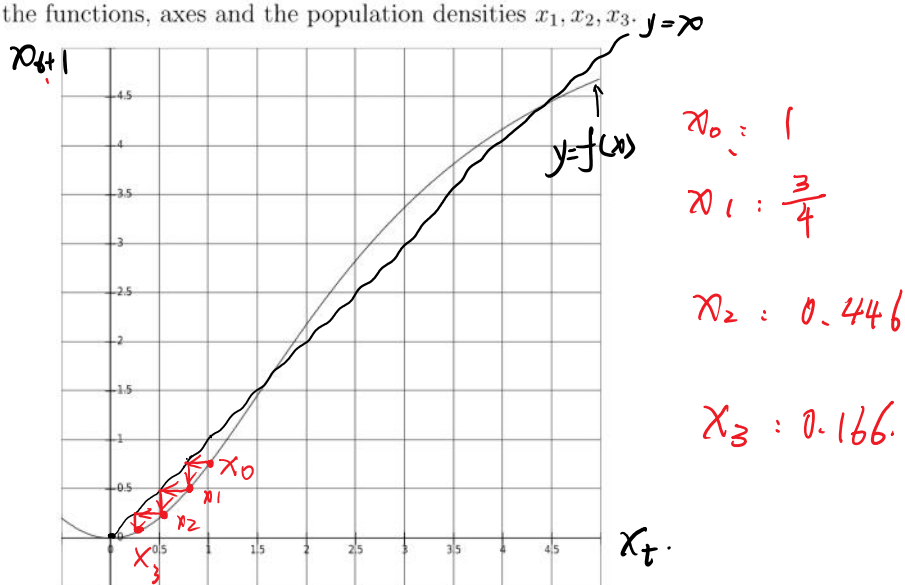
For each of x_1^* , x_2^* and x_3^* , determine whether this fixed point is stable or unstable, and justify your answers with the appropriate calculations and the Stability Theorem.

$$x_1^* : |f'(c-1)| = \left| \frac{-81}{(7+1)^2} \right| > 1 \quad \text{--- unstable}$$

$$x_2^* : |f'(c0)| = \left| \frac{0}{(7+0^2)^2} \right| = 0 < 1 \quad \text{--- stable}$$

$$x_3^* : |f'(c7)| = \left| \frac{84 \cdot 7}{(7+7^2)^2} \right| < 1 \quad \text{--- stable}$$

(d) The graph of the updating function for the hyena population is given below. Starting with an initial population density of $x_0 = 1$, **draw a cobweb diagram** for at least 3 steps. **Label** the functions, axes and the population densities x_1, x_2, x_3 .



(e) Write a sentence to explain what happens to this hyena population in the long-run, if the population density is initially $x_0 = 1$:

The population will keep decreasing until 0.

Question 3. (6 points) Show your work and simplify your answers.

(a) If $g(x) = \frac{1 - \sin(x)}{\sqrt{x}}$ then $g'(x) = \frac{-2 \cos x \cdot x - 1 + \sin x \cdot}{2x\sqrt{x}}$

$$g'(x) = \frac{-\cos x \cdot \sqrt{x} - \frac{1}{2\sqrt{x}} \cdot (1 - \sin x)}{(\sqrt{x})^2}$$

(b) If $y = 2^x \ln(x)$ then $y' = 2^x \cdot \ln 2 \cdot \ln x + \frac{2^x}{x}$

$$y' = 2^x \cdot \ln 2 \cdot \ln x + 2^x \cdot \frac{1}{x}$$

(c) If $e^{f(x)} + (f(x))^2 = x^2$, then $f'(x) = \frac{2x}{e^{f(x)} + 2f(x)}$

The answer will depend on x and on $f(x)$.

$$e^{f(x)} \cdot f'(x) + 2f(x) \cdot f'(x) = 2x$$

$$f'(x) = \frac{2x}{e^{f(x)} + 2f(x)}$$

Question 4. (4 points) Evaluate each of the following limits using the methods from the course, that is, without using sequences of numerical values. Show your work.

$$(a) \lim_{x \rightarrow -3^-} \frac{x+3|(5+x)}{x+3} = \boxed{-2}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow -3^-} \frac{-(x+3)(5+x)}{x+3} = \lim_{x \rightarrow -3^-} -(5+x) \\
 &= -(5-3) \\
 &= -2
 \end{aligned}$$

$$(b) \lim_{x \rightarrow \infty} \left(x \cos\left(\frac{1}{x}\right) - x \right) = \boxed{0}$$

$$= \lim_{x \rightarrow \infty} \left(x \left(\cos\frac{1}{x} - 1 \right) \right)$$

$$= \lim_{x \rightarrow \infty} \frac{\cos\frac{1}{x} - 1}{\frac{1}{x}}, \quad \text{let } u = \frac{1}{x}.$$

$$= \lim_{u \rightarrow 0^+} \frac{\cos u - 1}{u} \stackrel{\text{L'H}}{=} \lim_{u \rightarrow 0^+} \frac{-\sin u}{1} = 0$$

[Also OK if directly use L'Hôpital's Rule without the substitution. More work]

Question 5. (8 points) In this question you will study the graph of $f(x) = x \ln(x)$.

This question continues on the next page.

(a) Give the domain of definition of $f(x)$: $x > 0$ or $(0, +\infty)$

(b) Give all x -intercepts of $y = f(x)$: $x = 0, 1$

$x = 0$
or $\ln x = 0 \Rightarrow$
 $x = 1$

Mistake: 0 is not in the domain.

(c) Determine the limit: $\lim_{x \rightarrow \infty} f(x) = \infty$

(d) Using methods from the course (that is, not sequences of numerical values), compute :

$\lim_{x \rightarrow 0^+} f(x) = 0$

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x = 0$$

(e) Compute $f'(x) = \ln x + 1$

(f) Find all critical points of f , and determine where f is increasing or decreasing. Make a table, for example.

$f'(x) = 0 \Rightarrow x = e^{-1}$	
$x:$	$(0, e^{-1}) \quad \quad (e^{-1}, +\infty)$
$f'(x):$	$< 0 \quad \quad > 0$
$f(x):$	$\searrow \quad \quad \nearrow$

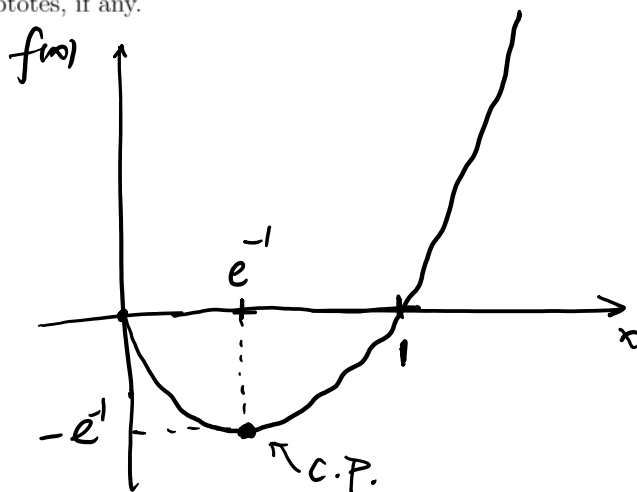
Question 5 continued.

(g) Compute $f''(x) = \frac{1}{x^2}$.

(h) Determine if there are any inflection points of the graph of f , and determine where f is concave up or concave down.

$\frac{1}{x^2} \Big|_{x>0} > 0$. no inflection points.
 $f(x)$ is concave up on $(0, +\infty)$

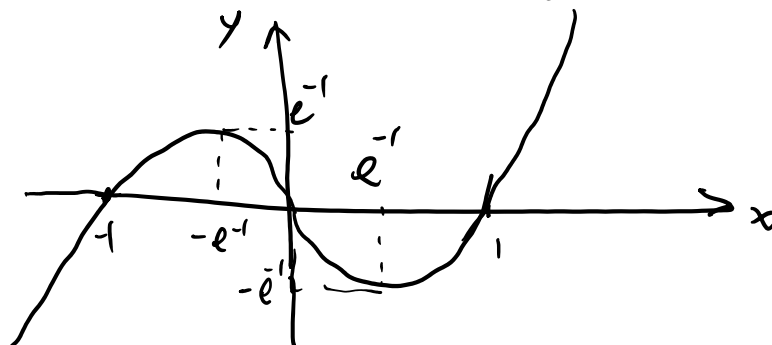
(i) In the space below, sketch the graph of $f(x) = x \ln(x)$. Label all critical points, inflection points and asymptotes, if any.



(j) Now, without further calculations, sketch the graph of $g(x) = x \ln|x|$, justifying briefly what property of the function $g(x)$ you are using.

Since $g(-x) = -x \ln|x| = -g(x)$.

$g(x)$ is odd. symmetry about the origin.



Question 6. (4 points) The nonzero equilibria of the DTDS $N_{t+1} = N_t^2(2 + N_t^3)$ are the values x which satisfy the equation

$$x(2 + x^3) = 1.$$

- (a) Using the Intermediate Value Theorem, explain why there exists a solution $x = c$ to this equation in the interval $[0, 1]$.

$$x(2 + x^3) - 1 = 0$$

Let $f(x) = x(2 + x^3) - 1$, which is continuous.

$$f(0) = 0 - 1 < 0, \quad f(1) = 1 \cdot 3 - 1 > 0.$$

By I.V.T. a solution exists.

- (b) Use Newton's method with an appropriate function f and the initial guess $x_0 = 0$ to estimate this value c . Do 3 iterations and give your final answer with a precision of 4 decimal places. Write down $f(x)$ and the formula you used.

Answers:

$$f(x) = x - \frac{x(2 + x^3) - 1}{2 + 4x^3}$$

$$x_{n+1} = x_n - \frac{x_n(2 + x_n^3) - 1}{2 + 4x_n^3}$$

$$f'(x) = 2 + x^3 + 3x^3 \cdot x \\ = 2 + 4x^3$$

$$x_0 = 0$$

$$x_1 = 0 - \frac{0 \cdot (2 + 0^3) - 1}{2 + 4 \cdot 0^3} = 0.5$$

$$x_2 = f(x_1) = 0.475$$

$$x_3 = f(x_2) \approx 0.4746$$

$$c \approx \boxed{0.4746}$$

Question 7. (4 points) Let

$$f(x) = \arcsin(x^2 - 1)$$

(a) Compute $f'(x)$. Simplify your answer.

$$f'(x) = \frac{2x}{|x|\sqrt{2-x^2}} \text{ or } \frac{2 \cdot \text{sign}(x)}{\sqrt{2-x^2}}$$

$$\begin{aligned} f'(x) &= \frac{2x}{\sqrt{1-(x^2-1)^2}} = \frac{2x}{\sqrt{1-(x^4-2x^2+1)}} = \frac{2x}{\sqrt{2x^2-x^4}} \\ &= \frac{2x}{|x|\sqrt{2-x^2}} \end{aligned}$$

(b) Find the equation of the tangent line to $y = f(x)$ at $x = 1$. Show your work.

Answer:

$$y = 2 \cdot (x - 1)$$

$$f'(1) = \frac{2 \cdot 1}{1 \cdot \sqrt{1}} = 2, \quad f(1) = \arcsin 0 = 0$$

(c) We have computed for you: $f''(1) = 2$ and $f'''(1) = 8$. Give the Taylor polynomial of degree 3 of f centered at the base point $a = 1$.

$$T_3(x) = 0 + 2(x-1) + (x-1)^2 + \frac{4}{3}(x-1)^3$$

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

(d) Given that $f(1.1) = \arcsin(0.21)$, use your answer to (c) to give an approximation of $\arcsin(0.21)$.

$$\text{Answer: } 0 + 2 \cdot (1.1 - 1) + (1.1 - 1)^2 + \frac{4}{3}(1.1 - 1)^3 \approx 0.2113$$

Question 8. (9 points) Find the following indefinite integrals. Show your work and check your answers. You may use the backs of pages if necessary but then write a note where to find your work. This question continues on the next page.

$$(a) \int \frac{(\sqrt{x}+1)^2}{x^3} dx = \boxed{-x^{-1} - \frac{4}{3}x^{-\frac{3}{2}} - \frac{1}{2}x^{-2} + C}$$

$$\begin{aligned} \int \frac{(\sqrt{x}+1)^2}{x^3} dx &= \int \frac{x + 2x^{\frac{1}{2}} + 1}{x^3} dx \\ &= \int x^{-2} dx + 2 \int x^{-\frac{5}{2}} dx + \int x^{-3} dx \\ &= \frac{x^{-1}}{-2+1} + 2 \cdot \frac{x^{-\frac{3}{2}}}{-\frac{5}{2}+1} + \frac{x^{-2}}{-3+1} + C \end{aligned}$$

$$(b) \int \frac{e^{\sqrt[3]{x+1}}}{(x+1)^{2/3}} dx = \boxed{3e^{\sqrt[3]{x+1}} + C}$$

$$\begin{aligned} \text{Let } u &= \sqrt[3]{x+1} = (x+1)^{\frac{1}{3}} \\ du &= \frac{1}{3} (x+1)^{-\frac{2}{3}} dx \\ \int \frac{e^{\sqrt[3]{x+1}}}{(x+1)^{2/3}} dx &= \int e^u \cdot 3 du \\ &= 3e^u + C \end{aligned}$$

Question 8 continued.

$$(c) \int x^2 \cos(4x) dx = \frac{1}{4} x^2 \sin 4x + \frac{1}{8} x \cos 4x - \frac{1}{32} \sin 4x + C.$$

• $u = x^2, \quad dv = \cos 4x dx.$

$$du = 2x dx, \quad v = \frac{1}{4} \sin 4x.$$

$$= \frac{1}{4} x^2 \cdot \sin 4x - \int \frac{1}{4} \sin 4x \cdot 2x dx.$$

$$= \frac{1}{4} x^2 \sin 4x - \frac{1}{2} \int x \cdot \sin 4x dx$$

... another. Inter. by. part.

let $u = x, \quad dv = \sin 4x dx$

$$du = dx, \quad v = -\frac{1}{4} \cos 4x$$

$$\int x \sin 4x dx = -\frac{1}{4} x \cos 4x + \frac{1}{4} \int \cos 4x dx.$$

$$= -\frac{1}{4} x \cos 4x + \frac{1}{16} \sin 4x + C.$$

Question 9. (3 points) Determine if the following statements are *always true* or *sometimes false*. Circle the correct option.

- (a) Suppose $f(x) = rx + c$ is a linear updating function for a DTDS $x_{t+1} = f(x_t)$. Then the general solution is $x_t = r^t x_0 + c$. **true** / **false**
- (b) If $(a, f(a))$ is a critical point of f , then it is always true that $f'(a) = 0$. **true** / **false**
- (c) If $f(x)$ is continuous everywhere on \mathbb{R} , then it is guaranteed that $f(x)$ is differentiable at every point of \mathbb{R} . **true** / **false**
- (d) It can happen that $f''(a) = 0$ but that the graph of f does not have an inflection point at $(a, f(a))$. **true** / **false**
- (e) The Extreme Value Theorem states that every function attains a global maximum and a global minimum. **true** / **false**
- (f) The definition of the derivative of f at x is $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ **true** / **false**

(a) $x_t = r^t(x_0 - x^*) + x^*$, where $x^* = \frac{c}{1-r}$ (b) $f'(a)$ may be undefined.

(c) e.g. $|x|$ at $x=0$. (d) e.g. $y=x$. or $y=x^4$ at $x=0$.

(e) has to be Continuous function on closed interval.