Midterm I Solutions

1. Find the derivative of

$$f(x) = \frac{1}{2x+1}$$

using the definition. *You may not use any of the differentiation rules from class, only the definition.*

[2pts]

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h}$$

$$= \lim_{h\to 0} \frac{(2x+1)-(2(x+h)+1)}{h}$$

$$= \lim_{h\to 0} \frac{2x+1-(2x+2h+1)}{h(2x+1)(2(x+h)+1)}$$

$$= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}(2x+1)(2(x+h)+1)}$$

$$= \frac{-2}{(2x+1)(2x+1)}$$

$$= \frac{-2}{(2x+1)^2}$$

2. Determine all values of b such that the function

$$g(x) = \begin{cases} (x-1)^2 + b & \text{if } x < 2 \\ 2^x + bx & \text{if } x \geq 2 \end{cases}$$

+[1pts] is continuous everywhere.

For g to be continuous at $x=2$, we need:

$$\lim_{x\to 2} g(x) = g(2)$$

Now for $\lim_{x\to 2} g(x)$ to exist, we need:

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x)$$

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}\left((x-1)^2+b\right) = (2-1)^2+b = 1+b$$

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}\left(2^x+bx\right) = 2^2+b(2) = 4+2b$$

Hence, we need $1+b = 4+2b$

that is, $b=-3$.

Thus $\lim_{x\to 2^-} g(x) = 1+b = 1+(-3) = -2 = \lim_{x\to 2^+} g(x)$

i.e., $\lim_{x\to 2} g(x) = -2$

We also have $g(2) = 2^2 + (-3)(2) = -2$

So $\lim_{x\to 2} g(x) = g(2)$ meaning that g is continuous at 2.

Next, since g is defined as an elementary function on $(-\infty,2)$ and $(2,\infty)$, g is continuous there. So g is continuous everywhere for $b=-3$.

3. Determine the following limits. *You may use any technique we have seen so far in the course.*

[4pts] (a) $\lim_{x\to\infty} \frac{x^2+\sqrt{x^6+x^2}}{-4x^3+3x}$ $= \lim_{x\to\infty} \frac{x^2+\sqrt{x^6\left(1+\frac{1}{x^4}\right)}}{x^3\left(-4+\frac{3}{x^2}\right)}$

$= \lim_{x\to\infty} \frac{x^2+|x^3|\sqrt{1+\frac{1}{x^4}}}{x^3\left(-4+\frac{3}{x^2}\right)} = \lim_{x\to\infty} \frac{x^2+x^3\sqrt{1+\frac{1}{x^4}}}{x^3\left(-4+\frac{3}{x^2}\right)}$ $|x|=x$ as $x\to\infty$

$= \lim_{x\to\infty} \frac{\cancel{x^3}\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^4}}\right)}{\cancel{x^3}\left(-4+\frac{3}{x^2}\right)}$

$= \frac{1}{-4} = -\frac{1}{4}$

(b) $\lim_{x\to 2} \frac{(x^2-4)\sin(x-2)}{(x-2)^2}$ $= \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}\sin(x-2)}{\cancel{(x-2)}(x-2)}$

$= \lim_{x\to 2}(x+2) \cdot \lim_{x\to 2} \frac{\sin(x-2)}{x-2}$

$= (4) \cdot \lim_{t\to 0} \frac{\sin(t)}{t}$

$= (4)\cdot(1) = 4$

4. Determine each of the following derivatives. *You may use any technique we have seen so far in the course. You do not need to simplify your answer.*

[4pts]

(a) $\dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{re^r + 2\pi^3}{r^3+1}\right)$

$$= \frac{(re^r + 2\pi^3)'(r^3+1) - (re^r + 2\pi^3)(r^3+1)'}{(r^3+1)^2}$$

$$= \frac{(e^r + re^r)(r^3+1) - (re^r + 2\pi^3)(3r^2)}{(r^3+1)^2}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos(e^{x^4})\right)$

$$= \left[-\sin\left(e^{x^4}\right)\right] \cdot \left(e^{x^4}\right)'$$

$$= \left[-\sin\left(e^{x^4}\right)\right]\left(4x^3 e^{x^4}\right)$$

5. Determine the equations of all lines that are tangent to the curve $f(x) = 2x^2$ and pass through the point $(0, -2)$. *Hint: find the slope of the tangent line to the curve at $x = b$, then find the equation of the tangent line at $x = b$, and then find the values of b that make this line pass through the given point.*

[2pts]

$f'(x) = 4x$

$f'(b) = 4b$

Equation of tangent at $(b, 2b^2)$: $4b = \frac{y-2b^2}{x-b}$

i.e., $4b(x-b) = y - 2b^2 \Rightarrow y = 4bx - 4b^2 + 2b^2$

$\Rightarrow y = 4bx - 2b^2$

Since the tangent passes through $(0,-2)$:

$-2 = 4b(0) - 2b^2 \Rightarrow -2 = -2b^2 \Rightarrow b^2 = 1$

$\Rightarrow \underline{\underline{b = \pm 1}}$

6. A function $y = f(x)$ is defined implicitly using the equation

$$y^3 = e^x + x^2.$$

[2pts] Differentiating implicitly, find an expression for y' in terms of x and y.

$$3y^2y' = e^x + 2x$$

$$y' = \frac{e^x + 2x}{3y^2}$$

7. Consider the function

$$f(x) = x^{(e^x)}.$$

[2pts] Find an expression for $f'(x)$ purely in terms of x. *Hint: logarithmic differentiation.*

$$y = x^{e^x}$$

$$\ln y = \ln x^{e^x}$$

$$\ln y = e^x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(e^x \ln x)$$

$$\frac{y'}{y} = e^x \ln x + e^x \cdot \frac{1}{x}$$

$$y' = y\left[e^x\left(\ln x + \frac{1}{x}\right)\right]$$

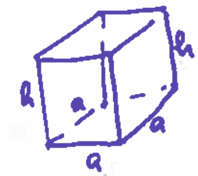
$$= x^{e^x}\left[e^x\left(\ln x + \frac{1}{x}\right)\right]$$

8. A box with a square base is changing its height (h) and the side (a) of its base continuously. When the side (a) measures 2 cm and height (h) measures 8 cm, the side (a) is growing at a rate of 1 cm/min while the height (h) is decreasing at a rate of 2 cm/min. What is the rate of change of the volume at that moment?

[3pts]

side of base : a

Height : h



Given : $\frac{da}{dt} = 1\,cm/min$ when $(a,h)=(2,8)$

$\frac{dh}{dt} = -2\,cm/min$ when $(a,h)=(2,8)$

We find $\frac{dV}{dt}$ when $(a,h)=(2,8)$

Equation: $V = a^2 h$

$$\frac{dV}{dt} = 2a\frac{da}{dt}h + a^2\frac{dh}{dt}$$

when $(a,h)=(2,8)$: $\frac{dV}{dt} = 2(2)(1)(8)+(2)^2(-2)$

$= 32 - 8 = \underline{\underline{24}}\ cm^3/min$

Answer: The volume is increasing at a rate of $24\ cm^3/min$