

VI. Derivatives

- In this section we will study derivatives. Derivative is a contract written on an asset which can be another security or commodity.
- There are many types of derivatives. We will study futures contracts, forward contracts, and options
- Derivatives are traded on exchanges and in “over-the-counter” (OTC) markets.
- The benefit of an exchange is that it facilitates meeting of people who want to trade. It regulates the contracts and provides solvency
- The benefit of OTC market is that it allows to write a derivative providing a better fit for the needs of trading parties
- Derivatives are traded by three types of investors: hedgers, speculators, and arbitrageurs
 - Hedgers use derivatives to reduce the risk they face from potential future movements in a market variable. Example: protective put strategy
 - Speculators use derivatives to bet on the future direction of a market variable. Example: a call option written on a stock
 - Arbitrageurs take offsetting positions in two or more instruments to lock in a profit. Example: trading based on violation of put–call parity
- We start from futures and forward contracts.

1. Futures and Forward Contracts

- A *futures contract* is an agreement to buy or sell an asset at a certain time in the future for a certain price.
 - The price of an asset to be sold in the future is called futures price. It is different from a spot price which a current price on an asset.
 - The time at which an asset has to be sold or purchased is called maturity time
 - The futures price for the given asset and maturity changes over time but it remains fixed for a given contract

- A trader who agrees to buy an asset has a long futures position. A trader who agrees to sell has a short futures position
 - Futures contracts are traded on exchanges (for example, on CME, Eurex, Tokyo Financial Exchange)
 - Futures contract requires opening margin account and following daily settlement procedures which will be discussed later
- Example:
- *Forward contract* is an agreement to buy or sell an asset at a certain time in the future for a certain price. Forward contracts are traded in OTC markets. This contract has terms similar to the futures contract. However, it does not require opening margin accounts and daily settlements like futures contracts do
 - Taking a long or short position in a futures or a forward contract is free

Futures

- Futures contract is usually referred by its delivery month. Actual day of a delivery is chosen by a party with a short position from a range of delivery days
 - Most of futures contracts are closed without delivery. Closing a position involves entering into an opposite trade to the original one. For example, an investor who buys ten July contracts on February can close her position in June by selling 10 July contracts
 - As the delivery period of the futures contract is approached, the futures price converges to the spot price of the underlying asset.
Question: Suppose that the futures price is below the spot price during the delivery period. What would be an arbitrage strategy in this market?
- Trading futures contract involves credit risk. Therefore, an exchange requires investors on the two sides of a trade to maintain a margin account
 - To understand the logic behind a margin account one has to remember that an investor who is long (short) in the futures contract is better off when the futures price becomes higher (lower) than the original one at which the contract was

opened. Moreover, an investor with a position in a futures can instantly realize her profit by closing a position

- Suppose that on February 1st an investor buys a futures contract to buy 100 ounces of gold for \$1,500 per ounce. Her total liability is \$150,000. An investor with a long position may not honor the contract if the price of gold will go down.
- At the time of opening the contract a buyer and a seller are required to make a deposit called the *initial margin* in their margin accounts. Suppose it is \$10,000 per contract.
- Suppose that the futures price of this contract has dropped to \$149,000 on February 2nd. Then the buyer of the contract has virtually lost \$1,000 while the seller has virtually gain \$1,000. Consequently, the margin account is reduced by \$1,000 which are passed to an investor with a short position. If the futures price increases then the difference between positions is passed to the buyer from the margin account of the seller
- This kind of settlements are made every day and are called *marking to market*
- To make sure that the balance on the margin account does not become negative, each investor is required to maintain a *maintenance margin* which is a minimal balance in the margin account. If margin account falls below the maintenance margin then an investor receives a margin call and must increase margin account to initial margin
- If contract is closed or expires, an investor walks away with his margin account
- The table below illustrates the operation of the margin account of a long investor for one possible sequence of futures prices. The maintenance margin is assumed to be \$5,000 per contract. The table shows that on day 6 the margin account drops below the maintenance margin which triggers the margin call to deposit an additional \$5,100 which have to be provided by the end of the day. On day 8 the investor closes her position by selling a futures contract with the futures price of \$1,453.90. The total loss of an investor is

...

- Brokers pay interest on the balance in a margin account.
- Margin requirements depend on variability of an asset price and types of traders

Day	Trade price	Settlement price	Daily gain	Margin account balance	Margin call
1	1,500			10,000	
2		1,490.00	-1,000	9,000	
3		1,478.20	-1,180	7,820	
4		1,461.50	-1,670	6,150	
5		1,455.40	-610	5,540	
6		1,449.00	-640	4,900	5,100
7		1,451.80	280	10,280	
8	1,453.90		210	10,490	

- In the OTC markets margin are replaced by collateral
- Finally we summarize the differences between futures and forward contracts:

Forward	Futures
Private	Traded on exchange
Not standardized	Standardized
One delivery day	Range of delivery dates
Settled at the end of contract	Settled daily
Delivery or cash settlement	Contract is usually is closed out
Some credit risk	No credit risk

Strategies with Futures

- Futures and forward contracts can be used in arbitrage strategies. The examples of such strategies will be considered in later
- Futures and forward contracts can also be used for speculations. Example:
 - Companies hedge because they have no skills in predicting commodity prices, interest rate or exchange rates. Moreover, most of the companies discount profits from the future price movement more than losses from these movements causing them to hedge

- Hedging using futures fixes the future prices but it does not necessarily remove the risk from the overall position. It is important to realize that hedging can result in a decrease or an increase of a company profit relatively to the case when it does not hedge
- The simplest hedging in buying is taking a long position in a futures contract while hedging in selling is taking a short position in a futures contract and waiting until delivery day. Example:

Pricing Forward and Futures Contracts

- Now we want to find forward and futures prices
- If we consider a forward and a futures contract written on the same asset and with the same maturity then generally the forward and futures prices are going to be different. The difference is due the existence of margin account for futures contracts. However, typically the difference between these two prices is rather small. Therefore, from now on **we assume that the futures prices and forward prices are the same** and we do not distinguish between forward and futures contracts while considering the prices.
- Price of a forward contract depends if an underlying is investment or consumption asset. Examples of investment assets are financial securities. Examples of consumption assets are commodities
- We will start from pricing forward contracts written on investment assets. The key feature of these assets is that they can be sold short
- To price a forward contract we will use the assumptions that
 - There are no transaction costs
 - Borrowing and lending rates are the same
 - Arbitrage opportunities do not exist (quickly disappear)

and the following notations

- T : time until delivery date in a contract

- S_0 : spot price of an underlying asset
- F_0 : spot price of a forward contract
- r_f : annual risk-free interest rate

Forward Price For An Investment Asset

- Assume that underlying investment asset provides the holder with no income

- The current price of a forward contract on this asset must be given by

$$F_0 = S_0(1 + r_f)^T \tag{6-1}$$

- Indeed, let us suppose that (6-1) does not hold and $F_0 < S_0(1 + r_f)^T$. Let us follow the following strategy: Short one share of underlying asset, invest the proceeds into risk free bonds, and take a long position in a forward contract on this asset with delivery at time T . Value of the strategy today is zero dollars. At time T : close a forward contract by buying an underlying asset for F_0 dollars, then close your short position. Final proceeds: $-F_0 + S_0(1 + r_f)^T > 0$.
- Let us suppose that $F_0 > S_0(1 + r_f)^T$. Let us follow the following strategy: Short a forward contract, borrow S_0 dollars and buy an underlying asset. Value of the strategy today is zero dollars. Its value at time T : close a forward contract by selling an underlying asset for F_0 dollars, return the borrowed funds with an interest, $S_0(1 + r_f)^T$. Final proceeds: $F_0 - S_0(1 + r_f)^T > 0$.
- Both strategies are examples of an arbitrage. Equation (6-1) must hold since arbitrage is not allowed.

- Now let us assume that an investment asset provides a perfectly predictable income whose present value during the life of a forward contract is I . The price of this forward contract must be

$$F_0 = (S_0 - I)(1 + r_f)^T \tag{6-2}$$

- Indeed, if $F_0 > (S_0 - I)(1 + r_f)^T$ an arbitrageur can lock in a profit by borrowing S_0 dollars, buying the asset, and shorting a forward contract. In the borrowed S_0 dollars, I dollars have to be borrowed with the maturity equal to the time of the cash flow which will be paid off by the cash flow of the asset. If $F_0 < (S_0 - I)(1 + r_f)^T$ an arbitrageur can lock in a profit

by shorting the asset for S_0 dollars, lending S_0 dollars, and taking a long position in a forward contract. In lending S_0 dollars, I dollars have to be lent until the time of the cash flow from the asset which will be used to forward to the lender who gave the asset for shorting. Therefore, equation (6-2) must hold since arbitrage is not allowed.

- Example: Consider a long forward contract to buy a coupon-bearing bond whose current price is \$1,000. Bond pays coupon of \$50 in 1 month and expires in one year while the forward contract matures in half a year. Suppose the annual risk-free rate is 1%. It follows that the price of the forward contract must be

Suppose that the futures price in the market is \$990.00. Then an investor shorts the contract, borrows and buys the bond. The present value of a coupon is Therefore, an investor borrows for one month and borrows for 6 months. She repays \$50 in one month after receiving the coupon. In 6 months, she sells the bond for \$990.00 and repays The arbitrage profit=

Forward and Futures on Commodities

- Now we want to price forward contracts on commodities. One of the main difference of the commodity from an investment asset is that it cannot be sold short. First we consider commodities that are also investment assets (gold and silver) and then we move to consumption assets
- Contrary to investment assets, holding commodities involves facing storage costs
 - Let the present value of the storage cost be U
 - Storage cost can be treated as a negative income
- Investment commodities cannot be shorted (except for gold?). However, a lot of investors hold them for investment purposes. Therefore, they can sell them (instead of shorting) if the forward contract is underpriced with respect to underlying commodity
 - It follows that the forward price is given by

$$F_0 = (S_0 + U)(1 + r_f)^T \tag{6-3}$$

- Indeed, suppose that $F_0 < (S_0 + U)(1 + r_f)^T$. Then a lot of investors will sell the investment commodity today, save on storage costs and take a long position in the forward contract to buy the commodity for F_0 dollars at time T. Since the forward contract has no default risk, the present value of this strategy is $S_0 + U - F_0(1 + r_f)^{-T} > 0$. This strategy is too good to be overlooked
- Question: Is above strategy an arbitrage? How investor can loose money in this strategy?
- Now suppose that $F_0 > (S_0 + U)(1 + r_f)^T$. What should be an arbitrage strategy in this case?
- Example. Consider one year futures contract on gold. It costs \$2 per ounce per year to store gold with payment *being made at the end of the year*. The risk free rate is 5% and the spot price is \$1,600. It follows that the futures price must be

$$F_0 =$$

Suppose that the observed price is \$1,650. What investor should do? Suppose that an investor has 100 ounces of gold. He should sell all gold for, lend these money plus present value of the storage cost and enter a long futures to buy 100 ounces of gold. In one year he makes

- Now let us consider pricing of a forward contract written on a consumption commodity. Investors hold these commodities for the purpose of consumption (production) and may refuse to sell it if investment opportunity arises

- It follows that the forward price should satisfy the following relation

$$F_0 \leq (S_0 + U)(1 + r_f)^T, \quad (6-4)$$

- Indeed, suppose that $F_0 > (S_0 + U)(1 + r_f)^T$. Then a lot of investors will take short position in the forward contract on the commodity, borrow $S_0 + U$ dollars and buy the commodity for the price S_0 . At time T, they will receive F_0 dollars and return $(S_0 + U)(1 + r_f)^T$ dollars to a lender. The riskless profit is $F_0 - (S_0 + U)(1 + r_f)^T > 0$ implying that $F_0 \leq (S_0 + U)(1 + r_f)^T$

- Now assume that $F_0 < (S_0 + U)(1 + r_f)^T$. A profitable strategy dictates that an investor should sell the commodity and take a long position in the forward contract. However, consumers will not trade commodities for forward contract because they need them for consumption. Therefore, the last inequality can hold in efficient market. We conclude that equation (6-4) holds.

2. Options

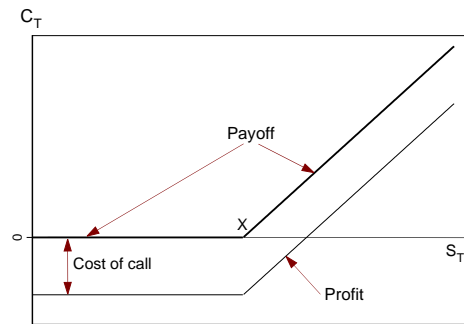
- In the next sections we will study different types of options, their pricing, and strategies
- Options allow a better risk management and make market more efficient
- In the following sections we will consider the two major types of options: calls and puts
- Call or put option is typically written on a stock share (underlying security)

Definitions

- A **call option** gives the holder the right to *purchase* a stock share for a specified price, called the *exercise* or *strike price*, on or before a given maturity date T.
- We say that an option is exercised when a long investor buys stock using the option
- Holder of a call option will buy a stock share for the strike price X only if its price S_t at the exercise time t exceeds X .
- Therefore, the value of the option C_T for its holder at expiration (also called the option payoff) is (also see the figure below)

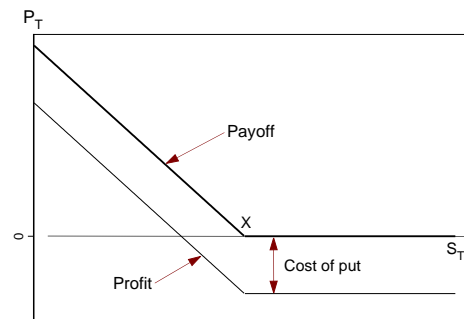
$$C_T = (S_T - X)^+ \equiv \begin{cases} 0 & \text{if } S_T \leq X \\ S_T - X & \text{if } S_T > X. \end{cases}$$

- Investors who buy call options believe that the stock price will grow



- A **put option** gives the holder the right to *sell* a stock share for the exercise price, on or before maturity date T.
- The value of the put option P_T at expiration is (also see the figure below)

$$P_T = (X - S_T)^+ \equiv \begin{cases} X - S_T & \text{if } S_T < X \\ 0 & \text{if } S_T \geq X. \end{cases}$$



- Investors who buy put options believe that the stock price will fall
- Profit on an option held to expiration is its value at expiration less the cost of purchase

Practice Problem

Show the payoff and profit for a writer of a (a) call option, (b) put option

- Question: What are the expectations of a call (put) *seller* on the future of the underlying stock price?

- Call and Put options can be of two types: an **American option** and a **European option**
 - An American option allows its holder to exercise it before or at its expiration
 - A European option can be exercised only at its expiration
 - All options in Canada, except for stock index options are American options

- Each option is characterized by its moneyness
 - An option is *in-the-money* if its exercising is beneficial (apart from the cost of purchase)
 - * It follows that a call option is in-the-money if $S > X$
 - * A put option is in-the-money if $S < X$
 - An option is *out-of-the-money* if its exercising is not beneficial
 - * It follows that a call option is out-of-the-money if $S < X$
 - * A put option is out-of-the-money if $S > X$
 - An option is *at-the-money* if price of underlying stock is equal to the strike price ($S = X$)
 - Most of the options are issued at-the-money

- An option can be written on an individual stock, stock index, foreign currency, on bonds and other financial instruments

- The typical maturity date for an option is a few months

- Typically, a vanilla option is rather cheap and has a very high expected return. However, it is also very risky

- Question: What should be the relation between the Sharpe ratios of option and underlying security in the efficient market?

Option Strategies

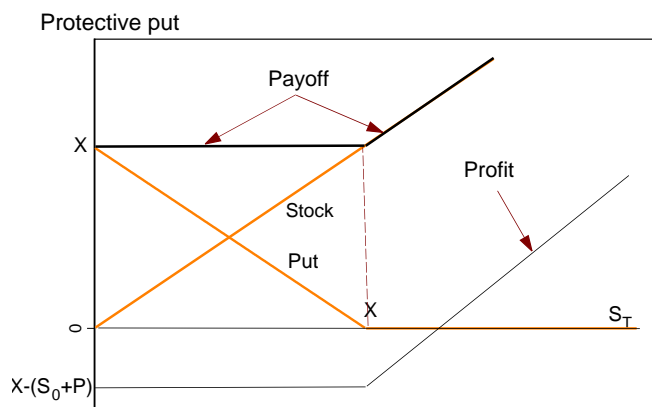
- Options could be used for speculations, taking advantage of mispricing in the markets and better risk management
- First, we consider the last application of options
- Due to market frictions such as transaction costs, illiquidity, margin requirements and so on, trading only bonds and stocks may not be sufficient to efficiently control the risk exposure of an investor
- By combining options and underlying security, investor can control her risk exposure and expected returns

Protective Put

- If an investor bets on the stock growth but still wants to protect herself from downside risk then she buys a put while taking a long position in the underlying stock. This strategy is called *protective put*.
- Payoff of this strategy is

	$S_T < X$	$S_T > X$
Stock	S_T	S_T
Put	$X - S_T$	0
Total	X	S_T

- The graph of this payoff is shown below



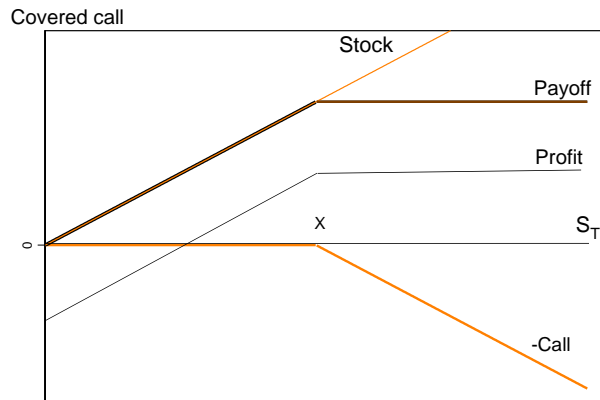
- The cost of insurance against downside risk is the put price
- Typically, an investor must roll over her position in a put since its maturity is much shorter than her investment horizon

Covered Call

- A *covered call* position is the purchase of a share of stock with simultaneous short-sale of a call on that stock
- Payoff of this strategy is

	$S_T < X$	$S_T > X$
Stock	S_T	S_T
-Payoff of call	0	$-(S_T - X)$
Total	S_T	X

- The graph of this payoff is shown below



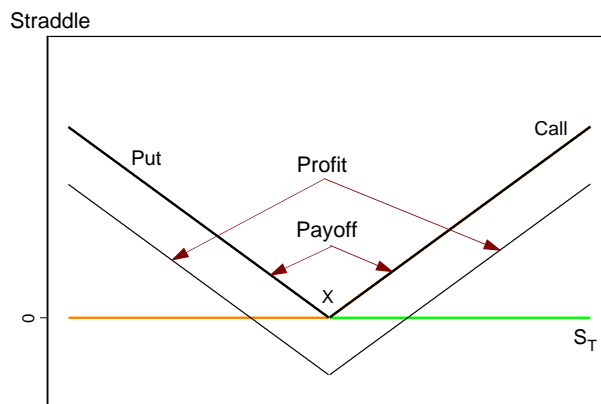
- This strategy is used by institutional investors who plan to sell the stock at price X
- These investors forfeit potential gains if stock rises above X but collect the premium for the call

Straddle

- A long *straddle* is established by buying both a call and a put written on the same stock, each with the same exercise price, X , and the same expiration date, T .
- Payoff of this strategy is

	$S_T < X$	$S_T > X$
Call	0	$S_T - X$
Put	$X - S_T$	0
Total	$X - S_T$	$S_T - X$

- The graph of this payoff is shown below



- This strategy is used by investors who bet on volatility of a stock and uncertain about overall direction of the move

The Put–Call Parity

- Before discussing option pricing let us establish a simple relation between prices of European put, call, underlying stock and riskless bond
- Recall that payoff of the protective put strategy is

	$S_T < X$	$S_T > X$
Stock	S_T	S_T
Put	$X - S_T$	0
Total	X	S_T

where T is a maturity date of a put option

- Now let us replicate the payoff of the protective put strategy by using T-bills and call options. Assume that call options are written on the same stock and have the same strike price and maturity as the put option does. T-bill pays one dollar at time T . The table below shows the number of calls and T-bills required to replicate protective put.

	$S_T < X$	$S_T > X$
_____ Call		
_____ T-bills	_____	_____
Total	X	S_T

- Assume that no arbitrage opportunities exist in the market
- Because the payoffs of the two strategies are the same and arbitrage strategies are absent, the costs to establish each portfolio must be equal
- Therefore

$$P_0 + S_0 = C_0 + PV(X)$$

or

$$P_0 + S_0 = C_0 + \frac{X}{1 + r_f},$$

where P_0 , S_0 , C_0 are the current price of put, call, and underlying stock, and r_f is risk-free rate per time-period T

- The last equation is called the **put-call parity relationship**
- If the interest rate is effective annual then

$$PV(X) = \frac{X}{(1 + r_f)^T}$$

- We emphasize that the put-call parity assumes that both call and put are written on the same stock, have the same expiration and strike price.

- The above put–call parity holds only if the stock does not pay dividends. With dividends, it becomes

$$P_0 + S_0 = C + \frac{X}{1 + r_f} + PV(D),$$

where $PV(D)$ is the present value of the dividends that will be paid by the stock during the life of the options.

Practice Problem

Suppose that the stock does not pay dividends and its current price is \$100, the call price is \$11, while the put price is \$3. The expiration date of both options is in 6 months, their strike price is \$100 and the annual effective risk–free interest rate is 10.25%. Check if the put–call parity is violated for these data. Do you have an access to an arbitrage opportunity here? If yes, then how do you trade to take advantage of it?

- Note that **American Options do not satisfy the put–call parity**

Option Pricing

- First, let us discuss the factors that affect option prices

Determinants of option values

The following table presents the effects of the relevant factors on call option prices

Increase in	Value of a Call Option
Stock price, S	Increases
Strike price, X	Decreases
Stock volatility, σ	Increases
Time to maturity, T	Increases
Interest rate, r_f	Increases
Dividends	Decreases

- Comments:

- The higher the stock price today, the more likely that it will be high at the time of exercise
- The higher X , the more likely that the difference $S - X$ will be lower at the time of exercise
- Consider at-the-money call option with exercise price of \$50 written on the stock that could be equal to \$25 and \$75 with probabilities 0.5 at maturity date T compared with situation where the stock could be equal to \$40 and \$60 with probabilities 0.5. In both cases the expected stock price is \$50.

The expected payoff of the option in the case of a more volatile stock is

$$E(C_T) = 0.5 \times 0 + 0.5 \times (75 - 50) = \$12.5$$

while the expected payoff of the option in the case of the other stock is

$$E(C_T) = 0.5 \times 0 + 0.5 \times (60 - 50) = \$5$$

- * In summary, the price of a call option increases with higher volatility of stock because a stock fall has limited effect on the option payoff (C_T cannot be less than 0), while its rise increases the payoff proportionally.
- If T increases then volatility of stock increases too causing increase in the option price. Moreover, the present value of the strike price falls
- If r_f increases then the present value of the strike price falls causing increase in the option price.
- The stock price is a PV of its dividends paid during its life-time. The higher the stock price at option maturity, the higher the option price.

Because the stock price today includes the PV of the dividends paid before option maturity and the stock value at maturity, the higher this PV the smaller the stock price would be at maturity date. Therefore, the option price falls with increasing PV of dividends paid before maturity

Practice Problem

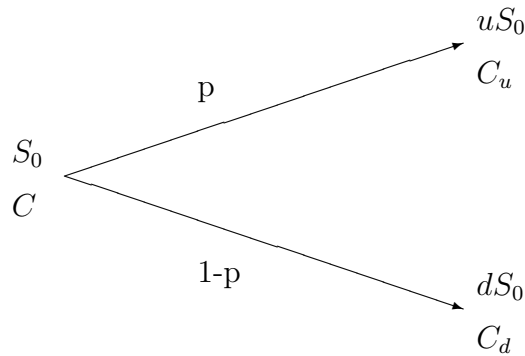
Fill out the same table (see below) for a put option

Increase in	Value of a Put Option
Stock price, S	
Strike price, X	
Stock return volatility, σ	
Time to maturity, T	
Interest rate, r_f	
Dividends	

Binomial Pricing of European Option

- We consider valuation of a European option written on a stock that does not pay dividends
- We can develop insight into option valuation by considering a simple special case when a stock can take only two possible states at option expiration and no states between today and option expiration. This model for the stock price is called a single-period binomial model
- Assume that the stock price today is S_0 and it can increase at expiration date T to $S_u = u \times S_0$ (where $u > 1$) with probability p or fall to $S_d = d \times S_0$ (where $d < 1$) with probability $1 - p$
- Consider the call option that expires at T with an exercise price X . Suppose that the current price of a T-bill is B_0 and the riskless interest rate per time period T is r_f .
- The payoff of this option is shown below

	$S_T = d S_0$	$S_T = u S_0$
Payoff	$C_d = (d S_0 - X)^+$	$C_u = (u S_0 - X)^+$



- Suppose that an investor writes the option, sells it at the market price c and then uses the proceeds to create a synthetic option comprised from H stock shares and Z dollars in cash. We require that the synthetic option must replicate the payoff of the call option. The payoff table for the synthetic option is shown below

	$S_T = dS_0$	$S_T = uS_0$
H shares of stock	$dS_0 \times H$	$uS_0 \times H$
Z T-bills	$Z B_0(1 + r_f)$	$Z B_0(1 + r_f)$
Total	C_d	C_u

- This table allows us to find a synthetic option, that is H shares and Z T-bills
- Requirement that the payoffs of the synthetic option and the option be identical implies

$$dS_0 H + Z B_0(1 + r_f) = C_d$$

$$uS_0 H + Z B_0(1 + r_f) = C_u$$

- We find the synthetic option after solving the system of two equations above

$$H = \frac{C_u - C_d}{S_0(u - d)}$$

$$Z = \frac{C_d u - C_u d}{B_0(u - d)(1 + r_f)}$$

- We assume that the market does not allow arbitrage opportunities. Therefore, the price of the call option is equal to the value of the synthetic option:

$$C = H S_0 + Z B_0 = \frac{C_u - C_d}{(u - d)} + \frac{C_d u - C_u d}{(u - d)(1 + r_f)} = \frac{C_u(1 - d + r_f) + C_d(u - 1 - r_f)}{(u - d)(1 + r_f)} \quad (6-5)$$

- The last formula holds for any strike price. In the special case when $dS_0 < X < uS_0$ we find

$$C = \frac{C_u(1 - d + r_f) + C_d(u - 1 - r_f)}{(u - d)(1 + r_f)} = \frac{(uS_0 - X)(1 - d + r_f)}{(u - d)(1 + r_f)}$$

- Question: How the formula for the call price will change if the option is put?
- The number of the stock shares in the synthetic option, H , is called the hedge ratio.
- Question: What is the sign of H for a call (put) option?
- Notice that the option price does not depend explicitly on distribution of the stock returns (probability p)!

- It follows from formula (6-5) that the price of a European option can be written as

$$C = E^Q(C_T)/(1 + r_f) = (C_uQ_u + C_dQ_d)/(1 + r_f),$$

where $Q_u = (1 - d + r_f)/[(u - d)]$ and $Q_d = (u - 1 - r_f)/[(u - d)]$. Q_u and Q_d can be interpreted as probabilities of u and d -states, respectively, under a new probability measure Q . Probability measure Q is different than a real one (so p and Q_u are not the same) and is called risk-neutral probability measure. Hence, the price of an option is simply its expected payoff under risk-neutral measure discounted by a risk-free rate.

- More generally, the price of any security can be found by calculating the expected payoff of this security under the risk neutral measure discounted by a risk-free rate

Example: the stock price today is

$$E^Q(S_T)/(1 + r_f) = \frac{S_u(1 - d + r_f) + S_d(u - 1 - r_f)}{(u - d)(1 + r_f)} = S_0 \frac{u(1 - d + r_f) + d(u - 1 - r_f)}{(u - d)(1 + r_f)} = S_0$$

It follows that the expected return on the stock is a risk-free rate under risk-neutral probabilities

Example of option pricing

Suppose that $S_0 = \$50$, $d = 0.8$, $u = 1.25$, $X = \$50$, $T = 0.25$, and the effective annual risk-free rate is $r_f = 5\%$. Find the price of the call option and the hedge ratio

The riskless rate of return for $T = 0.25$ is $(1 + r_f)^{1/4} - 1 = 1.05^{1/4} - 1 = 1.227\%$. Given that $C_d = 0$ ($S_0 d = 50 \times 0.8 = 40 < X = 50$) and $C_u = S_0 u - X = 50 \times 1.25 - 50 = \12.5 , we find

$$C = \frac{C_u(1 - d + r_f) - C_d(1 - u + r_f)}{(u - d)(1 + r_f)} =$$
$$\frac{12.5(1 - 0.8 + 0.01227) - 0(1 - 1.25 + 0.01227)}{(1.25 - 0.8)(1 + 0.01227)} = \$5.82$$

The hedge ratio:

$$H = \frac{C_u - C_d}{S_0(u - d)} = \frac{12.5 - 0}{50(1.25 - 0.8)} = 0.56$$

It follows that an investor takes a long position in the stock to replicate the payoff of the call option. Moreover, her position in the synthetic option is highly leveraged: $\frac{HS}{C} = \frac{28}{5.82} = 4.81$

The dollar position in T-bills

$$ZB_0 = \frac{C_d u - C_u d}{(u - d)(1 + r_f)} = C - H \times S_0 = 5.82 - 28 = -\$22.18$$

Question: Assume that the probability for the stock to go up is 60%. What are expected returns of the stock and the call option?

Question: What are the standard deviations of returns of the stock and the call option?

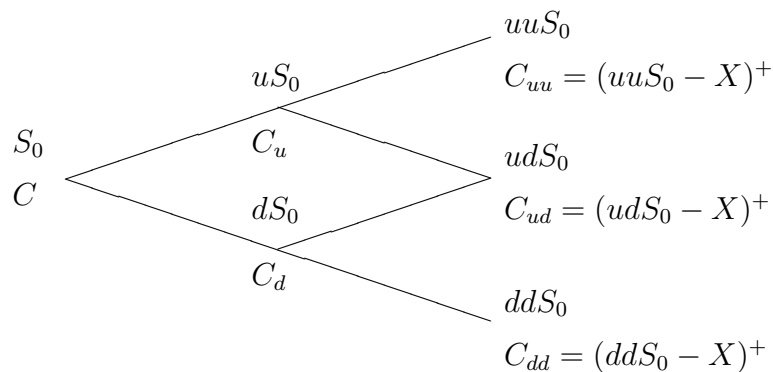
Practice Problem. In the example above, what would be your arbitrage strategy if the call option were selling for \$3?

- Notice that the interest rate usually has a negligible effect on an option price since the life of an option is relatively short

Practice Problem

Find the price of a put option in the example above assuming that it has the same expiration date and the exercise price as the call option and written on the same stock

- The two-state stock price model can be generalized to a multi-period model where the stock price can have multiple states at option expiration
- For example, consider the binomial tree for the two trading dates found by dividing the maturity date by 2
- The tree below shows the possible stock prices as well as the option prices



- The option price at each node of the tree is found by using backward induction:
 - Consider the node where $S = uS_0$ and find the option price C_u by using the the formula for C above in which C_u , C_d , uS_0 , and dS_0 are replaced by C_{uu} , C_{dd} , uuS_0 , and udS_0 , respectively. Use the interest rate per period in the formulas for C_u
 - Repeat the calculations for the node where $S = dS_0$ to find the option price C_d
 - Find the option price C for the node S_0 by using C_u and C_d found in the previous two steps

Practice Problem

Let r_f be a risk-free rate per period. Write the formula for the price of a call option, C , at the beginning of period 1 (see the diagram above) by using the risk-neutral probabilities, Q_u , Q_d , defined above.

Additional Practice Problems

1. Which of the following is true (circle one)
 - (a) Both forward and futures contracts are traded on exchanges
 - (b) Forward contracts are traded on exchanges, but futures contracts are not.
 - (c) Futures contracts are traded on exchanges, but forward contracts are not.
 - (d) Neither futures contracts nor forward contracts are traded on exchanges.

2. Suppose that on October 24, 2013, a company sells one April 2014 live-cattle futures contract. It closes out its position on January 21, 2014. The futures price (per pound) is 91.20 cents when it enters into the contract, 88.30 cents when it closes out the position and 88.80 cents at the end of December 2013. One contract is for the delivery of 40,000 pounds of cattle. What is the profit? Assume that the company has a December 31 year end.

3. A trader buys two July futures contracts on frozen orange juice. Each contract is for the delivery of 15,000 pounds. The current futures price is 160 cents per pound, the initial margin is \$6,000 per contract, and the maintenance margin is \$4,500 per contract. What price change would lead to a margin call? Under what circumstances could \$2,000 be withdrawn from the margin account?
4. A one-year long forward contract on a non-dividend-paying stock is entered into when the stock price is \$240 and the risk-free rate of interest is 10% per annum.
- a) What are the forward price and the initial value of the forward contract?
- b) Six months later, the price of the stock is \$245 and the risk-free interest rate is still 10%. What are the forward price and the value of the forward contract?
5. The spot price of oil is \$80 per barrel and the cost of storing a barrel of oil for one year is \$3, payable at the end of the year. The risk-free interest rate is 5% per annum. What is an upper bound for the one-year futures price of oil?
6. What is meant by (a) an investment asset and (b) a consumption asset. Why is the distinction between investment and consumption assets important in the determination of forward and futures prices?
7. A short forward contract that was negotiated some time ago will expire in three months and has a delivery price of \$40. The current forward price for three-month forward contract is \$42. The three month risk-free interest rate is 8%. What to the nearest cent is the fair value of the short forward contract?
8. Consider one year futures contract on silver. It costs \$1 per ounce per year to store silver with payment being made at the beginning of the year. The risk free rate is 3% and the spot price is \$20. What is a fair futures price? Suppose that the observed price is \$22. What would be an arbitrage strategy?
9. Buyers of put options anticipate the value of the underlying asset will and sellers of call options anticipate the value of the underlying asset will
- A) increase; increase
- B) decrease; increase
- C) increase; decrease

D) decrease; decrease

E) cannot tell without further information

10. HighFlyer Stock currently sells for \$48. A one-year call option with strike price of \$55 sells for \$9, and the risk free interest rate is 6%. What is the price of a one-year put with strike price of \$55?

A) \$9.00

B) \$12.89

C) \$16.00

D) \$18.72

E) \$15.60

11. The prices of European call and put options on a non-dividend-paying stock with 12 months to maturity, a strike price of \$120, and an expiration date in 12 months are \$20 and \$5, respectively. The current stock price is \$130. What is the implied risk-free rate?

12. Use the two-state put option value in this problem. $S_0 = 100$; $X = 120$; the two possibilities for S_T are \$150 and \$80. The range of P across the two states is _____; the hedge ratio is _____

A) \$0 and \$40; $-4/7$

B) \$0 and \$50; $+4/7$

C) \$0 and \$40; $+4/7$

D) \$0 and \$50; $-4/7$

E) \$20 and \$40; $+1/2$

13. A portfolio consists of 400 shares of stock and 200 calls on that stock. If the hedge ratio for the call is 0.6, what would be the dollar change in the value of the portfolio in response to a one dollar decline in the stock price?

A) +\$700

B) +\$500

C) -\$580

D) -\$520

E) none of the above