MAT 2384B-Fall 2017-Assignment #6

To be submitted in class on Friday, December 1 10:00 AM

Family Name Solutions

Given Name

Student Number

- **Please print the assignment and write solutions in the space provided.**
- You can use the back of the pages or additional pages if necessary but be sure to indicate it clearly.
- There are five questions in this assignment.
- You have to answer all the questions.
- Please write your arguments in a clear and readable manner.

Question 1. [12 points] Find the Laplace Transform of each of the following functions:

(a) $e^{-3t}\cos(\pi t)$

(b) $u(t-2)\,(2t^2 - 6t + 5)$

(c) $u(t-\pi)\,(2\cos(t) - 3\sin(3t) - 6t + 5)$

(d) $te^t \sin(2t)$

(e) $e^{3t} * (u(t-2)(t^3+1))$ (Here $*$ represents "convolution")

(f) $t^2 e^{-2t}\sin(2t)$.

(a) we know that $\mathcal{L}\{\cos(\pi t)\} = F(s) = \frac{s}{s^2+\pi^2}$, By the First shifting Theorem

$\mathcal{L}\{e^{-3t}\cos(\pi t)\} = F(s+3) = \frac{s+3}{(s+3)^2+\pi^2} = \frac{s+3}{s^2+6s+9+\pi^2}$

(b) we use the second shifting Theorem with $f(t-2) = 2t^2-6t+5 \Rightarrow$

$f(t) = 2(t+2)^2 - 6(t+2)+5 = 2t^2+2t+1$, so

$\mathcal{L}\{u(t-2)(2t^2-6t+5)\} = \mathcal{L}\{u(t-2)f(t-2)\} = e^{-2s}\mathcal{L}\{2t^2+2t+1\}$

$= e^{-2s}\left[\frac{4}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right]$

c) we use the second shifting Theorem with $f(t-\pi) = 2\cos(t) - 3\sin(3t) - 6t+5$.

so $f(t) = 2\cos(t+\pi) - 3\sin(3(t+\pi)) - 6(t+\pi)+5$

$= 2\underbrace{\cos(t+\pi)}_{-\cos(t)} - 3\underbrace{\sin(3t+3\pi)}_{-\sin(t)} - 6t - 6\pi + 5$

$= -2\cos t + 3\sin(3t) - 6t - 6\pi + 5$

$\mathcal{L}\{u(t-\pi)(2\cos t - 3\sin(3t) - 6t+5)\} = e^{-\pi s}\mathcal{L}\{-2\cos t + 3\sin(3t) - 6t - 6\pi + 5\}$

$= e^{-\pi s}\left[-2\frac{s}{s^2+1} + \frac{9}{s^2+9} - \frac{6}{s^2} - \frac{6\pi-5}{s}\right]$

(d) This can be done in 2 different ways:

Method 1 Multiplication by t^n Rule:

$F(s) = \mathcal{L}\{e^t \sin(2t)\} = \frac{2}{(s-1)^2+4}$ (by the first shifting Theorem)

$F(s) = \frac{2}{s^2-2s+5}$. By the multiplication by t Rule:

$\mathcal{L}\{t e^t \sin(2t)\} = (-1)^1 \frac{d}{ds} F(s) = -\frac{d}{ds}\left(\frac{2}{s^2-2s+5}\right) = -\frac{0-2(2s-2)}{(s^2-2s+5)^2}$

So $\mathcal{L}\{t e^t \sin(2t)\} = \frac{4(s-1)}{(s^2-2s+5)^2}$

Method 2 First shifting Theorem:

$\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4} \Rightarrow \mathcal{L}\{t\sin(2t)\} = (-1)^1 \frac{d}{ds}\left(\frac{2}{s^2+4}\right)$ (by the multiplication by t^n Rule) $= -\frac{0-4s}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2} = F(s)$

By the first shifting Theorem:

$\mathcal{L}\{e^t t \sin(2t)\} = F(s-1) = \frac{4(s-1)}{((s-1)^2+4)^2} = \frac{4(s-1)}{(s^2-2s+5)^2}$

(e) We know that $F(s) = \mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$ and $G(s) = \mathcal{L}\{u(t-2)(t^3+1)\}$

$= e^{-2s}\mathcal{L}\{(t+2)^3+1\} = e^{-2s}\mathcal{L}\{t^3+6t^2+12t+9\} = e^{-2s}\left(\frac{6}{s^4}+\frac{12}{s^3}+\frac{12}{s^2}+\frac{9}{s}\right)$

So $\mathcal{L}\{e^{3t} * u(t-2)(t^3+1)\} = F(s).G(s) = \frac{e^{-2s}}{s-3}\left(\frac{6}{s^4}+\frac{12}{s^3}+\frac{12}{s^2}+\frac{9}{s}\right)$

(f) Here is one way using multiplication by t^n Rule:

$\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4} \Rightarrow \mathcal{L}\{e^{-2t}\sin(2t)\} = \frac{2}{(s+2)^2+4}$ (by the first shifting Theorem) $= \frac{2}{s^2+4s+8}$

$\mathcal{L}\{t^2 e^{-2t}\sin(2t)\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4s+8}\right) = \frac{d^2}{ds^2}\left(\frac{2}{s^2+4s+8}\right)$

$\frac{d}{ds}\left(\frac{2}{s^2+4s+8}\right) = \frac{d}{ds}\left[2(s^2+4s+8)^{-1}\right] = -2(s^2+4s+8)^{-2}.(2s+4) = -\frac{4(s+2)}{(s^2+4s+8)^2}$

Work space

$$\Rightarrow \frac{d^2}{ds^2}\left(\frac{2}{s^2+4s+8}\right) = \frac{d}{ds}\left[\frac{-4(s+2)}{(s^2+4s+8)^2}\right] = \frac{-4(s^2+4s+8)^2+8(s+2)(s^2+4s+8)(2s+4)}{(s^2+4s+8)^4}$$

$$= \frac{-4(s^2+4s+8)+16(s+2)^2}{(s^2+4s+8)^3} = \frac{4[-s^2-4s-8+4s^2+16s+16]}{(s^2+4s+8)^3} = \frac{4(3s^2+12s+8)}{(s^2+4s+8)^3}$$

Question 2. **[8 points]** Find the Inverse Laplace Transform of the following functions:

(a) $\frac{4}{s^2+2s+5}$

(b) $e^{-3s}\,\frac{4s+5}{s^2+5s+6}$

(c) $e^{-2s}\left(\frac{8}{s^3}+\frac{4}{s^2}\right)$

(d) $\frac{8s+8}{(s^2+2s+5)^2}$

(a) $\mathcal{L}^{-1}\left\{\frac{4}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{(s+1)^2+4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} = \boxed{2e^{-t}\sin(2t)}$

(b) Note that $\frac{4s+5}{s^2+5s+6} = \frac{4s+5}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} = \frac{(A+B)s+3A+2B}{(s+2)(s+3)}$

$\Rightarrow \begin{cases} A+B=4 & ① \\ 3A+2B=5 & ② \end{cases}$ $\quad -3①+② \Rightarrow -B=-7 \Rightarrow B=7$

$① \Rightarrow A=-3$

$\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{4s+5}{s^2+5s+6}\right)\right\} = \mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{-3}{s+2}+\frac{7}{s+3}\right)\right\}$

$= -3\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+2}\right\} + 7\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+3}\right\}$

$= -3u(t-3)\,e^{-2(t-3)} + 7u(t-3)\,e^{-3(t-3)} = \boxed{u(t-3)\left[-3e^{-2t+6}+7e^{-3t+9}\right]}$

(c) $\mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{8}{s^3}+\frac{4}{s^2}\right)\right\} = 8\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} + 4\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$

$= \frac{8}{2!}u(t-2)(t-2)^2 + 4u(t-2)(t-2) = \boxed{4u(t-2)(t^2-3t+2)}$

(d) Consider the function $F(s) = \frac{1}{s^2+2s+5}$. We know that $\frac{d}{ds}F(s) = \frac{-2s-2}{(s^2+2s+5)^2}$

$\frac{dF}{ds} = -2\frac{s+1}{(s^2+2s+5)^2}$. So $\frac{8s+8}{(s^2+2s+5)^2} = 8\frac{s+1}{(s^2+2s+5)^2} = -4\frac{dF}{ds}$

$\mathcal{L}^{-1}\left\{\frac{8s+8}{(s^2+2s+5)^2}\right\} = 4\mathcal{L}^{-1}\left\{(-1)^1\frac{dF(s)}{ds}\right\} = 4t\,\mathcal{L}^{-1}\{F(s)\} = 4t\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$

$= 4t\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = 4t\,\frac{1}{2}e^{-t}\sin(2t) = \boxed{2t\,e^{-t}\sin(2t)}$

Question 3. [12 points] Use the **Laplace Transform** to solve the following initial value problems:

(a) $y'' - y' - 6y = \begin{cases} 0 & \text{for } 0 < t < \pi \\ \cos(t) & \text{for } t \geq \pi \end{cases}$, $y(0) = 3$, $y'(0) = 4$

(b) $y'' + 9y = \delta(t-2)$, $y(0) = 0$, $y'(0) = 9$

(c) $y'' + y' - 2y = -4e^{t}$, $y(0) = 4$, $y'(0) = -9$.

(a) Note that the right side of the ODE is $\cos t \cdot u(t-\pi)$ so the ODE is

$y'' - y' - 6y = u(t-\pi)\cos(t)$. Let $Y = \mathcal{L}\{y\}$ and apply $\mathcal{L}$ to the ODE:

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{u(t-\pi)\cos t\} = e^{-\pi s}\mathcal{L}\{\cos(t+\pi)\}$$

(By the second shifting theorem) $= e^{-\pi s}\mathcal{L}\{\cos t\}$ $\quad (\cos(t+\pi) = -\cos t) \Rightarrow$

$$s^2 Y - s y(0) - y'(0) - [sY - y(0)] - 6Y = -e^{-\pi s}\frac{s}{s^2+1} \Rightarrow$$

$$(s^2 - s - 6)Y = 3s + 4 - 3 - e^{-\pi s}\frac{s}{s^2+1} = 3s + 1 - e^{-\pi s}\frac{s}{s^2+1} \Rightarrow$$

$$Y = \frac{3s+1}{s^2-s-6} - e^{-\pi s}\frac{s}{(s^2+1)(s^2-s-6)} = \frac{3s+1}{(s-3)(s+2)} - e^{-\pi s}\frac{s}{(s^2+1)(s-3)(s+2)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3s+1}{(s-3)(s+2)}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{(s^2+1)(s-3)(s+2)}\right\}$$

- $\dfrac{3s+1}{(s-3)(s+2)} = \dfrac{A}{s-3} + \dfrac{B}{s+2} = \dfrac{(A+B)s + 2A - 3B}{(s-3)(s+2)} \Rightarrow \begin{cases} A + B = 3 & (1) \\ 2A - 3B = 1 & (2) \end{cases}$

$-2(1) + (2) \Rightarrow -5B = -5 \Rightarrow B = 1$; $(1) \Rightarrow A = 2$.

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-3)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-3} + \frac{1}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = 2e^{3t} + e^{-2t}$$

- $\dfrac{s}{(s^2+1)(s-3)(s+2)} = \dfrac{As+B}{s^2+1} + \dfrac{C}{s-3} + \dfrac{D}{s+2} = \dfrac{(As+B)(s^2-s-6) + C(s^2+1)(s+2) + D(s^2+1)(s-3)}{(s^2+1)(s-3)(s+2)}$

$$= \frac{As^3 - As^2 - 6As + Bs^2 - Bs - 6B + Cs^3 + 2Cs^2 + Cs + 2C + Ds^3 - 3Ds^2 + Ds - 3D}{(s^2+1)(s-3)(s+2)} \Rightarrow$$

$$\begin{cases} A + C + D = 0 \\ -A + B + 2C - 3D = 0 \\ -6A - B + C + D = 1 \\ -6B + 2C - 3D = 0 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ -1 & 1 & 2 & -3 & 0 \\ -6 & -1 & 1 & 1 & 1 \\ 0 & -6 & 2 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & -1 & 7 & 7 & 1 \\ 0 & -6 & 2 & -3 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 10 & 5 & 1 \\ 0 & 0 & 20 & -15 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/10 \\ 0 & 0 & 0 & -25 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/10 \\ 0 & 0 & 0 & 1 & 2/25 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -2/25 \\ 0 & 1 & 3 & 0 & 4/25 \\ 0 & 0 & 1 & 0 & 3/50 \\ 0 & 0 & 0 & 1 & 2/25 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -7/50 \\ 0 & 1 & 0 & 0 & -1/50 \\ 0 & 0 & 1 & 0 & 3/50 \\ 0 & 0 & 0 & 1 & 2/25 \end{array}\right] \quad A = -7/50, \; B = -\frac{1}{50}, \quad C = \frac{3}{50}, \quad D = \frac{2}{25}$$

$$\frac{s}{(s^2+1)(s-3)(s+2)} = \frac{-7/50\, s - 1/50}{s^2+1} + \frac{3/50}{s-3} + \frac{2/25}{s+2} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}\, s}{(s^2+1)(s-3)(s+2)}\right\} = -\frac{7}{50}\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2+1}\right\} - \frac{1}{50}\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} + \frac{3}{50}\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s-3}\right\} +$$

$$\frac{2}{25}\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s+2}\right\} = -\frac{7}{50}u(t-\pi)\cos(t-\pi) - \frac{1}{50}u(t-\pi)\sin(t-\pi) +$$

$$\frac{3}{50}u(t-\pi)\,e^{3(t-\pi)} + \frac{2}{25}u(t-\pi)\,e^{-2(t-\pi)}$$

$$= -\frac{1}{50}u(t-\pi)\left[7\cos(t-\pi) + \sin(t-\pi) - 3e^{3(t-\pi)} - 4e^{-2(t-\pi)}\right]$$

$$= -\frac{1}{50}u(t-\pi)\left(-7\cos(t) - \sin(t) - 3e^{3(t-\pi)} - 4e^{-2(t-\pi)}\right)$$

We conclude that the solution to the IVP is

$$y = \mathcal{L}^{-1}\left\{\frac{3s+1}{(s-3)(s+2)}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{(s^2+1)(s-3)(s+2)}\right\} =$$

$$\boxed{2e^{3t} + e^{-2t} + \frac{1}{50}u(t-\pi)\left[-7\cos(t) - \sin(t) - 3e^{3(t-\pi)} - 4e^{-2(t-\pi)}\right]}$$

(b) let $Y = \mathcal{L}\{y\}$. Apply the Laplace Transform to both sides of the ODE: $\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2)\} \Rightarrow$

$$s^2Y - sy(0) - y'(0) + 9Y = e^{-2s} \Rightarrow (s^2+9)Y = 9 + e^{-2s} \Rightarrow Y = \frac{9}{s^2+9} + \frac{e^{-2s}}{s^2+9}.$$

So $y = \mathcal{L}^{-1}\left\{\frac{9}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+9}\right\} = 3\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3e^{-2s}}{s^2+9}\right\}$

$$\boxed{y = 3\sin(3t) + \frac{1}{3}u(t-2)\sin 3(t-2)}$$

(c) Let $Y = \mathcal{L}\{y\}$ and apply the Laplace transform to the ODE:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = -4\mathcal{L}\{e^{t}\} \Rightarrow$$

$$s^2Y - sy(0) - y'(0) + sY - y(0) - 2Y = -\frac{4}{s-1} \Rightarrow (s^2+s-2)Y = 4s - 5 - \frac{4}{s-1} \Rightarrow$$

$$Y = \frac{4s-5}{s^2+s-2} - \frac{4}{(s-1)(s^2+s-2)} = \frac{4s-5}{(s-1)(s+2)} - \frac{4}{(s-1)^2(s+2)} \Rightarrow$$

$$y = \mathcal{L}^{-1}\left\{\frac{4s-5}{(s-1)(s+2)}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2(s+2)}\right\}$$

- $\frac{4s-5}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} = \frac{(A+B)s + 2A - B}{(s-1)(s+2)} \Rightarrow \begin{cases} A+B=4 & (1) \\ 2A-B=-5 & (2) \end{cases}$

$(1)+(2) \Rightarrow 3A = -1 \Rightarrow A = -1/3$; $(1) \Rightarrow B = 4 + \frac{1}{3} = \frac{13}{3}$

$$\frac{4s-5}{(s-1)(s+2)} = \frac{-1/3}{s-1} + \frac{13/3}{s+2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{4s-5}{(s-1)(s+2)}\right\} = -\frac{1}{3}e^{t} + \frac{13}{3}e^{-2t}$$

- $\frac{1}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2} = \frac{A(s^2+s-2) + B(s+2) + C(s^2-2s+1)}{(s-1)^2(s+2)}$

$$\Rightarrow \begin{cases} A + C = 0 \\ A + B - 2C = 0 \\ -2A + 2B + C = 1 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ -2 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 9 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1/9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/9 \end{array}\right] \Rightarrow \begin{array}{l} A = -\frac{1}{9} \\ B = \frac{1}{3} \\ C = 1/9 \end{array}$$

$$\frac{1}{(s-1)^2(s+2)} = \frac{-1/9}{s-1} + \frac{1/3}{(s-1)^2} + \frac{1/9}{s+2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2(s+2)}\right\} =$$

$$-\frac{1}{9}e^{t} + \frac{1}{3}e^{t}t + \frac{1}{9}e^{-2t}$$

The solution to the IVP is

$$y = \mathcal{L}^{-1}\left\{\frac{4s-5}{(s-1)(s+2)}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2(s+2)}\right\}$$

$$= -\frac{1}{3}e^{t} + \frac{13}{3}e^{-2t} + \frac{4}{9}e^{t} - \frac{4}{3}te^{t} - \frac{4}{9}e^{-2t}$$

$$\boxed{y = \frac{1}{9}e^{t} + \frac{35}{9}e^{-2t} - \frac{4}{3}te^{t}}$$

Question 4. [5 points] Use **the Improved Euler's Method** with $h = 0.2$ to approximate to 4 decimal places the solution of the first-order ODE

$$y' = 3x + 2y, \quad y(0) = 0$$

on the interval $[0, 1]$. Then find the exact solution and make a table to compare your approximations with the exact values .

$f(x,y) = 3x + 2y \;;\; x_0 = 0 \;;\; y_0 = 0 \;;\; h = 0.2$

$x_1 = x_0 + h = 0.2$

$y_1^P = y_0 + h\,f(x_0, y_0) = 0 + 0.2[3(0) + 2(0)] = 0$

$y_1^c = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^P)] = 0 + \frac{0.2}{2}[3(0) + 2(0) + 3(0.2) + 2(0)] = 0.06$

$\boxed{(x_1, y_1) = (0.2,\ 0.06)}$

$x_2 = x_1 + h = 0.4$

$y_2^P = y_1^c + h\,f(x_1, y_1^c) = 0.06 + 0.2[3(0.2) + 2(0.06)] = 0.2640$

$y_2^c = y_1^c + \frac{h}{2}[f(x_1, y_1^c) + f(x_2, y_2^P)] = 0.06 + \frac{0.2}{2}[3(0.2) + 2(0.06) + 3(0.4) + 2(0.264)] = 0.2928$

$\boxed{(x_2, y_2) = (0.4,\ 0.2928)}$

$x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

$y_3^P = y_2^c + h\,f(x_2, y_2^c) = 0.2928 + 0.2[3(0.4) + 2(0.2928)] = 0.6499$

$y_3^c = y_2^c + \frac{h}{2}[f(x_2, y_2^c) + f(x_3, y_3^P)] = 0.2808 + \frac{0.2}{2}[3(0.4) + 2(0.2808) + 3(0.4 + 2(0.6499)] = 0.7813$

$\boxed{(x_3, y_3) = (0.6,\ 0.7813)}$

$x_4 = x_3 + h = 0.6 + 0.2 = 0.8$

$y_4^P = y_3^c + h\,f(x_3, y_3^c) = 0.7813 + 0.2[3(0.6) + 2(0.7813)] = 1.4539$

$$y_4^c = y_3^c + \frac{h}{2}\left[f(x_3,y_3^c) + f(x_4,y_4^p)\right] = 0.7813 + \frac{0.2}{2}\left[3(0.6)+2(0.7813)+3(0.8)+2(1.4539)\right] = 1.6484$$

$$\boxed{(x_4, y_4) = (0.8, 1.6484)}$$

$$x_5 = x_4 + h = 0.8 + 0.2 = 1$$

$$y_5^p = y_4^c + h f(x_4, y_4^c) = 1.6484 + 0.2\left[3(0.8)+2(1.6484)\right] = 2.7877$$

$$y_5^c = y_4^c + \frac{h}{2}\left[f(x_4,y_4^c) + f(x_5,y_5^p)\right] = 1.6484 + \frac{0.2}{2}\left[3(0.8)+2(1.6484)+3(1)+2(2.7877)\right] = 3.0756$$

$$\boxed{(x_5, y_5) = (1, 3.0756)}$$

Now we proceed to solve the ODE. Note that the ODE can be rewritten under the form: $y' - 2y = 3x$: This is a linear first-order ODE with $f(x) = -2$ and $r(x) = 3x$. The general solution is given by:

$$y = \frac{\int e^{\int f(x)dx} r(x)\,dx + C}{e^{\int f(x)dx}} = \frac{\int e^{\int -2dx}(3x)dx + C}{e^{\int -2dx}} = \frac{3\int x e^{-2x} dx + C}{e^{-2x}}$$

We compute $\int x e^{-2x} dx$ by parts: $u = x$, $v' = e^{-2x} \Rightarrow u' = 1, v = -\frac{1}{2}e^{-2x}$

$$\int x e^{-2x} dx = -\frac{x}{2}e^{-2x} - \int\left(-\frac{1}{2}\right)e^{-2x}dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}. \text{ So}$$

$$y = \frac{-\frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x} + C}{e^{-2x}} = -\frac{3}{2}x - \frac{3}{4} + Ce^{2x}.$$

$y(0) = 0 \Rightarrow -\frac{3}{4} + C = 0 \Rightarrow C = 3/4$ and the solution to the IVP is

$$\boxed{y = -\frac{3}{2}x - \frac{3}{4} + \frac{3}{4}e^{2x}}.$$

The following Table compares True values of the solution (using the function $y = -\frac{3}{2}x - \frac{3}{4} + \frac{3}{4}e^{2x}$) with estimated values using

Work space

Improved Euler Method:

x	Exact value using $y = -\frac{3}{2}x - \frac{3}{4} + \frac{3}{4}e^{2x}$	Estimated value using Improved Euler Method
0	0	0
0.2	0.0689	0.0600
0.4	0.3192	0.2928
0.6	0.8401	0.7813
0.8	1.7648	1.6484
1.0	3.2918	3.0756

Question 5. **[6 points]** Use the **Runge-Kutta Method of order 4** with $h = 0.25$ to approximate (to 5 decimal places) the solution of $y' = 2xe^{-y}$, $y(0) = 1$ on $0 \le x \le 1$. Compare your approximations with the true values by calculating the errors.

$f(x,y) = 2xe^{-y}$; $x_0 = 0$, $y_0 = 1$; $h = 0.25$

$x_1 = x_0 + h = 0.25$

$k_1 = h f(x_0, y_0) = 0.25\left(2(0)e^{-1}\right) = 0$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.25 f(0.125, 1) = 0.25(2)(0.125)e^{-1} = 0.02299$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.25 f(0.125, 1.01150) = 0.25(2)(0.125)e^{-1.01150} = 0.02273$

$k_4 = h f(x_0 + h, y_0 + k_3) = 0.25 f(0.25, 1.02273) = 0.25(2)(0.25)e^{-1.02273} = 0.04495$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}(0 + 2(0.02299) + 2(0.02273) + 0.04495) = 1.02273$

$\boxed{(x_1, y_1) = (0.25, 1.02273)}$

$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

$k_1 = h f(x_1, y_1) = 0.25 f(0.25, 1.02273) = 0.25(2)(0.25)e^{-1.02273} = 0.04495$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.25 f(0.375, 1.04521) = 0.25(2)(0.375)e^{-1.04521} = 0.06593$

$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.25 f(0.375, 1.05570) = 0.25(2)(0.375)e^{-1.05570} = 0.06524$

$k_4 = h f(x_1 + h, y_1 + k_3) = 0.25 f(0.5, 1.08797) = 0.25(2)(0.5)e^{-1.08797} = 0.08422$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.02273 + \frac{1}{6}(0.04495 + 2(0.06593) + 2(0.06524) + 0.08422)$

$= 1.08798$

$\boxed{(x_2, y_2) = (0.5, 1.08798)}$

$x_3 = x_2 + h = 0.5 + 0.25 = 0.75$

$k_1 = h f(x_2, y_2) = 0.25 f(0.5, 1.08798) = 0.25(2)(0.5)e^{-1.08798} = 0.08422$

$k_2 = h f(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}) = 0.25 f(0.625, 1.13010) = 0.25(2)(0.625)e^{-1.13010} = 0.10094$

$k_3 = h f(x_2 + h/2, y_2 + \frac{k_2}{2}) = 0.25 f(0.625, 1.13845) = 0.25(2)(0.625)e^{-1.13845} = 0.10010$

$k_4 = hf(x_2+h, y_2+k_3) = hf(0.75, 1.18808) = 0.25(2)(0.75)e^{-1.18808} = 0.11430$

$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.08798 + \frac{1}{6}(0.08422 + 2(0.10094) + 2(0.10010) + 0.11430)$

$= 1.18808$

$\boxed{(x_3, y_3) = (0.75, 1.18808)}$

$x_4 = x_3 + h = 0.75 + 0.25 = 1$

$k_1 = hf(x_3, y_3) = 0.25(2)(0.75)e^{-1.18808} = 0.11430$

$k_2 = hf(x_3+\frac{h}{2}, y_3+\frac{k_1}{2}) = 0.25f(0.875, 1.24524) = 0.25(2)(0.875)e^{-1.24524} = 0.12594$

$k_3 = hf(x_3+\frac{h}{2}, y_3+\frac{k_2}{2}) = 0.25f(0.875, 1.25106) = 0.25(2)(0.875)e^{-1.25106} = 0.12521$

$k_4 = hf(x_3+h, y_3+k_3) = 0.25f(1, 1.31330) = 0.25(2)(1)e^{-1.31330} = 0.13447$

$y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.18808 + \frac{1}{6}(0.11430 + 2(0.12594) + 2(0.12521) + 0.13447)$

$= 1.31327$

$\boxed{(x_4, y_4) = (1, 1.31327)}$

To find the exact values (hence the errors), we need to solve the IVP completely: $y' = 2xe^{-y} \Rightarrow \frac{dy}{dx} = 2xe^{-y} \Rightarrow e^{y}dy = 2x\,dx \Rightarrow$
$\int e^{y}dy = \int 2x\,dx \Rightarrow e^{y} = x^2 + C$

$y(0) = 1 \Rightarrow e = 0^2 + C \Rightarrow C = e$. The solution to the IVP is

$\boxed{y = \ln(x^2 + e)}$

The following Table gives Exact values of the solution, the approximate values using Runge-Kutta, and the error (which is Exact value - approximated value)

x	Exact value of the solution	Estimated value	Error
0	1	1	0
0.25	1.022732121	1.02273	0.000002121
0.5	1.087983276	1.08798	0.000003276
0.75	1.188081756	1.18808	0.000001756
1	1.313261687	1.31327	-0.000008313