Linear Algebra Review

Complex Numbers

$\sqrt{-1} = i \rightarrow i^2 = -1$

Standard Form

$z = a + bi$

real part of z — imaginary part of z

$Re(z) = a$ $\quad Im(z) = b$ * doesn't include i

Properties:

① Addition

$(a+bi)+(c+di) = (a+c)+(b+d)i$

$\hookrightarrow$ closed under addition

② α, a scalar

$\alpha(a+bi) = (\alpha a) + (\alpha b)i$

③ Multiplication

$(a+bi)(c+di)$

$= ac + cbi + adi + bdi^2$

$= ac + bd(-1) + adi + bci$

$= (ac-bd) + (ad+bc)i$

Complex Conjugate

- if $z = a + bi$, then the conjugate is: $\bar{z} = a - bi$

* s.t. $|z|^2 = (z)(\bar{z}) = a^2 + b^2$, (a^2+b^2) is a real number

Norm of a Complex Number

$|z| = \sqrt{a^2+b^2}$

Rationalizing the Denominator

Ex: $(3+2i)/(2+5i)$ * we want form $a+bi$

* Multiply by the conjugate of the denominator

$$= \frac{3+2i}{2+5i} \cdot \frac{2-5i}{2-5i}$$

$$= \frac{6+4i-15i-10i^2}{2^2+5^2}$$ * Remember $i^2 = -1$

$$= \frac{6+10+4i-15i}{29}$$

$$= \frac{16-11i}{29} = \frac{16}{29} - \frac{11}{29}i$$

Vectors

Operations:

① Addition

$\vec{v} + \vec{w} = \binom{2}{3} + \binom{3}{1} = \binom{5}{4}$

② Subtraction

$\vec{v} - \vec{w} = \binom{2}{3} - \binom{3}{1} = \binom{-1}{2}$

③ Scalar Multiplication

$\alpha \vec{v} = 3\vec{v} = 3\binom{2}{3} = \binom{6}{9}$

Norm

- length of a vector

$|\vec{v}| = \sqrt{(v_1)^2 + (v_2)^2}$

Normalization

- Making a vector with a length of one

$|\vec{v}| = \sqrt{(v_1)^2 + (v_2)^2}$

$\vec{e}_{\vec{v}} = \frac{\vec{v}}{|\vec{v}|}$ or $\frac{1}{|\vec{v}|}\vec{v}$

Dot Product

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

* where θ is the angle between $\vec{u}$ & $\vec{v}$

Orthogonal (Perpindicular)

- vectors are orthogonal when $\vec{u} \cdot \vec{v} = 0$

The Matrix

eg: $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix}_{2\times3}$

- the dimensions of a matrix is expressed as m rows by n columns $(A_{m\times n})$
- Matrices have enteries at a_{ij}

Ex: $a_{11} = 1$ or $a_{23} = 6$

Matrix Operations & Properties

① Scalar Multiplication

$\alpha A = 2\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 10 & 12 \end{bmatrix}$

② Addition

* A matrix is closed under addition provided the given enteries are the same dimension

Ex: $E = \begin{bmatrix} 7 & 6 & -6 \\ 1 & 3 & -2 \end{bmatrix}$

$A + E = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 6 & -6 \\ 1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 8 & -7 \\ 1 & 8 & 4 \end{bmatrix}$

* $A + E = E + A$

Transpose

- Swap your rows & columns

Ex: $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix}_{2\times3}$, then $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ -1 & 6 \end{bmatrix}$

Multiplication

- In order to multiply two matrices the internal dimensions must match
- The external dimension's give the dimension of the solution

Ex: $A_{2\times3}$, $B_{2\times2}$, $C_{3\times1}$, $D_{2\times2}$

$$B_{2\times2} \cdot D_{2\times2} = [\,?\,]_{2\times2}$$

$$A_{2\times3} \cdot C_{3\times1} = [\,?\,]_{2\times1}$$

$$C_{3\times1} \cdot A_{2\times3} = \text{dimension mismatch}$$

Method to Multiply

- suppose A & B are matrices which can be multiplied,

s.t. $A \cdot B = C$, then $C_{ij} = \vec{a}_i \cdot \vec{b}_j$

row ↑ ↑ column

Ex: $B \cdot D = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ -1 & 1 \end{pmatrix}$

$(bd)_{11} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2+1 = 3$

$(bd)_{12} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = 5-1 = 4$

$(bd)_{21} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0-2 = -2$

$(bd)_{22} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = 0+2 = 2$

$\therefore BD = \begin{pmatrix} 3 & 4 \\ -2 & 2 \end{pmatrix}$

Powers

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$A^2 = AA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

Identity Matrix

Ex: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Diagonal Matrix

$\begin{pmatrix} d_1 & & \varnothing \\ & d_2 & \\ \varnothing & & d_3 \end{pmatrix}$

Upper triangle

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

Lower triangle

$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$

Both

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{pmatrix}$

Properties

- where A, B, C are nice matrices

① $A + B = B + A$

② $A + B + C = (A+B) + C = A + (B+C)$

③ $A + [0] = A$

④ $A + [-A] = [0]$

⑤ $AB \neq BA$

⑥ $ABC = (AB)C = A(BC)$

⑦ $AI = IA = A$

⑧ $(A + B + C)^T = A^T + B^T + C^T$

⑨ $(ABC)^T = C^T B^T A^T$

Gaussian Elimination

- We solve a system of equations by Gaussian elimination by performing elementary row operations

Ex: solve the system $\begin{cases} x + y + z = 3 \\ y - 2z = -1 \\ 2x + 3y - z = 4 \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & -1 \\ 2 & 3 & -1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -3 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Now draw out the system

$$\begin{cases} x + y + z = 3 \\ y - 2z = -1 \\ z = 1 \end{cases} \qquad \begin{aligned} y &= 2z - 1 \\ &= 2(1) - 1 \\ &= 1 \end{aligned} \qquad \begin{aligned} x &= 3 - y - z \\ &= 3 - 1 - 1 \\ &= 1 \end{aligned}$$

$\therefore$ the solution is $\begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$ or $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

RRRF

Rules:

① Every leading value in each row is 1

② Each leading 1 is to the right of all leading 1's in rows above

③ Each leading 1 is the only non-zero entery in its row

④ All rows of zero are at the bottom

Solutions

$\left.\begin{array}{l}\text{unique solution} \\ \text{multiple solution}\end{array}\right\}$ consistent

no solution $\}$ inconsistent

Examples

① $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right)$ $\begin{cases} x = 29 \\ y = 16 \\ z = 3 \end{cases}$ - Unique - Consistent

② $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{cases} x = 29 \\ \underline{y = 16} \\ z = t \end{cases}$ - multiple - consistent

③ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 3 \end{array}\right)$ $\begin{cases} x = 29 \\ y = 16 \\ 0 = 3 \end{cases}$ ← inconsistent

Inverse of a 2x2 (Adjoint Method)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Terminology: $A^{-1} = \frac{1}{\det A}\,\text{Adj}\,A$

determinate of A ↗ ↖ Adjoint of A

Ex: $A = \begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix}$ $A^{-1} = \frac{1}{(2\times 8)-(4\times -3)}\begin{pmatrix} 8 & -4 \\ 3 & 2 \end{pmatrix}$

$= \frac{1}{28}\begin{pmatrix} 8 & -4 \\ 3 & 2 \end{pmatrix}$

* Always check your Answer

Check $A^{-1}A = I$

$= \frac{1}{28}\begin{pmatrix} 8 & -4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix}$

$= \frac{1}{28}\begin{pmatrix} 28 & 0 \\ 0 & 28 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ ✓

Inverse of a 3x3 (or bigger)

- sometimes called the RREF method
- Property: $(A \,|\, I) \sim (I \,|\, A^{-1})$

Ex: Find the inverse of C

$C = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 2 & 0 \end{pmatrix}$

$\left(\begin{array}{ccc|ccc} 1/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 1/3 & 0 & 1 & 0 \end{array}\right) \begin{matrix} 2R_1 \\ 1/2\,R_3 \\ R_2 \end{matrix}$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/3 & 0 & 1 & -1/2 \end{array}\right) R_3 - R_2 \sim \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} & \underbrace{\begin{matrix} 2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 3 & -3/2 \end{matrix}}_{C^{-1}} \end{array}\right) 3R_3$

Check: $C^{-1}C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 3 & -3/2 \end{pmatrix}\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Determinants

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Note:

① $A_{1\times1}$, $\det A = a_{11}$

② $A_{n\times n}$, where A is upper/lower triangular, then $\det A$ is just the product of the main diagonal

Ex: ① $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \to \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = (1)(3) = 3$

② $\begin{pmatrix} 1 & & & & \emptyset \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ \emptyset & & & & 5 \end{pmatrix} = 1\cdot2\cdot3\cdot4\cdot5 = 120$

Co-Factor Expansion

Rules:

① You can expand along any row/col

* a row/col with lots of zeros is good

② You must follow the correct +/- pattern

2×2 $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$

Ex: solve for $\det C$

$C = \begin{pmatrix} 1 & 5 & -1 \\ 0 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix}$

$\det C = \begin{vmatrix} 1^{+} & 5 & -1 \\ 0^{-} & 2 & 1 \\ 1^{+} & 3 & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - 0\begin{vmatrix} 5 & -1 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 5 & -1 \\ 2 & 1 \end{vmatrix}$

$= 1((2\times4)-(3\times1)) - 0((5\times4)-(-1\times3)) + 1((5\times1)-(-1\times2)) = 12$

Properties

① $\det(M^{-1}) = \frac{1}{\det M}$

② $\det(M^{T}) = \det M$

③ $\det(MM^{-1}) = \det(I) = 1$

④ $\det A \det B = \det(AB)$

⑤ $\det(A^{n}) = (\det A)^{n}$

⑥ If $A_{n\times n}$, & α is a constant, then: $\det(\alpha A) = \alpha^{n} \det A$

Ex: Let $|A| = 2$, $|B| = 5$, $|C| = -1$, $|D| = 7$

① Solve the $\det(A^3 B^T C D^{-1})$

$= \det(AAAB^TCD^{-1})$

$= \det(A)\det(A)\det(A)\det(B)\det(C)\left(\frac{1}{\det D}\right)$

$= (2)(2)(2)(5)(-1)\left(\frac{1}{7}\right) = -40/7$

② Let A & B be $A_{4\times4}$, $B_{4\times4}$

Solve $|3AB^T|$

$= 3^4 \det A \det B$

$= 81 \det A \det B$

$= 81(2)(5) = 810$

A Clever Co-factor Expansion

- Expand / attack a row / col of almost all zeros

Ex: $\begin{vmatrix} 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 3 \\ 2 & 1 & 3 & 6 \\ 1 & 0 & 7 & 6 \end{vmatrix} = -0 + 0 - 0 + 3 \begin{vmatrix} 2 & 0 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 7 \end{vmatrix} = 3\left[-0 + 1 \begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} - 0\right]$

$= 3\left[1(2\times 7 - 4\times 1)\right] = 30$

Rules of Row / Col Operations

Recall from elementary row operations:

1. Interchange 2 rows
2. Multiply a row by a scalar
3. Add one row to another

For determinants we must add these clauses

IF ① then: you must multiply the det by (-1)

If ② then: you must divide the determinate by that scalar

If ③ then: the operation must be of the form ($R_i' = R_i + \alpha R_j$)

Ex:

Valid	Invalid
$R_1' = R_1 - R_2$	$R_1' = R_2 - R_1$ (not allowed)
$R_3' = R_3 - 4R_1$	$R_3' = 4R_3 - 4R_1$ (not allowed)

Ex: $\begin{vmatrix} 2 & 1 & 4 & 4 \\ 2 & 1 & 4 & 3 \\ 2 & 1 & 3 & 6 \\ 1 & 6 & 7 & 5 \end{vmatrix} \sim \begin{vmatrix} 0 & 0 & 0 & 1 \\ 2 & 1 & 4 & 3 \\ 0 & 0 & -1 & 3 \\ 1 & 6 & 7 & 5 \end{vmatrix}$ $\quad R_1' = R_1 - R_2$, $R_3' = R_3 - R_2$

$= -1 \begin{vmatrix} 2 & 1 & 4 \\ 0 & 0 & -1 \\ 1 & 6 & 7 \end{vmatrix} = (-1)(1) \begin{vmatrix} 2 & 1 \\ 1 & 6 \end{vmatrix} = (-1)(1)[12 - 1] = -11$

George Costanza Rule

$A\vec{x} = \vec{b}$

$\vec{x} = A^{-1}\vec{b}$

This gives the unique solution, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix}$

$A = \begin{pmatrix} 5 & -3 \\ 7 & 4 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \vec{b} = \begin{pmatrix} -4 \\ 8 \end{pmatrix} \quad \therefore \quad A^{-1} = \frac{1}{41}\begin{pmatrix} 4 & 3 \\ -7 & 5 \end{pmatrix}$

$\vec{x} = \frac{1}{41}\begin{pmatrix} 4 & 3 \\ -7 & 5 \end{pmatrix}\begin{pmatrix} -4 \\ 8 \end{pmatrix}$

$= \frac{1}{41}\begin{pmatrix} 8 \\ 68 \end{pmatrix} = \begin{pmatrix} 8/41 \\ 68/41 \end{pmatrix}$

Cramer's Rule

- Recall: $A\vec{x} = \vec{b}$
- Suppose $\det A \neq 0$

WLOG $A_{(1)} = (\vec{b}, \vec{a}_2, \vec{a}_3 \dots)$

$A_{(2)} = (\vec{a}_1, \vec{b}, \vec{a}_3 \dots)$

$A_{(3)} = (\vec{a}_1, \vec{a}_2, \vec{b} \dots)$

Using determinants we solve the system

$x_1 = \det(A_{(1)}) / \det A$

$x_2 = \det(A_{(2)}) / \det A$

$x_3 = \det(A_{(3)}) / \det A$

$\vdots$

$x_n = \det(A_{(n)}) / \det A$

Ex: $A = \begin{pmatrix} 5 & -3 \\ 7 & 4 \end{pmatrix}$ $\quad b = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$ $\quad x = \begin{pmatrix} x \\ y \end{pmatrix}$

$\det A = (5)(4) - (-3)(7) = 41$

$A_{(1)} = \begin{pmatrix} -4 & -3 \\ 8 & 4 \end{pmatrix} \rightarrow |A_{(1)}| = 8$

$A_{(2)} = \begin{pmatrix} 5 & -4 \\ 7 & 8 \end{pmatrix} \rightarrow |A_{(2)}| = 68$

$\therefore \quad x = \dfrac{\det A_{(1)}}{\det A} = \dfrac{8}{41}$ $\qquad$ * Same as before

$\qquad y = \dfrac{\det A_{(2)}}{\det A} = \dfrac{68}{41}$

Eigenvalues & Eigenvectors

- An eigenvector for $A_{n\times n}$, with associated eigenvalue satisfied the condition.

$$A\vec{x} = \lambda \vec{x}$$

Ex: $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \lambda = 2$

L.S $= A\vec{x}$

$= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

R.S $= \lambda \vec{x}$

$= 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$=$ L.S ✓

Characteristic Polynomial

- A scalar λ is an eigenvalue of $A_{n\times n}$ iff λ satisfies

$C_A = (\lambda) = \det(A - \lambda I) = 0$ $\qquad$ where $C_A(\lambda) = 0$

is called the characteristic polynomial of A.

Ex: Find $C_A(\lambda) = 0$ for $A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$

$$0 = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 5 \\ 1 & -1-\lambda \end{vmatrix}$$

$= (3-\lambda)(-1-\lambda) - 5(1)$

$= -3 + \lambda - 3\lambda + \lambda^2$

$= \lambda^2 - 2\lambda - 8$

$= (\lambda - 4)(\lambda + 2)$

$\therefore \lambda = 4, -2$

Eigenvectors

Recall $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}$ $\lambda = 2, 1, -1$

The eigenvectors solve $(B - \lambda I)\vec{x} = 0$

$$\begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{pmatrix}$$

$$\lambda = 2 \left(\begin{array}{ccc|c} 2-2 & 0 & 0 & 0 \\ 1 & 2-2 & -1 & 0 \\ 1 & 3 & -2-2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 3 & -4 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x - z = 0 & \rightarrow x = t \\ y - z = 0 & \rightarrow y = t \\ z = t & \rightarrow z = t \end{cases} \quad \vec{x} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

* let $t = 1$ then, $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

*we note that for $\lambda = -1$, we had a 2-D eigenspace (ie: $\dim\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\} = 2$), so we say $\lambda = -1$ has geometric multiplicity $= 2$

- Looking back for $\lambda = 2$, the eigenspace was $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}^3$ which is 1-D. Thus for $\lambda = 2$, alg·m $= 1$ & geo·m $= 1$

Summary: $\begin{cases} \lambda = 2 & \vec{x} = \left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \\ \lambda = -1, -1 & \vec{x} = \left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\} \end{cases}$

$\therefore C = PDP^{-1}$ where $D = \begin{pmatrix} 2 & & \emptyset \\ & -1 & \\ \emptyset & & -1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

* For a matrix $A_{m \times n}$ to be diagonalizable for every eigen value: alg·m = geo·m

* If $A_{n \times n}$ has a distinct eigenvalue, then A is diagonalizable

* Geo·m can never be greater than alg·m $\geq$ geo·m

Rank of a Matrix

- row-rank: the number of leading 1's in the rows $A_{m \times n}$

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{row-rank} = 2$$

- col-rank: the number of leading 1's in the cols of $A_{m \times n}$

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{col-rank} = 3$$

- The rank(A) = row-rank(A) = col-rank(A)

Diagonalization

$A = PDP^{-1}$

* where D is a diagonal matrix (with ordered eigenvalues) & P is a square invertable matrix whose columns are the associated eigenvalues ordered w.r.t D.

$A^n = (PDP^{-1})^n \longrightarrow A^n = PD^nP^{-1}$

Ex: Diagonalize $B = \begin{pmatrix} 6 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$$0 = C_B(\lambda) = \begin{vmatrix} 6-\lambda & 1 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (6-\lambda)(1-\lambda)(3-\lambda)$$

$\therefore \lambda = 6, 1, 3$

Solve $(B - \lambda I)$

$\lambda = 6$ $\begin{pmatrix} 0 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} y=0 \\ z=0 \\ x=t \end{cases}$ $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ * where $t=1$

$\lambda = 1$ $\begin{pmatrix} 5 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} x + \frac{1}{5}y = 0 \\ z = 0 \\ y = t \end{cases}$ $x = -\frac{1}{5}t$, $z = 0$, $y = t$ $\vec{x} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$ * where $t=5$

$\lambda = 3$ $\begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} 3x + 2z = 0 \\ y - z = 0 \\ z = t \end{cases}$ $x = \frac{2}{3}t$, $y = -t$, $z = t$ $\vec{x} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$ * where $t=3$

$\therefore B = PDP^{-1}$ where $D = \begin{pmatrix} 6 & & \emptyset \\ & 1 & \\ \emptyset & & 3 \end{pmatrix}$

$P = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 5 & -3 \\ 0 & 0 & 3 \end{pmatrix}$

Linear Transformations

$T: R^n \to R^m$ is called a linear transformation, L.T provided

① $T(\vec{x}) + T(\vec{y}) = T(\vec{x} + \vec{y})$

② $T(\alpha\vec{x}) = \alpha T(\vec{x})$

* $T(\vec{0}) = \vec{0}$

* $T(-\vec{b}) = -T(\vec{b})$

$T = \underset{\text{domain}}{\mathbb{R}^n} \to \underset{\text{range}}{\mathbb{R}^m}$

L.T's can be shown in 2 ways

① As a Formula $T(\underset{\text{[domain] element}}{\vec{x}}) = \underset{\text{image}}{\vec{b}}$

Note: $\vec{x} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^m$

② $T: \mathbb{R}^n \to \mathbb{R}^m$

$A\vec{x} = \vec{b}$

Linear Combination of Vectors

-consider the vectors $\vec{u}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ & $\vec{u}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$ (belong to $\mathbb{R}^2$). Express $\vec{b} = \begin{pmatrix}4\\0\end{pmatrix}$ as a linear combination of $\vec{u}_1$ & $\vec{u}_2$. We will use RREF & leading 1's.

* take the vectors $\vec{u}_1$ & $\vec{u}_2$ & compress them into a matrix augmented with $\vec{b}$

$$\left(\begin{array}{cc|c}1 & 1 & 4\\1 & -1 & 0\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 1 & 4\\0 & -2 & -4\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 1 & 4\\0 & 1 & 2\end{array}\right) \sim \begin{array}{c}\left(\begin{array}{cc|c}1 & 0 & 2\\0 & 1 & 2\end{array}\right)\\ u_1 \; u_2 \qquad\end{array}$$

$$\therefore \begin{pmatrix}4\\0\end{pmatrix} = 2\vec{u}_1 + 2\vec{u}_2$$
$$= 2\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix} \checkmark$$

Linear Transformations

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\4\end{pmatrix}, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\5\\-2\end{pmatrix}, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix}$$

Ex: Find $T\begin{pmatrix}1\\1\\0\end{pmatrix}$

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = T\begin{pmatrix}1\\0\\0\end{pmatrix} + T\begin{pmatrix}0\\1\\0\end{pmatrix}$$
$$= \begin{pmatrix}2\\-1\\4\end{pmatrix} + \begin{pmatrix}1\\5\\-2\end{pmatrix} = \begin{pmatrix}3\\4\\2\end{pmatrix}$$

Ex: $\mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}, \quad T\begin{pmatrix}0\\-1\\2\end{pmatrix} = \begin{pmatrix}-3\\2\\-1\end{pmatrix}, \quad T\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

Find $T\begin{pmatrix}2\\1\\0\end{pmatrix}$ $\left(\vec{b} = \begin{pmatrix}2\\1\\0\end{pmatrix}\right)$

$$\left(\begin{array}{ccc|c}1 & 0 & 1 & 2\\1 & -1 & 0 & 1\\1 & 2 & 1 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & 1 & 2\\0 & -1 & -1 & -1\\0 & 2 & 0 & -2\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & 1 & 2\\0 & 1 & 0 & -1\\0 & -1 & -1 & -1\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & 1 & 2\\0 & 1 & 0 & -1\\0 & 0 & -1 & -2\end{array}\right) \sim \begin{array}{c}\left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & -1\\0 & 0 & 1 & 2\end{array}\right)\\ T_1 \; T_2 \; T_3 \qquad\end{array}$$

$$T\left[\begin{pmatrix}2\\1\\0\end{pmatrix}\right] = 0\,T\begin{pmatrix}1\\1\\1\end{pmatrix} + (-1)\,T\begin{pmatrix}0\\-1\\2\end{pmatrix} + 2\,T\begin{pmatrix}1\\0\\1\end{pmatrix}$$
$$= 0\begin{pmatrix}2\\0\\-1\end{pmatrix} + (-1)\begin{pmatrix}-3\\2\\-1\end{pmatrix} + 2\begin{pmatrix}1\\1\\0\end{pmatrix}$$
$$= \vec{0} + \begin{pmatrix}3\\-2\\1\end{pmatrix} + \begin{pmatrix}2\\2\\0\end{pmatrix} = \begin{pmatrix}5\\0\\1\end{pmatrix}$$

Standard Transformation Matrix

$T: \mathbb{R}^n \to \mathbb{R}^m$

$A_{m \times n}\ \vec{x} = \vec{b}$ * where $\vec{x} \in \mathbb{R}^n$ & $\vec{b} \in \mathbb{R}^m$

Note: Given $T(\vec{x}) = A\vec{x} = \vec{b}$, we realize A is the coefficient matrix of a system of equations.

Ex: Given $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x-y\end{pmatrix}$, find the std-matrix A & define T

$$\begin{cases}x+y\\x-y\end{cases} \to A\vec{x} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}_{2\times 2}$$

$\therefore A = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$ & $T: \mathbb{R}^2 \to \mathbb{R}^2$

Ex: Given $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\2x-y\\y\end{pmatrix}$ Find A & define T

$$A = \begin{pmatrix}1 & 1\\2 & -1\\0 & 1\end{pmatrix}_{3\times 2} \qquad T: \mathbb{R}^2 \to \mathbb{R}^3$$

Rotation

$$A = R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$$

Ex: $\vec{x}\begin{pmatrix}1\\1\end{pmatrix}$ Rotate $\vec{x}$ by $\frac{\pi}{2}$

$$A = R_{\frac{\pi}{2}} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\therefore \vec{x}\begin{pmatrix}1\\1\end{pmatrix} \to b = A\vec{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

Theorm: $T: \mathbb{R}^n \to \mathbb{R}^m$ is a L.T, with std-matrix, A then

① T is onto provided row-rank$(A) = m$

② T is one-to-one provided col-rank$(A) = n$

③ If T is both onto & 1-to-1, then A^{-1} exists & thus T is an invertible L.T.

④ A vector $\vec{b}$ is a valid image of T provided $(A \,\vdots\, \vec{b}) \sim (I^* \,\vdots\, x)$ is consistent

Ex: is $\vec{b} = \begin{pmatrix}4\\3\\2\end{pmatrix}$ a valid image? is T onto or 1-to-1?

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

row-rank$(A) = 3$ $\therefore$ onto

col-rank$(A) = 3$ $\therefore$ not 1-to-1

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

consistent

$\hookrightarrow$ multiple solutions

$\therefore \vec{b}$ is a valid image

Linear Dependence & Independence

- A set of vectors $V = \{\vec{N}_1, \vec{N}_2, \vec{N}_3 \dots\}$ is said to be linearly independent (LI) if the LC:

$$a_1\vec{N}_1 + a_2\vec{N}_2 + a_3\vec{N}_3 + \dots = \vec{0} \quad \text{(trivial solution)}$$

- A set is linearly dependent (LD) if there exists a non-trivial solution, such that one of the vectors can be expressed as a LC of one/some/all of the others

Ex: Consider the set $V = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\2\end{pmatrix} \right\}$, is it LI or LD?

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ only col 1 & col 2 have leading 1's so the set is L.D.

* A set of vectors in a matrix are L.I. if the matrix has Full col-rank.

Ex: Identify the LI vectors (using V from above)

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ the LI vectors are $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix} \right\}$

* This answer is non-unique

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ now the LI vectors are $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\2\end{pmatrix} \right\}$

Basis & Sets

- The basis is a set which "covers" a space. This basis has n elements which makes up the minimum-set needed to "cover" the given space.
- Thus, a basis set is also a LI set

* Look for the leading 1's

- A spanning set is a set which "covers" some space, w.

$w = \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3 \ldots\}$

Ex: Find a basis for $w = \text{span}\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}-2\\2\\4\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$

$$\begin{pmatrix}1 & -2 & 1 & 1\\ -1 & 2 & 0 & 1\\ 2 & 4 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\end{pmatrix}$$

$\therefore \text{basis}(w) = \left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}-2\\2\\4\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

- Col A: The original column vectors that become columns with leading 1's
- Row A: The original row vectors that become rows with leading 1's

Ex: Find Col A & Row A

$$A = \begin{pmatrix}1 & -2 & 1 & 1\\ -1 & 2 & 0 & 1\\ 2 & 4 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\end{pmatrix}$$

$\text{col} A = \left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}-2\\2\\4\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

$\text{row} A = \left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}^T, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}^T, \begin{pmatrix}0\\0\\1\\2\end{pmatrix}^T\right\}$

- nul: is the solution of the homogeneous system $(A\vec{x} = 0)$
- nullity: The number of dependent columns / free variables

* $\text{rank}(A) + \text{nullity}(A) = n$ where $A_{m \times n}$

Ex: Solve for rank B, nullity and nul B

$$B = \begin{pmatrix}1 & -2 & 1 & 1\\ -1 & 2 & 0 & 1\\ 2 & 4 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\end{pmatrix}$$

$\therefore \text{rank} B = 3$ & nullity $= 1$

$$\left\{\begin{array}{l} x_1 - x_4 = 0\\ x_2 = 0\\ x_3 + 2x_4 = 0\\ \hline x_4 = t\end{array}\right. \rightarrow \begin{array}{l} x_1 = t\\ x_2 = 0\\ x_3 = -2t\\ x_4 = t\end{array} \qquad \vec{x} = \begin{pmatrix}1\\0\\-2\\1\end{pmatrix}$$

$\text{nul} B = \left\{\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}\right\}$

Vector Sub-space

- w is a sub-space of v (where v is a complete vector space), provided:

1. if $\vec{x} \in W$, & $\vec{y} \in W$, then $(\overrightarrow{x+y}) \in W$

   * closed under addition

2. if $\vec{x} \in W$, α is a scalar, then $(\alpha\vec{x}) \in W$

   * closed scalar multiplication

3. $\vec{0} \in W$

   * identity element $\rightarrow \vec{0}$ under addition

4. if $\vec{x} \in W$, $(-\vec{x}) \in W$

   * inverse element $\rightarrow (-\vec{x})$ under addition

Note: we say that w must be non-empty

Ex: Show that w is a sub-space of $\mathbb{R}^3$

$w = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$

③ $x = y = 0 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

① let $a, b, c, d \in \mathbb{R}$

$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} c \\ d \\ 0 \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ 0 \end{pmatrix}$ Let $x = a + c$ & $y = b + d$ $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ ✓

② $\alpha \in \mathbb{R}$

$\alpha \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \\ 0 \end{pmatrix}$ Let $x = \alpha a$ & $y = \alpha b$ $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ ✓

④ $\begin{pmatrix} \alpha a \\ \alpha b \\ 0 \end{pmatrix}$ let $\alpha = -1$ $= \begin{pmatrix} -a \\ -b \\ 0 \end{pmatrix}$ Let $x = -a$ & $y = -b$ $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ ✓

Ex 2: $w = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x \geq 0, \; x, y \in \mathbb{R} \right\}$

③ $x = y = 0$ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

① $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} c \\ d \\ 0 \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ 0 \end{pmatrix}$ Let $x = a + c$ & $y = b + d$ $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ ✓

② $\alpha \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \\ 0 \end{pmatrix}$ Let $x = \alpha a$ & $y = \alpha b$ $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ ✓

④ $\begin{pmatrix} \alpha a \\ \alpha b \\ 0 \end{pmatrix}$ let $\alpha = -1$ $\begin{pmatrix} -a \\ -b \\ 0 \end{pmatrix}$ Let $x = -a$ & $y = -b$ $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ but only if $x > 0$ ✗

The 4 Most Important Things:

① How to find the norm of a vector

② For $\vec{u}, \vec{v}$, provided $\vec{u} \cdot \vec{v} = 0$, iff $\vec{u} \perp \vec{v}$

③ Systems of equations: Matrix equation, plug into augmented matrix, RREF, solution (consistent, inconsistent; parametric/vector-parametric)

④ Unifying Matrix Theorem:

- Given $A_{n \times n}$. If one of the following is true, then they are all true

① A^{-1} exists

② $\det A \neq 0$

③ $\lambda = 0$ is not an eigenvalue of A

④ $A\vec{x} = \vec{b}$ has an unique solution

⑤ $A\vec{x} = \vec{0}$ (the homogeneous system)

⑥ $A \sim I$

⑦ All rows/cols of A are LI

⑧ Col A also forms a basis for $\mathbb{R}^n$

⑨ rank $A = n$

⑩ nullity $A = 0$