Faculty of Science »
Department of mathematics and statistics

# MAT 1341C Winter 2015 Final Exam

16 April, 2015.

Instructor - Barry Jessup

| | |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| sub-total | |
| 11 | |
| 12 | |
| 13 | |
| 14 | |
| 15 | |
| 16 [Bonus] | |
| Total | |

Family Name:______________________

First Name:________________________ Seat number: _________

Student number:___________________

## Some Advice

Take a few minutes to read the entire paper before you begin to write, and read each question carefully. The multiple choice questions are only worth 1 point and questions 11-15 are worth 6 points each. Make a note of the questions you feel confident you can do, and try those first: you do not have to do the questions in the order they are presented.

## Instructions

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are permitted. **The use of calculators, cell phones, or similar devices is not permitted.** All cyber devices not necessary for life-support must be disabled at the beginning of the exam.
3. Questions 1 to 10 are multiple choice. These questions have just one correct answer, are worth 1 point each and no part marks will be given. Please record your answers in the spaces opposite.
4. Questions 11 – 15 require a complete solution, and are worth 6 points each. Question 16 is a bonus question and should only be attempted after all other questions have been completed and checked.

   **Spend your time accordingly.**

   Answer questions 11 – 16 in the space provided, and use the backs of pages if necessary.
5. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
6. Where it is possible to check your work, do so.

**Good luck! Bonne chance!**

**1.** Let $X = \{(x, y, z) \in \mathbf{R}^3 \mid x - 2y + z = 0\}$. Which one of the following statements is true?

A. $X$ is a subspace of $\mathbf{R}^3$ and $\dim X = 3$

B. $X$ is a subspace of $\mathbf{R}^3$ and $\{(2, 1, 0), (-1, 0, 1)\}$ is a basis of $X$

C. $X$ is a subspace of $\mathbf{R}^3$ and $\{(2, 1, 0), (4, 2, 0)\}$ is a basis of $X$

D. $X$ is a not a subspace of $\mathbf{R}^3$.

E. $X$ is a line in $\mathbf{R}^3$ with direction vector $(1, -2, 1)$

F. $X$ is a plane in $\mathbf{R}^3$ with normal $(2, 1, 0)$

**2.** Which two of the following statements are true?

I. $\{1, x, x^2\}$ is linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$ .

II. A homogeneous linear system always has infinitely many solutions.

III. If $A$ and $B$ are $2 \times 2$ matrices, and $A$ is invertible then $AB = 0$ implies $B = 0$.

IV. If $u$ and $v$ are independent vectors in $\mathbf{R}^3$, then $\{u, v\} = \text{span}\{u, v\}$.

A. $I$ and $II$.
B. $I$ and $III$.
C. $I$ and $IV$.
D. $II$ and $III$.
E. $II$ and $IV$.
F. $III$ and $IV$.

**3.** Let $A$ be a square $n \times n$ matrix with $n \geq 2$. Which of the following statements is true?

I. If $\operatorname{rank} A = n - 1$, there is just one parameter in the general solution of $Ax = 0$.

II. If $\operatorname{rank} A = n - 1$, there are $n - 1$ parameters in the general solution of $Ax = 0$.

III. If $A$ is invertible, $Ax = 0$ has more than one solution

IV. If $Ax = 0$ has more than one solution, then $\operatorname{rank} A < n$.

A. I. only

B. II. only

C. III only

D. I. and III.

E. I and IV.

F. III and IV.

**4.** If $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $D$ is a $2 \times n$ matrix, then the second row of the matrix $AD$ is

A. not defined unless $n = 2$.

B. twice the first row of $D$.

C. the same as the first row of $D$.

D. the same as the second row of $D$.

E. the sum of the first and the second rows of $D$.

F. the sum of twice the first row of $D$ and the second row of $D$.

**5.** Let $S = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and consider the subset $W = \{A \in \mathbf{M}_{22} \mid SA = AS\}$. Which one of the following statements is true?

A. $W$ is not a subspace of $\mathbf{M}_{22}$
B. $W$ is a subspace of $\mathbf{M}_{22}$, and $\dim W = 4$
C. $W$ is a subspace of $\mathbf{M}_{22}$, and $\dim W = 3$
D. $W$ is a subspace of $\mathbf{M}_{22}$, and $\dim W = 2$
E. $W$ is a subspace of $\mathbf{M}_{22}$, and $\dim W = 1$
F. $W$ is a subspace of $\mathbf{M}_{22}$, and $\dim W = 0$

**6.** Find the value of $t$ for which $(1, 3, t)$ belongs to $\text{span}\{(1, 2, 1), (1, 1, 2)\}$.

A. -2
B. -1
C. 0
D. 1
E. 2
F. 7

**7.** For a non-homogeneous system of 15 equations in 12 unknowns, answer the following three questions:

∘ Can the system be inconsistent?
∘ Can the system have a unique solution?
∘ Can the system have infinitely many solutions?

A. No, Yes, Yes.
B. Yes, Yes, No.
C. Yes, No, Yes.
D. No, No, Yes.
E. Yes, Yes, Yes.
F. No, Yes, No.

**8.** Let $A$ be a square $n \times n$ matrix. Which one of the statements below **is not equivalent** to

"The rows of A are linearly independent"

A. 0 is an eigenvalue of $A$

B. The homogeneous system $Ax = 0$ has a unique solution

C. The columns of $A$ form a basis of $\mathbf{R}^n$

D. $\operatorname{rank} A = n$

E. $\det A \neq 0$

F. $A$ is invertible

**9 .** If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3$, find

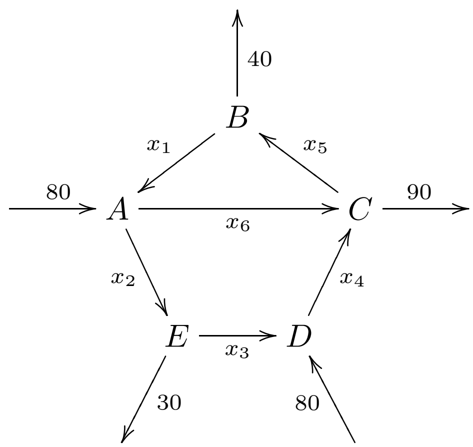
$$\begin{vmatrix} 3a & 3d & 3g \\ b+5c & e+5f & h+5i \\ -2c & -2f & -2i \end{vmatrix}$$

A. -64
B. -18
C. -16
D. 16
E. 18
F. 64

**10.** The vectors $u_1 = (1, -1, 2), u_2 = (-5, -1, 2)$, and $u_3 = (0, 2, 1)$ form an orthogonal basis of $\mathbf{R}^3$. If we write $v = (1, 0, -1) = a_1u_1 + a_2u_2 + a_3u_3$, what is $a_2$?

A. $-\frac{1}{3}$
B. $\frac{1}{3}$
C. $-\frac{7}{30}$
D. $\frac{7}{30}$
E. $-\frac{1}{5}$
F. $\frac{1}{5}$

**11.** Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each $x_i$ denotes the unknown number of cars which passed along the indicated streets during the same period.




a) Write down a system of linear equations which describes the the traffic flow, together with all the constraints on the variables $x_i$, $i = 1, \ldots, 6$.

(**Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.**)

**11(b).** The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 40 \\ 0 & 0 & 1 & 0 & -1 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 1 & 90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

c) If $\overline{AC}$ were closed due to roadwork, find the minimum flow along $\overline{ED}$, **using your results from (b).**

*(You must justify all your answers.)*

**12.** Let $X = \text{span}\{(1,-1,1,0),(0,1,1,1),(1,2,4,3),(1,0,2,2)\}$.

a) Find any basis for $X$, and hence find $\dim X$.

b) Find a basis for $X$ which is a **<u>subset</u>** of the given spanning set above.

c) Extend your basis for $X$ in part (b) to a basis of $\mathbf{R}^4$.

d) If $X$ were the rowspace of a $4 \times 4$ matrix $A$, how many parameters would there be in the general solution to $Ax = 0$?

**13.** Let $W = \{(x, y, z, u) \in \mathbf{R}^4 \mid x - z - u = 0\}$.

a) *Without referring to the Subspace Test* briefly explain why $W$ is a subspace of $\mathbf{R}^4$.

b) Find a basis for $W$.

c) Use the **Gram-Schmidt algorithm** to find an orthogonal basis for $W$.

d) Find the best approximation by a vector in $W$ to the vector $(1, 0, 1, 1)$.

**14.** Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

a) Find the characteristic polynomial of $A$, and use this to show that the eigenvalues of $A$ are 2 and $-1$.

b) Find a basis of $E_2 = \{v \in \mathbf{R}^3 \mid Av = 2v\}$.

c) Find a basis of $E_{-1} = \{v \in \mathbf{R}^3 \mid Av = -v\}$.

d) If possible, find an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$. If this is not possible, explain why.

**15.** State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers!**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or by giving a *proof valid for every case.*

a) If $A$ is an $4 \times 3$ matrix and if a row echelon form of $A$ has a row of zeros, then $\operatorname{rank} A < 3$.

ANSWER

b) The dimension of the kernel of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$ is 4.

ANSWER

**15** (cont.)

c) If $\{v_1, v_2\}$ is linearly independent in a vector space $V$, then $\{v_1 - v_2, v_1 + 2v_2\}$ is also linearly independent.

ANSWER

d) The funtion $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x^2, x + y)$ is a linear transformation.

ANSWER

**16. (Four bonus marks) Make sure you finish and check the rest of the paper before trying this. As you know, bonus marks are much harder to earn.**

In what follows, $A$ denotes an $n \times n$ matrix.

a) Prove that if $u$, $v$ and $w$ are eigenvectors of $A$ corresponding to three distinct eigenvalues, then $\{u, v, w\}$ is linearly independent.

b) Prove that if all the eigenvalues of $A$ are non-zero, then $A$ is invertible.