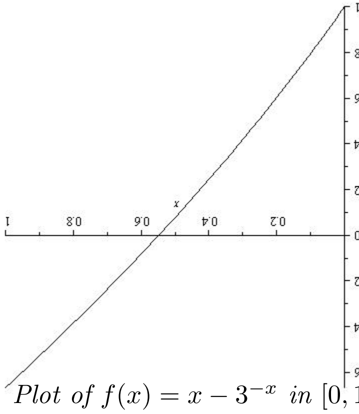
MATH-354, MAST-334, Section AA

Assignment 2, Solutions

Problem 2, (a),(b),(d), page 15. Locate **all solutions** of the given non-linear equation in intervals of length at most 0.5 such that: In each interval of location the given equation has a unique solution.

$$\begin{aligned} &\textbf{(a)} \quad x - 3^{-x} = 0; \\ &\textbf{(b)} \quad 3x^2 - e^{0.5x} = 0; \\ &\textbf{(d)} \quad x^3 + 4.001x^2 + 4.002x + 1.101 = 0 \end{aligned}$$

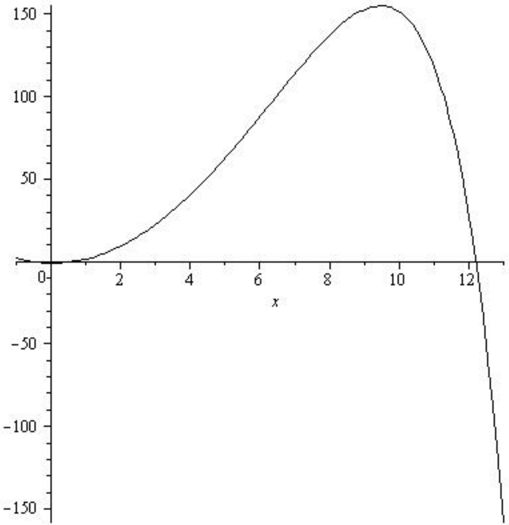
Solution. (a) Existence of solutions. Denote $f(x) = x - 3^{-x}$. We compute $f(0.4) = -0.2443940150 < 0$, $f(0.6) = 0.0827181420 > 0$ and by the IVT $f(x) = 0$ has a solution located in $[0.4, 0.6]$.
Uniqueness of the solution. Suppose that $f(x) = 0$ has at least 2 solutions $p_1 < p_2$. Then by Rolle's Theorem there is a number $c \in (p_1, p_2)$ such that $f^{'}(c) = 0$. However, $f^{'}(x) = 1 + 3^{-x}\ln(3) > 0$ for all $x \in (-\infty, \infty)$ hence, we got a contradiction. From here, $f(x) = 0$ has at most one solution but we know that it has one solution. Hence, $f(x) = 0$ has exactly one solution, located in $[0, 4, 0.6]$.



Plot of $f(x) = x - 3^{-x}$ *in* $[0, 1]$.

(b) Existence of solutions. Denote $f(x) = 3x^2 - e^{0.5x}$. We compute $f(-0.6) = 0.3391817793 > 0$, $f(-0.4) = -0.3387307531 < 0$, $f(0.6) = -0.269858808 < 0$, $f(0.8) = 0.428175302 > 0$, $f(12.2) = 0.6622299 > 0$, $f(12.4) = -31.4690411 < 0$ and by the IVT $f(x) = 0$ has three solutions located in $[-0.6, -0.4]$, $[0.6, 0.8]$ and $[12.2, 12.4]$.
Counting of the solutions. Suppose that $f(x) = 0$ has at least 4 solutions $p_1 < p_2 <$

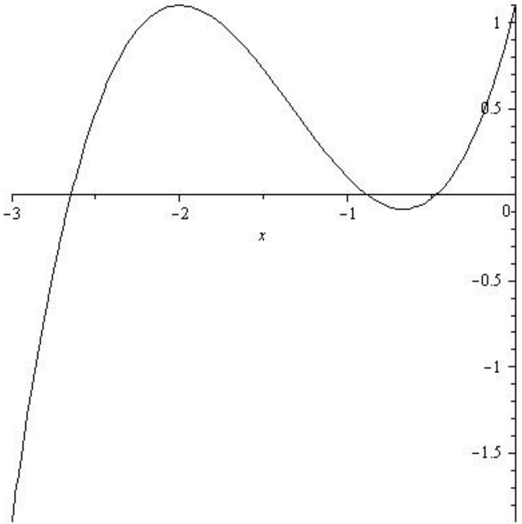
$p_3 < p_4$. Then by the Generalized Rolle's Theorem there is a number $c \in (p_1, p_4)$ such that $f^{(3)}(c) = 0$. However, $f^{(3)}(x) = -(0.5)^3 e^{0.5x} < 0$ for all $x \in (-\infty, \infty)$ hence, we got a contradiction. From here, $f(x) = 0$ has at most three solutions but we know that it has three solutions. Hence, $f(x) = 0$ has exactly three solution, located in the intervals $[-0.6, -0.4]$, $[0.6, 0.8]$ and $[12.2, 12.4]$.



Plot of $f(x) = 3x^2 - e^{0.5x}$ *in* $[-1, 13]$.

(c) Existence of solutions. Denote $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$. First $f(x) > 0$ for $x \geq 0$ hence, no solutions for $f(x) = 0$ in $[0, \infty)$. Next, we compute $f(-3) = -1.89600 < 0$, $f(-2.5) = 0.47725 > 0$, $f(-1) = 0.1 > 0$, $f(-0.5) = -0.02475 < 0$, $f(0) = 1.101 > 0$ and by the IVT $f(x) = 0$ has three solutions located in $[-3, -2.5]$, $[-1, -0.5]$ and $[-0.5, 0]$.

Counting of the solutions. Suppose that $f(x) = 0$ has at least 4 solutions $p_1 < p_2 < p_3 < p_4$. Then by the Generalized Rolle's Theorem there is a number $c \in (p_1, p_4)$ such that $f^{(3)}(c) = 0$. However, $f^{(3)}(x) = 6 > 0$ for all $x \in (-\infty, \infty)$ hence, we got a contradiction. From here, $f(x) = 0$ has at most three solutions but we know that it has three solutions. Hence, $f(x) = 0$ has exactly three solution, located in the intervals $[-3, -2.5]$, $[-1, -0.5]$ and $[-0.5, 0]$.

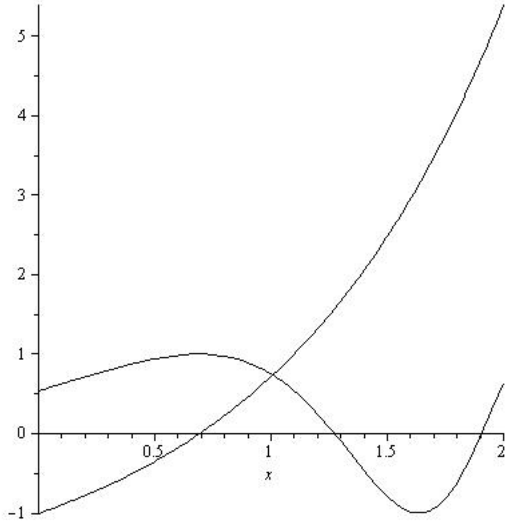


Plot of $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$ *in* $[-3, 0]$.

Problem 9, page 55.(a) Plot the graphs of $y = e^x - 2$ and $y = \cos(e^x - 2)$ in the interval $[0, 2]$ in order to visualize that the non-linear equation

$$e^x - 2 = \cos(e^x - 2)$$

has a unique solution p in this interval.
(b) Use the Bisectional method to find an approximation to within (absolute error) 10^{-5} to the solution p starting with the interval $[0.5, 1.5]$.
(c) Find an upper bound for the relative error of the approximation obtained in **(b)**. Use this upper bound to decide: How many true (significant) digits does the approximation obtained in **(b)** have?
Solution. (a)



Plot of $y = e^x - 2$ *and* $y = \cos(e^x - 2)$ *in* $[0, 2]$.

The plot shows that there is a unique $p \in [0.5, 1.5]$ such that $e^p - 2 = \cos(e^p - 2)$.
(b) Denote $f(x) = \cos(e^x - 2) - e^x + 2$, $f(0.5) = 1.29021220079 > 0$, $f(1,5) = -3.27174041290 < 0$ and we can start the Bisectional method in order to find the desired approximant to p. Solving for n:

$$\frac{1.5 - 0.5}{2^n} < 10^{-5} \Rightarrow n > \frac{5\ln(10)}{\ln(2)} = 16.6096404744 \Rightarrow \mathbf{n} = \mathbf{17}.$$

Hence, we need 17 iterations with the Bisectional method on $[0.5, 1.5]$ with $f(x) = \cos(e^x -$

$2)-e^x+2$ to obtain an approximation to the exact value p with accuracy 10^{-5}:

n	$a_n(+)$	$b_n(-)$	p_n	$f(p_n)$
1	0.5	1.5	1	0.03465572639
2	1.0	1.5	1.25	-1.40997635239
3	1.0	1.25	1.125	-0.60907974742
4	1.0	1.125	1.06250000000	-0.26698228760
5	1.0	1.06250000000	1.03125000000	-0.11114776428
6	1.0	1.03125000000	1.01562500000	-0.03700287465
7	1.0	1.01562500000	1.00781250000	-0.00086442520
8	1.0	1.00781250000	1.00390625000	0.01697271614
9	1.00390625000	1.00781250000	1.00585937500	0.00807344030
10	1.00585937500	1.00781250000	1.00683593750	0.00360933475
11	1.00683593750	1.00781250000	1.00732421875	0.00137366202
12	1.00732421875	1.00781250000	1.00756835938	0.00025492024
13	1.00756835938	1.00781250000	1.00769042969	-0.00030467701
14	1.00756835938	1.00769042969	1.00762939454	-0.00030467701
15	1.00756835938	1.00762939454	1.00759887696	0.00011503507
16	1.00759887696	1.00762939454	1.00761413575	0.00004508894
17	1.00761413575	1.00762939454	**1.00762176514**	0.00001011502

Hence, $\mathbf{p_{17}} = \mathbf{1.00762176514}$ is an approximation to p with absolute error $|p-p_{17}| < 10^{-5}$.
(c) Observing that $p_{17} < p$ we compute:

$$\text{rel. error} = \frac{|p-p_{17}|}{|p|} \leq \frac{|p-p_{17}|}{|p_{17}|} \leq \frac{10^{-5}}{1.00762176514} = 0.00000992435886755 = 0.992435886755 \times 10^{-5}$$

hence, the approximation p_{17} has at least 5 true (significant) digits.
Remark. The value of p with 20 digits rounding approximation is $p = 1.0076239716581366925$.
Problem 7, page 65. Consider the function:

$$g(x) = \pi + 0.5\sin(x/2).$$

(a) Show that $g(x)$ has a unique fixed point p in $[0, 2\pi]$.
(b) Construct a fixed-point method that is convergent to the fixed point p.
(c) By using the fixed-point method obtained in **(a)** find an approximation to the fixed point p with accuracy 10^{-2}.
(d) Compare the number of iterations from **(c)** with the number of iterations actually needed if the exact value of p is $p = 3.6269420148715736418$.
Solution. (a) Existence of a fixed point. We must have $p = g(p)$ hence, p is a solution of the non-linear equation:

$$x = \pi + 0.5\sin(x/2) \Rightarrow x - [\pi + 0.5\sin(x/2)] = 0.$$

Denote: $f(x) = x - [\pi + 0.5\sin(x/2)], f(0) = -\pi < 0, f(2\pi) = \pi > 0$ and by the IVT a fixed point $p \in [0, 2\pi]$ exists.
Another way to show existence of a fixed point: Obviously for each $x \in [0, 2\pi]$:

$$\pi \le g(x) = \pi + 0.5\sin(x/2) \le \pi + 0.5 < 2\pi$$

hence,

$$g : [0, 2\pi] \to [0, 2\pi]$$

and by Theorem 2.2 a fixed point p to $g(x)$ in $[0, 2\pi]$ exists.
Uniqueness of the fixed point.

$$f^{'}(x) = 1 - 0.25\cos(x/2) > 0$$

hence, $f(x)$ is increasing in $[0, 2\pi]$ and from here $f(x)$ has at most one solution in $[0, 2\pi]$ and from here $g(x)$ has at most one fixed point in $[0, 2\pi]$.
Another method to show uniqueness is to show that $g(x)$ is contractive in $[0, 2\pi]$ and to use Theorem 2.2.

$$|g^{'}(x)| = |0.25\cos(x/2)| \le 0.25 < 1, \; x \in [0, 2\pi]$$

hence, $g(x)$ is contractive in $[0, 2\pi]$ with $k = 0.25$ and according to Theorem 2.2, the uniqueness of p follows.
(b). We have obtained in **(a)** that:

$$g : [0, 2\pi] \to [0, 2\pi]; \quad |g^{'}(x)| \le 0.25, \; x \in [0, 2\pi]$$

hence, according to Theorem 2.3, with any $p_0 \in [0, 2\pi]$ the iterated sequence

$$\{p_{n+1} = g(p_n)\}_{n=0}^{\infty}, \quad p_0 \in [0, 2\pi]$$

is convergent to the fixed point p.

(c)
(1) According to Theorem 2.3, Estimate A):

$$|p_n - p| \le (2\pi - 0)(0.25)^n < 10^{-2} \quad \Rightarrow \quad n > \frac{\ln((2\pi)10^2)}{\ln(1/0.25)} = 4.65 \quad \Rightarrow \; \mathbf{n = 5}.$$

(2) According to Theorem 2.3, Estimate B): With $p_0 = \pi$, $p_1 = g(\pi) = \pi + 0.5$ $k = 0.25$:

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0| = \frac{(0.25)^n}{1 - 0.25}|(\pi + 0.5) - \pi)| = \frac{2}{3}(0.25)^n < 10^{-2}$$

$$\Rightarrow n > \frac{\ln((2/3)10^2)}{\ln(1/0.25)} = 3.0294 \Rightarrow \mathbf{n = 4}.$$

We compute with $p_0 = \pi$, $p_1 = g(p_0) = 3.64159265358979323840$, $p_2 = g(p_1) = 3.62604886444511563050$, $p_3 = g(p_2) = 3.62699562243873547520$, $p_4 = g(p_3) = 3.62693879422541710030$, $p_5 = g(p_4) = 3.62694220835109469630$,
We can use the following stopping rule also in the process of computations:
(3) According to Theorem 2.3, Estimate C): With $p_0 = \pi$, $p_1 = g(\pi) = \pi + 0.5$ $k = 0.25$:

$$|p_n - p| \leq \frac{k}{1-k}|p_n - p_{n-1}| = \frac{0.25}{0.75}|p_n - p_{n-1}|.$$

For $n = 1$:

$$|p_1 - p| \leq \frac{0.25}{0.75}|p_1 - p_0| = \frac{0.25}{0.75}|3.64159265358979323840 - 3.14159265358979323840| = 0.16667 > 10^{-2}$$

For $n = 2$:

$$|p_2 - p| \leq \frac{0.25}{0.75}|p_2 - p_1| = \frac{0.25}{0.75}|3.62604886444511563050 - 3.64159265358979323840|$$

$$= 0.0051812630482258693000 < 10^{-2}$$

For $n = 3$:

$$|p_3 - p| \leq \frac{0.25}{0.75}|p_3 - p_2| = \frac{0.25}{0.75}|3.62699562243873547520 - 3.62604886444511563050|$$

$$= 0.00031558599787328156667 < 10^{-2}$$

Hence, by using Estimate C) we obtain $\mathbf{n = 2}$.
(d) The actual value: $p = 3.62694201487157364180$.

$$|p_0 - p| = 0.48534936128178040340 > 10^{-2}; \ |p_1 - p| = 0.01465063871821959660 > 10^{-2};$$

$$|p_2 - p| = 0.00089315042645801130 < 10^{-2}$$

hence actually, we need 2 iterations in order to get the desired accuracy. We see that by using the Estimate A) we needed 5 iterations; by using the Estimate B) we needed 4 iterations and by using Estimate C) we needed 2 iterations that coincides with the iterations that we actually need. Hence, the Estimate C) gives more economic number of iterations to achieve a prescribed accuracy.

Problem 9, page 65. Construct a convergent fixed point method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}.

Solution 1. We observe that $p = \sqrt{3}$ is the positive solution of $x^2 - 3 = 0$ that is equivalent to $x^2 - 4 = -1$ that is $(x-2)(x+2) = -1$ that is

$$x - 2 = -\frac{1}{x+2} \quad \Rightarrow \quad x = 2 - \frac{1}{x+2}$$

hence, $p = \sqrt{3}$ is a fixed point of

$$g(x) = 2 - \frac{1}{x+2}.$$

We apply Theorem 2.3:
(1) Obviously $1.7 < p < 1.8$. For $x \in [1.7, 1.8]$ by using the fact that $g(x)$ is increasing we obtain:

$$1.7 < 1.72972973 = g(1.7) \leq g(x) \leq g(1.8) = 1.736842105 < 1.8$$

to conclude that

$$g \ : \ [1.7, 1.8] \ \rightarrow \ [1.7, 1.8].$$

Another consideration here: g is contractive, $g^{'}(x) \geq 0$ in $[1.7, 1.8]$, and $g(x)$ has a fixed point. Then

$$g \ : \ [1.7, 1.8] \ \rightarrow \ [1.7, 1.8].$$

(2) For $x \in [1.7, 1.8]$:

$$g(x) = 2 - \frac{1}{x+2}, \quad g^{'}(x) = \frac{1}{(x+2)^2} \leq \frac{1}{(1.7+2)^2} < 0.074$$

and we conclude that $g(x)$ is contractive in $[1.7, 1.8]$ with $k = 0.074$.
In view of (1) and (2), by Theorem 2.3, for each initial iteration $p_0 \in [1.7, 1.8]$ the iterated sequence $p_{n+1} = g(p_n), n = 0, 1, 2, \ldots$ is convergent to the fixed point $p = \sqrt{3}$ and

$$|p_n - p| \leq (1.8 - 1.7)(0.074)^n < 10^{-4} \Rightarrow n > \frac{\ln((0.1)10^4)}{\ln(1/0.074)} = 2.653 \Rightarrow \mathbf{n = 3}.$$

In view of this with an arbitrary $p_0 \in [1.7, 1.8]$, the third iteration p_3 will approximate p with accuracy 10^{-4}. We compute with $p_0 = (1.7+1.8)/2 = 1.75$, $p_1 = g(p_0) = 1.73333333333333$, $p_2 = g(p_1) = 1.73214285714286$, $\mathbf{p_3 = g(p_2) = 1.73205741626794}$.

Solution 2 by using the $(\mathbf{m_1}, \mathbf{M_1})$ method. Construction of another convergent fixed-point method. $f(x) = x^2 - 3, \ f^{'}(x) = 2x > 0, \ x \in [1.7, 1.8]$. Then

$$\max_{x \in [1.7, 1.8]} f^{'}(x) = 3.6; \qquad \min_{x \in [1.7, 1.8]} f^{'}(x) = 3.4.$$

Consider:

$$x = x - \frac{f(x)}{\max_{x\in[1.7,1.8]} f^{'}(x)} = x - \frac{x^2-3}{3.6} \quad g_1(x) = x - \frac{x^2-3}{3.6}.$$

(1) Obviously, $p = \sqrt{3}$ is a fixed point of $g_1(x)$.
(2) $g_1(x)$ is contractive in $[1.7, 1.8]$:

$$0 \leq g_1^{'}(x) = 1 - \frac{f^{'}(x)}{3.6} = \leq 1 - \frac{\min_{x\in[1.7,1.8]} f^{'}(x)}{3.6} = 1 - \frac{3.4}{3.6} = 0.0556 \ \mathbf{k = 0.0556}.$$

(3) From the fact that $g_1^{'}(x) \geq 0$, by Theorem $2.3^{'}$, we conclude that with any $p_0 \in [1.7, 1.8]$, the iterated sequence $p_{n+1} = g(p_n), n = 0, 1, 2, \ldots$ will be in the interval $[1.7, 1.8]$, it will be convergent to p, and

$$|p_n - p| \leq (0.1)(0.0556)^n < 10^{-4} \Rightarrow n > \frac{\ln((0.1)10^4)}{\ln(1/0.056)} = 2.397 \Rightarrow \mathbf{n = 3}.$$

Hence, $p_3 = g_1(p_2)$ will give the desired approximation to p.

Problem 12,(b), page 65. **(a)** Use the given interval to determine a sub-interval on which to construct a fixed-point method that is convergent to the solution p of the given equation in the given interval:

$$x^3 - 2x - 5 = 0, \quad [2, 3].$$

(b) Estimate the number of iterations sufficient to obtain an approximation to p accurate to within 10^{-5} and perform the computations.
(c) Find an upper bound for the relative error of the approximation obtained in **(b)**.

Solution 1. **(a)** $f(x) = x^3 - 2x - 5, f(2) = -1 < 0, f(3) = 16 > 0$ and by the IVT $f(x) = 0$ has a solution in $[2, 3]$. In addition, $f^{'}(x) = 3x^2 - 2 > 0, \ x \in [2, 3]$ hence, $f(x) = 0$ has a unique solution $p \in [2, 3]$.

$$x = \sqrt[3]{2x+5}, \ g(x) = \sqrt[3]{2x+5}, \ g^{'}(x) = \frac{2}{3}(2x+5)^{-2/3}$$

(1) For all x in the interval $[2, 3]$, $g^{'}(x) > 0$, hence, $g(x)$ is increasing and

$$2 < 2.08008382305190 = g(2) \leq g(x) \leq g(3) = 2.22398009056932 < 3$$

$$\Rightarrow \quad g : [2, 3] \rightarrow [2, 3].$$

(2) $g^{'}(x)$ is decreasing on $[2,3]$ and from here

$$|g^{'}(x)| = g^{'}(x) \leq \frac{2}{3}(2 \times 2 + 5)^{-2/3} = \frac{2}{3}\frac{1}{9^{2/3}} = 0.1540802832 < 1$$

and in view of this, $g(x)$ is contractive in $[2,3]$ with $k = 0.1540802832$. According to Theorem 2.3, with any $p_0 \in [2,3]$ the iterated sequence $\{p_{n+1} = g(p_n)\}_{n=0}^{\infty}$ is convergent to p and

$$|p_n - p| \leq (3-2)(0.1540802832)^n \ < 10^{-5}$$

$$\Rightarrow n > \frac{\ln(10^5)}{\ln(1/0.1540802832)} = 6.15571800742551 \Rightarrow \mathbf{n = 7}.$$

We compute with $p_0 = 2$, $p_1 = 2.08008382305190$, $p_2 = 2.09235067779758$, $p_3 = 2.09421699601252$, $p_4 = 2.09450065219465$, $p_5 = 2.09454375753281$, $p_6 = 2.09455030780827$, $\mathbf{p_7 = 2.09455130318276}$.

(c) Find an upper bound for the relative error of the approximation obtained in **(b)**. Obviously, $p_0 = 2 < p$, $g^{'}(x) > 0, x \in [2,3]$ and by Theorem $2.3^{'}$:

$$2 = p_0 < p_1 < p_3 < \cdots < p_7 < p.$$

Then:

$$\frac{|p - p_7|}{|p|} < \frac{10^{-5}}{p_7} = 0.000004774292225 = 4.774292225 \times 10^{-6} < 5 \times 10^{-6}$$

and from here, the approximation p_7 has at least 6 true (significant) digits. The exact value of p: $\mathbf{p = 2.0945514815423265915}$.

Solution 2 by using the $(\mathbf{m_1}, \mathbf{M_1})$ method. (a) $f(x) = x^3 - 2x - 5, f(2) = -1 < 0, f(3) = 16 > 0$ and by the IVT $f(x) = 0$ has a solution in $[2,3]$. In addition, $f^{'}(x) = 3x^2 - 2 > 0,\ x \in [2,3]$ hence, $f(x) = 0$ has a unique solution $p \in [2,3]$.

In addition:

$$f^{'}(x) = 3x^2 - 2 >, x \in [2,3]; \quad \max_{x\in[2,3]} f^{'}(x) = f^{'}(3) = 25; \quad \min_{x\in[2,3]} f^{'}(x) = f^{'}(2) = 10.$$

Define:

$$x = x - \frac{f(x)}{25}, \quad g_1(x) = x - \frac{f(x)}{25}.$$

Obviously, p is a fixed point for $g(x)$. In addition,

$$0 \leq g_1^{'}(x) = 1 - \frac{f^{'}(x)}{25} \leq 1 - \frac{10}{25} = 0.6,\ x \in [2,3]$$

hence, $g_1(x)$ is contractive in $[2,3]$ with $k = 0.6$. It follows also that

$$g_1 \ : \ [2,3] \to [2,3]]$$

and by Theorem 2.3 the sequence $p_{n+1} = g_1(p_n), n = 0, 1, 2, \ldots$ is convergent to p.
(b) According to Theorem 2.3, Estimate A) we have

$$|p_n - p| \leq (3-2)(0.6)^n < 10^{-5} \Rightarrow n > \frac{\ln(10^5)}{\ln(1/0.6)} = 22.5379 \Rightarrow \mathbf{n = 23}.$$

Computations: Starting with $p_0 = 2$ and performing 22 iterations we shall obtain p_{22} that is the desired approximation. **Note the fixed-point method $x = g_1(x)$ is slower convergent than the fixed-point method $x = g(x)$ given in Solution 1. However, applying first Bisectional method we can take smaller interval containing p that will increase the speed of convergence of g_1.**
(c) We have $g_1^{'}(x) \geq 0$ in $[2, 3]$. Hence, starting with $p_0 = 2$ we obtain

$$2 = p_0 < p_1 < p_2 < \cdots < p_{22} < p$$

and then an upper bound for the relative error:

$$\frac{|p_{22} - p|}{|p|} < \frac{10^{-5}}{p_{22}}.$$

Perform the computations as exercise.

The next problem is not in the assignment problems. It is given to exercise. More precisely, the solution uses the Modified Bisectional method.

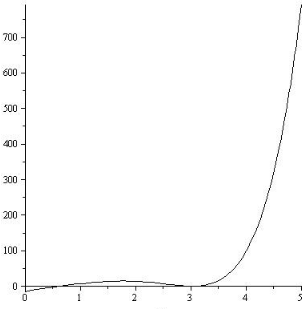
Problem*. Given the nonlinear equation:

$$f(x) = 0, \quad f(x) = -15.0851804866 + 34.4913399681x - 27.6648428963x^2$$
$$+25.0902699178x^3 - 11.6062706976x^4 + 1.7419344x^5.$$

(a). Use the Modified Bisectional method to approximate the solution p of $f(x) = 0$ located in the interval $[2, 5]$ with accuracy 10^{-2}.

(b) Use the interval location of the solution obtained in **(a)** to construct a convergent fixed-point method to approximate the solution p with accuracy 10^{-8}.
Hint: First, plot the graph of $f(x)$ on the given interval to see that the usual Bisectional method is not applicable. Then use the Modified Bisectional method.

Solution. Here is the plot of $f(x)$ in the interval $[0, 5]$.

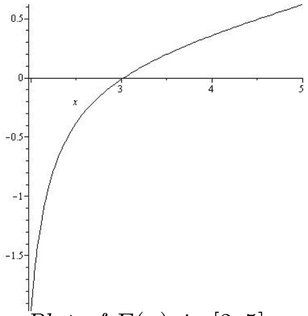


Plot of $f(x)$ *in* $[0,5]$.

The plot shows that the usual Bisectional method is not applicable. In fact p is a solution to $f(x) = 0$ of multiplicity 2. Then consider

$$F(x), \quad F(x) = \frac{f(x)}{f'(x)}, x \in [2.5].$$

Note that p is a solution to $F(x) = 0$ with multiplicity 1 (simple solution) and $F(x) < 0$ for $x < p$ and $F(x) > 0$ for $x > p$. Hence, Bisectional method is applicable for $F(x)$. In addition, the plot of $F(x)$ in $[2, 5]$ also shows that the Bisectional method is applicable for $F(x) = 0$ to approximate p:



Plot of $F(x)$ *in* $[2,5]$.

We start the Bisectional Method in $[2.5, 3.5]$: $F(2.5) = -0.377906673084 < 0$, $F(3.5) = 0.197874718952 > 0$, $(3.5 - 2.5)/2^n < 0.01$, $n > \ln(100)/\ln(2) = 6.64385618978$, $\mathbf{n = 7}$:

n	$a_n(-)$	$b_n(+)$	p_n	$F(p_n)$
1	2.5	3.5	3	-0.0120112091827
2	3	3.5	3.25	0.102500123588
3	3	3.25	3.125	0.0483031972085
4	3	3.125	3.0625	0.0190222945931
5	3	3.0625	3.03125	0.00374099763846
6	3	3.03125	3.015625	-0.00407401412003
7	3.015625	3.03125	3.0234375	-0.000151522529646

(b) From **(a)** the solution p is located in $[3.0234375, 3.03125]$. We observe that $F^{'}(x) > 0$ in $[3.0234375, 3.03125]$. We construct the (m_1, M_1) method. By using CAS MAPLE: *maximize*$(F^{'}(x), x = 3.0234375..3.03125)$;, *minimize*$(F^{'}(x), x = 3.0234375..3.03125)$; gives

$$\max_{x\in[3.0234375,3.03125]} F^{'}(x) = 0.500148728627; \quad \min_{x\in[3.0234375,3.03125]} F^{'}(x) = 0.496348308822.$$

$$x = x - \frac{F(x)}{\max_{x\in[3.0234375,3.03125]} F^{'}(x)}; x = x - \frac{F(x)}{0.500148728627}$$

Hence,

$$g(x) = x - \frac{F(x)}{0.500148728627}, \quad 0 \leq g^{'}(x) \leq 1 - \frac{0.496348308822}{0.500148728627} = 0.007598579357$$

$$\Rightarrow \mathbf{k} = \mathbf{0.007598579357}.$$

Hence, $g(x)$ is contractive with $\mathbf{k} = \mathbf{0.007598579357}$.
Now, $g^{'}(x) \geq 0$, $g(x)$ is contractive in $[3.0234375, 3.03125]$, and g has a fixed point in this interval. Then

$$g \,:\, [3.0234375, 3.03125] \to [3.0234375, 3.03125]\,.$$

By Theorem 2.3, with any p_0 in the interval $[3.0234375, 3.03125]$, the iterated sequence $p_{n+1} = g(p_n), n = 0, 1, 2, \ldots$ is convergent to p and

$$|p_n - p| \leq (3.03125 - 3.0234375)(0.007598579357)^n < 10^{-8}$$

$$\Rightarrow \; n > \frac{\ln[(3.03125 - 3.0234375)10^8]}{\ln(1/0.007598579357)} = 2.78057855456.$$

From here $\mathbf{n} = \mathbf{3}$. We compute with $p_0 = (3.0234375 + 3.03125)/2 = 3.02734375000$, $p_1 = g(p_0) = 3.02374792187$, $p_2 = g(p_1) = 3.02374050223$, $\mathbf{p_3} = \mathbf{g(p_2)} = \mathbf{3.02374050000}$.
Remark. The exact value of p is $p = 3.0237405$.