MATH-354 / MAST-334, Section AA, Fall 2018

Assignment 1, Solutions

Problem 11, page 15. (a) Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = (x-1)\ln(x)$ about $x_0 = 1$.
(b) Use $P_3(0.5)$ to approximate $f(0.5)$. Find an upper bound for the absolute approximation error $|f(0.5) - P_3(0.5)|$ using the error formula (Taylor's inequality) and compare it to the actual error.
(c) Apply Taylor's inequality to find an upper bound for the absolute approximation error $|f(x) - P - 3(x)|$ in using $P_3(x)$ to approximate $f(x)$ on the interval $[0.5, 1.5]$

Solution. (a) We compute $f(1) = 0$, $f^{'}(1) = 0$, $f^{''}(1) = 2$, $f^{(3)}(1) = -3$ and

$$P_3(x) = 0 + \frac{0}{1!}(x-1) + \frac{2}{2!}(x-1)^2 + \frac{-3}{3!}(x-1)^3$$

$$\mathbf{P_3(x) = (x-1)^2 - \frac{(x-1)^3}{2}} \quad \Rightarrow \quad \mathbf{P_3(x) = \frac{3}{2} - \frac{7}{2}x + \frac{5}{2}x^2 - \frac{1}{2}x^3}.$$

By using CAS MAPLE:

$$t3 := taylor((x-1)*ln(x), x = 1, 4): \quad convert(t3, polynom);$$

MAPLE returns:

$$(x-1)^2 - (1/2)*(x-1)^3$$

(b) $P_3(0.5) = 0.312500$ is an approximation to the actual value $f(0.5) = 0.3465735903$ with an actual absolute error $|f(0.5) - P_3(0.5)| = \mathbf{0.0340735903}$.
We apply the error formula

$$f(0.5) - P_3(0.5) = \frac{f^{(4)}(\xi)}{4!}(0.5-1)^4,$$

where ξ is between 0.5 and 1. The fourth derivative of $f(x)$

$$f^{(4)}(x) = 6x^{-4} + 2x^{-3}$$

is positive and decreasing on $[0.5, 1]$ and from here:

$$M_4 = \max_{x\in[0.5,1]} |f^{(4)}(x)| = f^{(4)}(0.5) = 6(2)^4 + 2(2)^3 = 112.$$

From here

$$|f(0.5) - P_3(0.5)| \le \frac{\max_{x\in[0.5,1]} |f^{(4)}(x)|}{4!}(0.5)^4$$

$$= \frac{M_4}{4!}(0.5)^4 = \frac{112}{4!}(0.5)^4 = \mathbf{0.291666667}.$$

Hence, **0.291666667** is an upper bound for the absolute error: $|f(0.5) - P_3(0.5)|$. As it was expected it is bigger than the actual upper bound **0.0340735903**.

(c) We apply the error formula:

$$f(x) - P_3(x) = \frac{f^{(4)}(\xi(x))}{4!}(x-1)^4,$$

where $\xi(x)$ is a number located between x and 1 and x is in the interval $[0.5, 1]$. The fourth derivative of $f(x)$

$$f^{(4)}(x) = 6x^{-4} + 2x^{-3}$$

is positive and decreasing on $[0.5, 1]$ and from here:

$$M_4 = \max_{x \in [0.5,1]} |f^{(4)}(x)| = f^{(4)}(0.5) = 6(2)^4 + 2(2)^3 = 112.$$

In view of this for all $x \in [0.5, 1.5]$ we have:

$$|f(x) - P_3(x)| \le \frac{\max_{x \in [0.5,1]} |f^{(4)}(x)|}{4!}[\max(1-0.5, 1.5-1)]^4$$

$$= \frac{112}{4!}(0.5)^4 = \frac{112}{4!}(0.5)^4 = \mathbf{0.291666667}$$

or equivalently

$$\max_{x \in [0.5,1.5]} |f(x) - P_3(x)| \le \mathbf{0.291666667}.$$

Problem A. Using Taylor polynomials find an approximant to $\int_{0.5}^{0.7} \sin(\cos(x^2))dx$ with an absolute error less than 10^{-2}.

Solution. Define

$$f(x) = \int_{0.5}^{x} \sin(\cos(t^2))dt.$$

We compute

$$f(0.5) = 0,\ f'(x) = \sin(\cos(x^2)), f'(0.5) = 0.824270418114272$$

$$f''(x) = -2x\sin(x^2)\cos(\cos(x^2)), f''(0.5) = -0.140079212384515$$

$$f^{(3)}(x) = -2\sin(x^2)\cos(\cos(x^2)) - 4x^2\cos(x^2)\cos(\cos(x^2)) - 4x^2\sin(x^2)^2\sin(\cos(x^2))\,.$$

Solution 1 by using the first Taylor polynomial centered at $x_0 = 0.5$:

$$P_1(x) = f(0.5) + \frac{f'(0.5)}{1!}(x - 0,5) = 0.824270418114272(x - 0.5)\,.$$

$$M_2 = \max_{x\in[0.5,0.7]} |f''(x)| = \max_{x\in[0.5,0.7]} 2x\sin(x^2)\cos(\cos(x^2))$$

$$= 2(0.7)\sin(0.7^2)\cos(\cos(0.7^2)) = \mathbf{0.418617935288276}\,.$$

Then

$$\left|\int_{0.5}^{0.7} \sin(\cos(x^2))dx - P_1(0.7)\right| = |f(0.7) - P_1(0.7)| \le \frac{M_2}{2!}(0.7 - 0.5)^2$$

$$= \frac{0.418617935288276}{2}(0.2)^2 = 0.837235870576552 \times 10^{-2} < 10^{-2}\,.$$

Hence,

$$P_1(0.7) = 0.824270418114272(0.7 - 0.5) = \mathbf{0.164854083622854}$$

is an approximant to the given definite integral with an absolute error less than 10^{-2}.

Solution 2 by using the second Taylor polynomial centered at $x_0 = 0.5$:

$$P_2(x) = f(0.5) + \frac{f'(0.5)}{1!}(x - 0.5) + \frac{f''(0.5)}{2!}(x - 0.5)^2$$

$$= 0.824270418114272(x - 0.5) + \frac{-0.140079212384515}{2}(x - 0.5)^2$$

$$= 0.824270418114272(x - 0.5) - 0.0700396061922575(x - 0.5)^2\,.$$

We compute applying the triangle inequality. For $x \in [0.5, 0.7]$ we have:

$$|f^{(3)}(x)| \le |2\sin(x^2)\cos(\cos(x^2))| + |4x^2\cos(x^2)\cos(\cos(x^2))| + |4x^2\sin(x^2)^2\sin(\cos(x^2))|$$

$$\le 2 + 4 \times 0.7^2 + 4 \times 0.7^2 = 5.92\,.$$

Remark. Better estimate can be found for M_3 by using the fact that sin is increasing in $[0.5, 0.7]$ and cos is decreasing in $[0.5, 0.7]$. However, it will be difficult and time-taking the optimal $M_3 = \max_{x\in[0.5,0.7]} |f^{(3)}(x)|$ to be found without CAS (maximize code; closed interval method by computng the critical numbers; or plotting). We continue with $M_3 = 5.92$:

$$\left|\int_{0.5}^{0.7} \sin(\cos(x^2))dx - P_2(0.7)\right| = |f(0.7) - P_2(0.7)| \le \frac{M_3}{3!}(0.7 - 0.5)^3$$

$$= \frac{5.92}{6}(0.2)^3 = 0.789333333333331 \times 10^{-2} < 10^{-2}\,.$$

Hence,

$P_2(0.7) = 0.824270418114272(0.7-0.5) - 0.0700396061922575(0.7-0.5)^2 = \mathbf{0.162052499375164}$

is an approximant to the given definite integral with an absolute error less than 10^{-2}.
Remark. The actual value of the integral with 15 true digits is **0.160569403351548**.

Problem 18, page 16. Let $f(x) = (1-x)^{-1}$ and $x_0 = 0$.
(a) Find the n-th Taylor polynomial $P_n(x)$ to $f(x)$ about x_0 (centered at x_0).
(b) Find a value of n such that $P_n(x)$ approximates $f(x)$ to within 10^{-6} for each x in the interval $[0, 0.5]$.

Solution. (a) We differentiate:

$$f(x) = (1-x)^{-1},\ f'(x) = (1-x)^{-2},\ f''(x) = 2!(1-x)^{-3},\ etc.$$

By induction:

$$f^{(k)}(x) = k!(1-x)^{-k-1} \quad \Rightarrow \quad f^{(k)}(0) = k!\ (0! = 1).$$

Then

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{k!}{k!} x^k = \sum_{k=0}^{n} x^k$$

or written in a bit different way

$$P_n(x) = 1 + x + x^2 + \cdots + x^{n-1} + x^n.$$

Another solution of **(a)**: We now from Calculus that the sum the geometric series is:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad -1 < x < 1$$

and

$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + x^{n+1} + x^{n+2} + \cdots$$

and from here the n-th Taylor polynomial

$$P_n(x) = \sum_{k=0}^{n} x^k$$

is simply the n-th partial sum of the Taylor series for $f(x) = 1/(1-x)$ about $x_0 = 0$.
(b) The solution of **(b)** is a bit tricky. Let us try first with the error formula:

$$|f(x) - P_n(x)| \le \frac{M_{n+1}}{(n+1)!} |x - 0|^{n+1}$$

$$M_{n+1} = \max_{x\in[0,0.5]} |f^{(n+1)}(x)| = \max_{x\in[0,0.5]} \left[(n+1)!(1-x)^{-n-2}\right] = (n+1)!2^{n+2}$$

and

$$|f(x) - P_n(x)| \leq \frac{(n+1)!2^{n+2}}{(n+1)!}|x|^{n+1} \leq \frac{(n+1)!2^{n+2}}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} = 2$$

and obviously we can not make 2 to be less than 10^{-6}. So, no solution based on Taylor's inequality.

Now by using the geometric series:

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \ \ -1 < x < 1$$

we conclude that

$$f(x) = P_n(x) + x^{n+1}\left[1 + x + x^2 + \cdots\right] = P_n(x) + \frac{x^{n+1}}{1-x}$$

and from here for each $x \in [0, 0.5]$ we have

$$|f(x) - P_n(x)| = \frac{x^{n+1}}{1-x} \leq \frac{(1/2)^{n+1}}{1-(1/2)} = \frac{1}{2^n} < 10^{-6}$$

and from here

$$2^n > 10^6 \Leftrightarrow n > \frac{\ln(10^6)}{\ln(2)}\frac{6\ln(10)}{\ln(2)} = 19.93$$

hence

$$\mathbf{n = 20}$$

and

$$\mathbf{P_{20}(x) = \sum_{k=0}^{20} x^k}$$

approximates $f(x)$ for each $x \in [0, 0.5]$ with accuracy 10^{-6} (with absolute error less than 10^{-6}).

Problem 20, page 16. Find the n-th MacLaurin polynomial $P_n(x)$ (Taylor polynomial centered at $x_0 = 0$) for the function $f(x) = \arctan(x)$.
(b) Find n such that

$$\max_{x\in[0,0.65]} |f(x) - P_n(x)| < 10^{-5}.$$

Hint: Use the Alternating Series Estimation Theorem.

Solution.

$$f(x) = \arctan(x), \quad f^{'}(x) = \frac{1}{1+x^2}.$$

Then by using the geometric series we obtain:

$$f^{'}(x) = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots + (-x^2)^n + \cdots$$

$$f^{'}(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

Integrating term by term the above series we obtain:

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots + C$$

that is

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots + C$$

In order to determine C we substitute in the above formula $x = 0$ to obtain:

$$0 = \arctan(0) = C$$

Hence,

$$P_{2n+1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1}$$

and in view of this: For **n odd**

$$\mathbf{P_n(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{(n-1)/2} \frac{x^n}{n}}$$

and for **n even** $\mathbf{P_n(x) = P_{n-1}(x)}$.

(b) The MacLaurin series of $f(x) = \arctan(x)$ is alternating series. Then, by using Alternating Series Estimation Theorem we obtain

$$|f(x) - P_{2n+1}(x)| \le \frac{|x|^{2n+3}}{2n+3} \le \frac{(0.65)^{2n+3}}{2n+3} = r(2n+3) < 10^{-5}.$$

We compute:

$$r(19) = 0.00001467574562 > 10^{-5}, \; r(21) = 0.5609978471 \times 10^{-5} < 10^{-5}$$

hence, $\mathbf{n = 21}$ and $\mathbf{P_{21}(x)}$ approximates $f(x)$ on the interval $[0, 0.65]$ with accuracy 10^{-5} (with an absolute error less than 10^{-5}).

Problem 22, page 16. The n-th Taylor polynomial $P_n(x)$ for $f(x)$ at x_0 is sometimes referred to as *the polynomial of degree at most n that best approximates the function $f(x)$ near x_0.*
(a) Explain why this description is accurate.
(b) Find a quadratic polynomial that best approximates a function $f(x)$ near $x_0 = 1$ if the tangent line to the graph of $f(x)$ at $x_0 = 1$ has an equation $y = 4x - 1$ and $f''(1) = 6$.

Solution. (a) Explanation 1. The n-th Taylor polynomial $P_n(x)$ for a sufficiently smooth function $f(x)$ about x_0 is the unique polynomial of degree at most n that interpolates f and its derivatives up to order n at x_0, i.e.:

$$P_n^{(k)}(x_0) \;=\; f^{(k)}(x_0), \;\; k = 0, 1, \ldots, n; \;\; (f^{(0)} = f). \qquad (*)$$

Hence, the shape of f is approximated by the shape of P_n in a best possible way near x_0 if we approximate f by polynomials of degree $\leq n$. From here, the approximation of by $P_n(x)$ will be the best possible near x_0 if we compare with all other polynomials of degree $\leq n$.

Explanation 2. We know that if $f(x)$ is $(n+1)$-time differentiable in $[x_0-h, x_0+h]$, $h > 0$, then

$$|f(x) - P_n(x)| \leq K|x - x_0|^{n+1}, \;\; x \in [x_0 - h, x_0 + h], , h > 0$$

where K does not depend on x and x_0. Suppose that $Q_n(x)$ is a polynomial of degree at most n such that

$$|f(x) - Q_n(x)| \leq K|x - x_0|^{n+1}, \;\; x \in [x_0 - h, x_0 + h], h > 0 \qquad (**)$$

Take the limit from both sides when $x \to x_0$ in (**)and we obtain that

$$\lim_{x \to x_0} |f(x) - Q_n(x)| = |f(x_0) - Q_n(x_0)| = 0$$

and from here $f(x_0) = Q_n(x_0)$.
Divide both sides (**) by $|x - x_0|$ and take again the limit when $x \to x_0$:

$$\lim_{x \to x_0} \left| \frac{f(x) - Q_n(x)}{x - x_0} \right| \leq \lim_{x \to x_0} K|x - x_0|^n = 0$$

and from here

$$\lim_{x \to x_0} \frac{f(x) - Q_n(x)}{x - x_0} = 0$$

By L'Hospital's rule:

$$\lim_{x \to x_0} \frac{f(x) - Q_n(x)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x) - Q_n'(x)}{1}$$

$$= f^{'}(x_0) - Q_n^{'}(x_0) = 0 \Rightarrow f^{'}(x_0) = {Q_n}'(x_0).$$

Analogously, dividing (**) by $(x-x_0)^2$ and taking the limit when $x \to x_0$ by using L'Hospital rule, we obtain: $f^{''}(x_0) = Q_n^{''}(x_0)$.
This procedure can be performed till a division of (**) by $|x-x_0|^n$ to obtain the interpolation condition $f^{(n)}(x_0) = Q_n^{(n)}(x_0)$. In view of this, $Q_n(x)$ satisfies the $(n+1)$ interpolation conditions (*):

$$Q_n^{(k)}(x_0) \ = \ f^{(k)}(x_0), \ \ k = 0, 1, \ldots, n.$$

The n-th Taylor polynomial $P_n(x)$ is uniquely determined by the $(n+1)$ interpolation conditions (*) and in view of this:

$$Q_n(x) = P_n(x).$$

Hence, the unique polynomial of degree $\leq n$ such that it approximates $f(x)$ near x_0 with order: $O\left[(x-x_0)^{n+1}\right]$ (or bigger order: $O\left[(x-x_0)^m\right], \ m \geq n+1$) is the n-th Taylor polynomial $P_n(x)$ for $f(x)$ about x_0 (centered at x_0).
Remark. In the solution of **Problem 22, (a)** given above we suppose that the function $f(x)$ is sufficiently smooth in an interval containing x_0.

(b) This is the second Taylor polynomial $P_2(x)$ for $f(x)$ about $x_0 = 1$. The first Taylor polynomial for $f(x)$ at $x_0 = 1$ is the tangent line to the graph of $f(x)$ at $x_0 = 1$:

$$P_1(x) = 4x - 3 = 3 + \frac{4}{1!}(x-1), \ f(1) = 3, \ f^{'}(1) = 4.$$

By using the recursive formula for Taylor polynomials we obtain:

$$P_2(x) = P_1(x) + \frac{f^{''}(x)}{2!}(x-1)^2 = 3 + \frac{4}{1!}(x-1) + \frac{6}{2!}(x-1)^2$$

that is

$$\mathbf{P_2(x) = 3 + 4(x-1) + 3(x-1)^2} \quad \Rightarrow \quad \mathbf{P_2(x) = 2 - 2x + 3x^2}.$$

Exercise problems, not in the assignment.

Problem B. The function $y(x)$ satisfies the differential equation:

$$y^{'}(x) \ - \ 2x\,[y(x)]^2 \ = \ 0$$

and the interpolation condition (initial value condition) $y(0) = 1$. Construct the fourth Taylor polynomial approximant $P_4(x)$ to $f(x)$ about $x_0 = 0$.
Hint: Use implicit differentiation.

Solution. With $x = 0$ in the given differential equation we obtain:

$$y^{'}(0) \,-\, 2(0)\,[y(0)]^2 \,=\, 0$$

$$y^{'}(0) \,-\, 2(0)\,[1]^2 \,=\, 0 \quad \Rightarrow \quad y^{'}(0) = 0.$$

Let us differentiate the given equation:

$$y^{''}(x) \,-\, 2\,[y(x)]^2 \,-2\,x\,[2y(x)y^{'}(x)] =\, 0$$

and substituting $x = 0$ we obtain:

$$y^{''}(0) \,-\, 2\,[y(0)]^2 \,-2\,(0)\,[2y(0)y^{'}(0)] =\, 0$$

$$y^{''}(0) \,-\, 2\,[1]^2 \,-2\,(0)\,[2(1)(0)] \,=\, 0 \quad \Rightarrow \quad y^{''}(0) = 2.$$

Now we differentiate the differential equation:

$$y^{''}(x) \,-\, 2\,[y(x)]^2 \,-4\,x\,y(x)\,y^{'}(x) =\, 0$$

$$y^{(3)}(x) \,-\, 4\,y(x)\,y^{'}(x) \,-4\,y(x)\,y^{'}(x) - 4\,x\,[y^{'}(x)]^2 - 4\,x\,y(x)\,y^{''}(x) =\, 0$$

and with $x = 0$ we obtain:

$$y^{(3)}(0) \,-\, 4\,y(0)\,y^{'}(0) \,-4\,y(0)\,y^{'}(0) - 4\,(0)\,[y^{'}(0)]^2 - 4\,(0)\,y(0)\,y^{''}(0) =\, 0$$

that is:

$$y^{(3)}(0) \,-\, 4\,(1)\,(0) \,-4\,(1)\,(0) - 4\,(0)\,(0)^2 - 4\,(0)\,(1)\,(2) \,=\, 0 \quad \Rightarrow \quad y^{(3)}(0) = 0.$$

Now we differentiate the differential equation:

$$y^{(3)}(x) \,-\, 8\,y(x)\,y^{'}(x) \,-4\,x\,[y^{'}(x)]^2 - 4\,x\,y(x)\,y^{''}(x) =\, 0$$

to obtain:

$$y^{(4)}(x) \,-\, 8\,y(x)\,y^{''}(x) \,-\, 8\,[y^{'}(x)]^2 - 4\,[y^{'}(x)]^2$$

$$-8\,x\,y^{'}(x)y^{''}(x) - 4\,y(x)\,y^{''}(x) - 4\,x\,y^{'}(x)\,y^{''}(x) - 4\,x\,y(x)\,y^{(3)}(x) =\, 0$$

and with $x = 0$ we obtain:

$$y^{(4)}(0)(x) \,-\, 8\,y(0)\,y^{''}(0) \,-\, 8\,[y^{'}(0)]^2 - 4\,[y^{'}(0)]^2 - 8\,(0)\,y^{'}(0)y^{''}(0) - 4\,y(0)\,y^{''}(0)$$

$$-4\,(0)\,y^{'}(0)\,y^{''}(0) - 4\,(0)\,y(0)\,y^{(3)}(0) =\, 0.$$

Taking into account that $y(0) = 1$, $y^{'}(0) = 0$, $y^{''}(0) = 2$, $y^{(3)}(0) = 0$ we obtain

$$y^{(4)}(0) = 8\,y(0)\,y^{''}(0) + 4\,y(0)\,y^{''}(0) = 12(1)(2) = 24.$$

Finally,

$$P_4(x) = y(0) + \frac{y^{'}(0)}{1!}x + \frac{y^{''}(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4$$

and taking into account that

$$y(0) = 1,\ y^{'}(0) = 0,\ y^{''}(0) = 2,\ y^{(3)}(0) = 0,\ y^{(4)}(0) = 24$$

we obtain:

$$P_4(x) = 1 + \frac{0}{1!}x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 + x^2 + x^4$$

$$\Rightarrow \quad \mathbf{P_4(x) = 1 + x^2 + x^4}.$$

Remark (not in the assignment). The exact solution $y(x)$ of the differential equation $y^{'}(x) - 2x\,[y(x)]^2 = 0$ with the initial value condition $y(0) = 1$ is

$$y(x) = \frac{1}{1 - x^2} = 1 + x^2 + x^4 + x^6 + \cdots,\ -1 < x < 1$$

and from here

$$|y(x) - P_4(x)| \leq \frac{x^6}{1 - x^2},\ -1 < x < 1.$$

Problem C. Given a two times continuously differentiable function $f(x)$ defined on the interval $[a, b]$. Let $x_0 \in (a, b)$. Suppose that the first degree polynomial $Q_1(x)$ approximates $f(x)$ on the interval $[a, b]$ with an upper-bound for the approximation error at each $x \in [a, b]$ given by:

$$|f(x) - Q_1(x)| \leq K\,(x - x_0)^2, \qquad (***)$$

where the constant K does not depend on x.
Then, show that $Q_1(x)$ is in fact the first Taylor polynomial $P_1(x)$ for the function $f(x)$ about $x = x_0$ (centered at $x = x_0$), i.e.,

$$Q_1(x) = f(x_0) + \frac{f^{'}(x_0)}{1!}(x - x_0).$$

Solution. Take the limit in (***) when $x \to x_0$:

$$0 \leq \lim_{x \to x_0} |f(x) - Q_1(x)| \leq K \lim_{x \to x_0} (x - x_0)^2 = 0.$$

By the Squeeze Theorem (Sandwich Theorem) from Calculus:

$$\lim_{x \to x_0} |f(x) - Q_1(x)| = 0$$

and obviously,

$$\lim_{x \to x_0} (f(x) - Q_1(x)) = f(x_0) - Q_1(x_0) = 0 \Rightarrow \mathbf{Q_1(x_0) = f(x_0)}.$$

Next, assuming $x \neq x_0$ divide both sides of (*) by $|x - x_0|$.

$$0 \leq \left| \frac{f(x) - Q_1(x)}{x - x_0} \right| \leq K\,|x - x_0|$$

and by the Squeeze Theorem, taking the limit in the above inequalities we obtain:

$$\lim_{x \to x_0} \left| \frac{f(x) - Q_1(x)}{x - x_0} \right| = 0$$

that is equivalent to

$$\lim_{x \to x_0} \frac{f(x) - Q_1(x)}{x - x_0} = 0.$$

Because $Q_1(x_0) = f(x_0)$ the above expression is L'Hospital form $0/0$ at x_0. Hence, by using L'Hospital rule:

$$\lim_{x \to x_0} \frac{f(x) - Q_1(x)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x) - Q_1'(x)}{1} = 0.$$

However, $f'(x) - Q_1'(x)$ is continuous at $x = x_0$ hence,

$$\lim_{x \to x_0} \frac{f'(x) - Q_1'(x)}{1} = f'(x_0) - Q_1'(x_0) = 0 \Rightarrow \mathbf{Q_1'(x_0) = f'(x_0)}.$$

Then, by using the interpolation relations:

$$\mathbf{f(x_0) = Q_1(x_0),\ f'(x_0) = Q'(x_0)}$$

we obtain

$$\mathbf{Q_1(x) = Q_1(x_0) + \frac{Q_1'(x_0)}{1!}(x - x_0) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) = P_1(x)}$$

$$\mathbf{Q_1(x) = P_1(x)},$$

where $P_1(x)$ is the first Taylor polynomial for $f(x)$ about $x = x_0$.