Exam version: 11

Part A: Provide your full solution on these pages. Write your work in a neat and organized format, and take the time to fully justify your steps. Do not simplify your answer.

Question 1 (5 points) The radius of a sphere is increasing at a constant rate. Recall that the surface area of a sphere is given by $S = 4\pi r^2$.

(a) (2 points) Find an equation that relates $\frac{dS}{dt}$ and $\frac{dr}{dt}$.

Differentiate $S = 4\pi r^2$ with respect to t:

$$\frac{dS}{dt} = \underbrace{4\pi \cdot 2r \cdot \frac{dr}{dt}}_{\text{chain rule}} = 8\pi r \frac{dr}{dt}$$

1 pt for correct use of implicit diff.
1 pt. for answer

(b) (3 points) Assuming the radius of the sphere is initially 0m and is increasing at a rate of 2m/s, find the rate at which the surface area is expanding when $S = 16\pi$ m^2.

We are given $\frac{dr}{dt} = 2$. Need r when $S = 16\pi$:

$$16\pi = 4\pi r^2 \iff r^2 = 4$$
$$\iff r = 2 \quad \text{since } r \geq 0.$$

1 pt.

Thus $\frac{dS}{dt} = 8\pi \cdot \underset{r}{(2)} \cdot \underset{\frac{dr}{dt}}{(2)} = 32\pi$ m^2/s.

2pts.

Exam version: 11

Question 2 (5 points) Let $f(x) = \ln(x^2 + 2x + 2)$. Note $x^2+2x+2>0$.

(a) (2 points) Calculate all singular points (if any) for f and all critical points (if any) for f.

$$f'(x) = \frac{2x+2}{x^2+2x+2} = 0 \iff 2x+2=0 \iff x=-1.$$

to find critical points

f' always exists, so there are no singular points.

Singular: none
Critical: $x=-1$

1 point for correct derivative.
0.5 point each for singular / critical

(b) (3 points) Determine the absolute minimum for f on the closed interval $[-2, 2]$.

Closed interval method:

Evaluate at endpoints

$f(2) = \ln(4+4+2) = \ln(10) > 0$

$f(-2) = \ln(4-4+2) = \ln(2) > 0$

Evaluate at critical/singular points within the desired interval

$f(-1) = \ln(1-2+2) = \ln 1 = 0$

2 points

The absolute min for f on $[-2,2]$ is $f(-1)=0$.

1 point.

"f has absolute min at $x=-1$" is also acceptable.

Exam version: 11

Part B: Choose the best answer to each of the following questions. Fill in the appropriate circle on the scantron sheet with a pencil AND circle your answer in the booklet. You may keep this booklet when the exam concludes. There are 12 problems.

1. If $f(x) = \arctan(2x)$, then

   (a) $f'(x) = \frac{1}{1+4x^2}$.
   (b) $f'(x) = \frac{2}{1+4x^2}$.
   (c) $f'(x) = \frac{1}{1+2x^2}$.
   (d) $f'(x) = \frac{2}{1+2x^2}$.
   (e) $f'(x) = \frac{8x}{1+4x^2}$.

$f'(x) = \frac{1}{1+(2x)^2} \cdot (2x)' = \frac{2}{1+4x^2}$

2. If $f(x) = (x^2+1)^x$ then

   (a) $f'(x) = x(x^2+1)^{x-1}$.
   (b) $f'(x) = \ln(x^2+1)(x^2+1)^x$.
   (c) $f'(x) = \left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)$.
   (d) $f'(x) = (x^2+1)^x \left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)$.
   (e) $f'(x) = (x^2+1)^x \left(\frac{2x^2}{x^2+1}\right)$.

$\ln(f(x)) = x\ln(x^2+1)$

$\Leftrightarrow \frac{f'(x)}{f(x)} = x \cdot \frac{2x}{x^2+1} + \ln(x^2+1)$

$\Leftrightarrow f'(x) = (x^2+1)^x \left[\frac{2x^2}{x^2+1} + \ln(x^2+1)\right]$

Exam version: 11

3. Suppose f is differentiable at every x in $\mathbb{R}$. Which of the following statements is true?

   (a) If f has a local maximum at $x = c$ then $f'(c) = 0$.

   ~~(b)~~ If f has a local maximum at $x = c$ then $f'(c)$ does not exist.

   ~~(c)~~ If $f'(c) = 0$, then f has a local maximum or local minimum at $x = c$. No. $f(x) = x^3$ at $x = 0$

   ~~(d)~~ If f has a local maximum at $x = c$, then $f(c)$ is the absolute maximum for f.

   ~~(e)~~ None of the above statements are true.

4. Let $f(x) = \frac{1}{x+1}$. The linear approximation for f at $a = 1$ is given by which of the following?

   (a) $L(x) = \frac{1}{2} - \frac{1}{4}(x - 1)$.

   (b) $L(x) = \frac{1}{2} - \frac{1}{2}(x - 1)$.

   (c) $L(x) = \frac{1}{2} + \frac{1}{4}(x - 1)$.

   (d) $L(x) = \frac{1}{2} + \frac{1}{2}(x - 1)$.

   (e) None of the above.

$$f(1) = \frac{1}{2}$$

$$f'(x) = \frac{-1}{(x+1)^2}$$

$$f'(1) = \frac{-1}{2^2} = -\frac{1}{4}$$

Exam version: 11

5. Let $T_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ denote the third degree Taylor polynomial for $f(x) = \sin x$ about the point $a = 0$ (also called the Maclaurin polynomial of degree 3).

   Which of the following is equal to a_3?

   (a) 0.
   (b) $\frac{1}{6}$.
   (c) $\frac{1}{3}$.
   (d) -$\frac{1}{6}$.
   (e) -$\frac{1}{3}$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

$f'''(0) = -1$

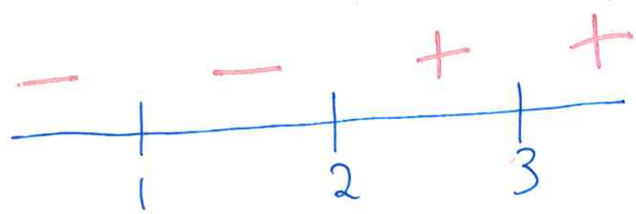
$a_3 = \frac{-1}{3!} = -\frac{1}{6}.$

6. Suppose f is a differentiable function and

$$f'(x) = (x-1)^2(x-2)(x-3)^4.$$

   Which of the following statements is true?

   ~~(a)~~ f is not decreasing on any interval.
   ~~(b)~~ f is not increasing on any interval.
   ~~(c)~~ f is increasing on the open interval $(1, 2)$.
   (d) f is increasing on the open interval $(2, 3)$.
   ~~(e)~~ None of the above statements are true.



Exam version: 11

7. Suppose $f(x) = \sqrt[3]{x^3+1}$. Which of the following statements is true?

   (a) f has one critical point and one singular point.
   (b) f has two distinct critical points and no singular points.
   (c) f has two distinct singular points and no critical points.
   (d) f has more than one critical point.
   (e) f has more than one singular point.

$$f'(x) = \frac{3x^2}{3\cdot(x^3+1)^{2/3}}$$

$f'(-1)$ DNE $\Rightarrow x=-1$ is singular

$f'(x)=0 \iff 3x^2=0 \iff x=0$ is critical

8. The total cost required to build a large cylindrical tank is given by

$$C = \pi r^2 + 4\pi r h,$$

where r is the radius of the tank in metres and h is the height of the tank in metres. If the total volume required is 4π m^3, what radius will minimize the total cost? The volume of a cylinder is given by $V = \pi r^2 h$.

   (a) $r = 1$m.
   (b) $r = \frac{3}{2}$m.
   (c) $r = 2$m.
   (d) $r = 8$m.
   (e) $r = 16$m.

We have the constraint $4\pi = \pi r^2 h$

$\Rightarrow \frac{4}{r^2} = h$.

Thus, $C = \pi r^2 + 4\pi r \cdot \frac{4}{r^2}$

$= \pi r^2 + \frac{16\pi}{r}$

$C'(r) = 2\pi r - \frac{16\pi}{r^2} = 0$

$\iff r^3 = \frac{16\pi}{2\pi} \iff r = 2.$

	$2\pi r - \frac{16\pi}{r^2}$
$0<r<2$	$-$
$r>2$	$+$

So C has an abs. min at $r=2$ by general interval method

Exam version: 11

9. The limit

$$\lim_{x\to 0} \frac{x^2 - \sin(2x)}{\sin(x) + \sin(x^2)}$$ " $\frac{0}{0}$ "

equals

(a) 0.
(b) $-\frac{1}{2}$.
(c) -1.
(d) -2.
(e) This limit does not exist.

$$\overset{l'H}{=} \lim_{x\to 0} \frac{2x - 2\cos(2x)}{\cos(x) + 2x\cos(x^2)}$$

$$= \frac{0-2}{1+0} = -2$$

10. Suppose A and B are two quantities depending on time t and that they are related by the equation

$$A^2 + 4AB = 12.$$

If $\frac{dA}{dt} = 3$, determine $\frac{dB}{dt}$ when $A = 2$.

(a) $\frac{dB}{dt} = 3$.
(b) $\frac{dB}{dt} = 2$.
(c) $\frac{dB}{dt} = -1$.
(d) $\frac{dB}{dt} = -2$.
(e) $\frac{dB}{dt} = -3$.

When $A=2$:
$4 + 8B = 12$
$\Rightarrow B = 1$

$$2A\cdot\frac{dA}{dt} + \overbrace{4A\cdot\frac{dB}{dt} + 4\frac{dA}{dt}B}^{\text{product rule}} = 0$$

$$\Leftrightarrow 2A\cdot 3 + 4A\cdot\frac{dB}{dt} + 4\cdot 3\,B = 0$$

Now sub in $A=2, B=1$:

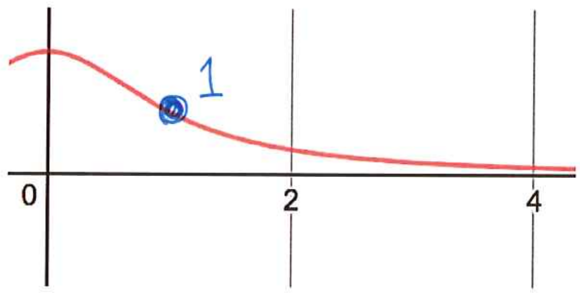
$$12 + 8\cdot\frac{dB}{dt} + 12 = 0$$

$$\Leftrightarrow \frac{dB}{dt} = -\frac{24}{8} = -3$$

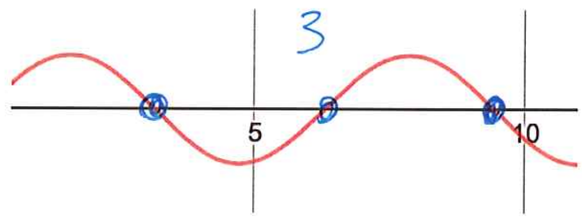
Exam version: 11

11. The following four images are graphs of twice differentiable functions. In which of the following graphs are exactly two distinct inflection points visible?

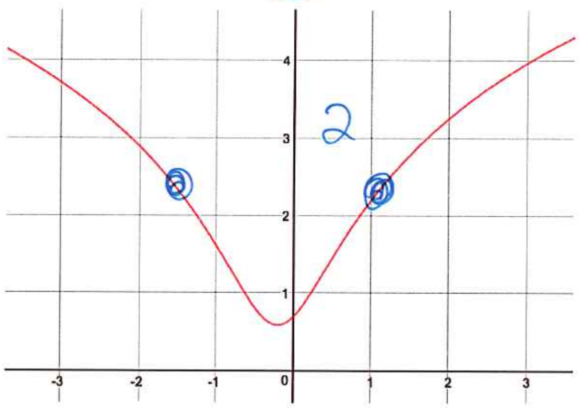
(a)



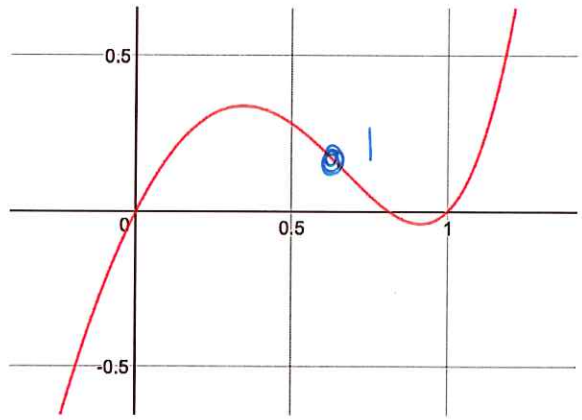
(b)



(c)



(d)



12. The limit $\lim_{x\to 0^+} \frac{\arctan(x)}{\sqrt{x}}$ is equal to

(a) 0.
(b) 1.
(c) $\frac{\pi}{2}$.
(d) ∞.
(e) This limit does not exist.

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x^2}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{2\sqrt{x}}{1+x^2} = \frac{0}{1+0} = 0$$

Exam version: 33

Part A: Provide your full solution on these pages. Write your work in a neat and organized format, and take the time to fully justify your steps. Do not simplify your answer.

Question 1 (5 points) The radius of a sphere is increasing at a constant rate. Recall that the surface area of a sphere is given by $S = 4\pi r^2$.

(a) (2 points) Find an equation that relates $\frac{dS}{dt}$ and $\frac{dr}{dt}$.

Differentiate $S = 4\pi r^2$ wrt t :

$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

1 pt for correct implicit diff

1 pt. for answer.

(b) (3 points) Assuming the radius of the sphere is initially 0m and is increasing at a rate of 5m/s, find the rate at which the surface area is expanding when $S = 4\pi$ m^2.

We must find r when $S = 4\pi$: $4\pi r^2 = 4\pi$

$\iff r^2 = 1$

$\iff r = 1$ (since $r > 0$)

1 point.

We are given $\frac{dr}{dt} = 5$

Thus, $\frac{dS}{dt} = 8\pi \cdot 1 \cdot 5$

$= 40\pi$ m^2/s.

2 points

Exam version: 33

Question 2 (5 points) Let $f(x) = \ln(x^2 + 4x + 5)$.

(a) (2 points) Calculate all singular points (if any) for f and all critical points (if any) for f.

$$f'(x) = \frac{2x+4}{x^2+4x+5}$$ 1 point.

Since x^2+4x+5, there are no singular points. 0.5

Set $f'(x)=0$ to find critical points:

$$\iff 2x+4=0$$

$$\iff x=-2$$

0.5

(b) (3 points) Determine the absolute minimum for f on the closed interval $[-4, 0]$.

Closed interval method:

check endpoints

$$f(-4) = \ln(16-16+5) = \ln 5$$

$$f(0) = \ln(0+0+5) = \ln 5$$

2 points

check critical/sing. points in the desired interval

$$f(-2) = \ln(4-8+5) = \ln 1 = 0$$

this is the minimum

1 point for stating the min (either the x or y coordinate)

Exam version: 33

Part B: Choose the best answer to each of the following questions. Fill in the appropriate circle on the scantron sheet with a pencil AND circle your answer in the booklet. You may keep this booklet when the exam concludes. There are 12 problems.

1. If $f(x) = \arctan(3x)$, then

   (a) $f'(x) = \frac{3}{1+3x^2}$.

   (b) $f'(x) = \frac{1}{1+9x^2}$.

   (c) $f'(x) = \frac{3}{1+9x^2}$.

   (d) $f'(x) = \frac{1}{1+3x^2}$.

   (e) $f'(x) = \frac{18x}{1+9x^2}$.

$$f'(x) = \frac{(3x)'}{1+(3x)^2} = \frac{3}{1+9x^2}$$

2. If $f(x) = (x^2+4)^x$ then

   (a) $f'(x) = x(x^2+4)^{x-1}$.

   (b) $f'(x) = \ln(x^2+4)(x^2+4)^x$.

   (c) $f'(x) = \ln(x^2+4) + \frac{2x^2}{x^2+4}$.

   (d) $f'(x) = (x^2+4)^x \left(\ln(x^2+4) + \frac{2x^2}{x^2+4}\right)$.

   (e) $f'(x) = (x^2+4)^x \left(\frac{2x^2}{x^2+4}\right)$.

Logarithmic:

$$\ln(f(x)) = x\ln(x^2+4)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln(x^2+4) + \frac{x \cdot 2x}{x^2+4}$$

Exam version: 33

3. Suppose f is differentiable at every x in $\mathbb{R}$. Which of the following statements is true?

   ~~(a)~~ If $f'(c) = 0$, then f has a local maximum or local minimum at $x = c$.

   ~~(b)~~ If f has a local maximum at $x = c$, then $f(c)$ is the absolute maximum for f.

   ~~(c) None of the above statements are true.~~

   c (d) If f has a local minimum at $x = c$ then $f'(c) = 0$.

   d (e) If f has a local minimum at $x = c$ then $f'(c)$ does not exist.

   (e) None of the above

4. Let $f(x) = \frac{1}{x+1}$. The linear approximation for f at $a = 2$ is given by which of the following?

   (a) $L(x) = \frac{1}{3} + \frac{1}{9}(x-2)$.

   (b) $L(x) = \frac{1}{3} + \frac{1}{3}(x-2)$.

   (c) $L(x) = \frac{1}{3} - \frac{1}{9}(x-2)$.

   (d) $L(x) = \frac{1}{3} - \frac{1}{3}(x-2)$.

   (e) None of the above.

$f(2) = \frac{1}{3}$

$f'(x) = -\frac{1}{(x+1)^2}$

$f'(2) = -\frac{1}{3^2} = \frac{-1}{9}$

Exam version: 33

5. Let $T_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ denote the third degree Taylor polynomial for $f(x) = \sin x$ about the point $a = 0$ (also called the Maclaurin polynomial of degree 3).

   Which of the following is equal to a_3?

   (a) 0.
   (b) $\frac{1}{6}$.
   (c) $\frac{1}{3}$.
   (d) -$\frac{1}{6}$.
   (e) -$\frac{1}{3}$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f'''(0) = -1$

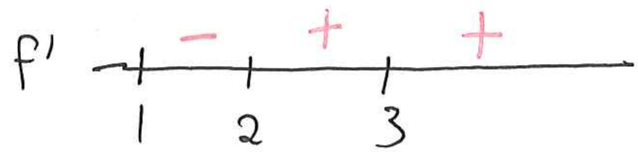
$a_3 = \frac{f'''(0)}{3!} = -\frac{1}{6}$

6. Suppose f is a differentiable function and

$$f'(x) = (x-1)(x-2)(x-3)^4.$$

   Which of the following statements is true?

   ~~(a)~~ f is not decreasing on any interval.
   ~~(b)~~ f is not increasing on any interval.
   ~~(c)~~ f is increasing on the open interval $(1, 2)$.
   (d) f is increasing on the open interval $(2, 3)$.
   ~~(e)~~ None of the above statements are true.



Exam version: 33

7. Suppose $f(x) = \sqrt[3]{x^3+1}$. Which of the following statements is true?

   (a) f has more than one critical point.

   (b) f has more than one singular point.

   (c) f has one critical point and one singular point.

   (d) f has two distinct critical points and no singular points.

   (e) f has two distinct singular points and no critical points.

$$f'(x) = \frac{3x^2}{3\sqrt{(x^3+1)^{2/3}}}$$

Critical: $x=0$

singular: $x=-1$

8. The total cost required to build a large cylindrical tank is given by

$$C = \pi r^2 + 4\pi r h,$$

where r is the radius of the tank in metres and h is the height of the tank in metres. If the total volume required is 4π m^3, what radius will minimize the total cost? The volume of a cylinder is given by $V = \pi r^2 h$.

   (a) $r = 1$m.

   (b) $r = \frac{3}{2}$m.

   (c) $r = 2$m.

   (d) $r = 4$m.

   (e) $r = 8$m.

The constraint is $4\pi = \pi r^2 h \Rightarrow h = \frac{4}{r^2}$

So $C = \pi r^2 + 4\pi r \cdot \frac{4}{r^2}$

$= \pi r^2 + \frac{16\pi}{r}$

$C'(r) = 2\pi r - \frac{16\pi}{r^2} = 0$

$\Rightarrow r = 2$

	$c'(r)$
$0<r<2$	$-$
$r>2$	$+$

absolute min at $r=2$ by genral interval method

Exam version: 33

9. The limit

$$\lim_{x \to 0} \frac{\sin(x^2 + 2x)}{\sin(x^2 - 3x)}$$

equals

(a) 0.
(b) $-\frac{2}{3}$.
(c) 1
(d) $\frac{3}{4}$.
(e) This limit does not exist.

$$\overset{l'H}{=} \lim_{x \to 0} \frac{(2x+2)\cos(x^2+2x)}{(2x-3)\cos(x^2-3x)} \qquad \to 1, \ \to 1$$

$$= \frac{2}{-3}$$

10. Suppose A and B are two quantities depending on time t and that they are related by the equation

$$A^2 + 4AB = 12.$$

If $\frac{dA}{dt} = 4$, determine $\frac{dB}{dt}$ when $A = 2$.

(a) $\frac{dB}{dt} = 4$.
(b) $\frac{dB}{dt} = 2$.
(c) $\frac{dB}{dt} = -1$.
(d) $\frac{dB}{dt} = -2$.
(e) $\frac{dB}{dt} = -4$.

Note when $A=2$: $4 + 8B = 12$, so $B = 1$.

We have $\frac{dA}{dt} \cdot 2A + 4A\frac{dB}{dt} + 4\frac{dA}{dt}B = 0$

$$\iff 4 \cdot 4 + 4 \cdot 2 \cdot \frac{dB}{dt} + 4 \cdot 4 \cdot 1 = 0$$

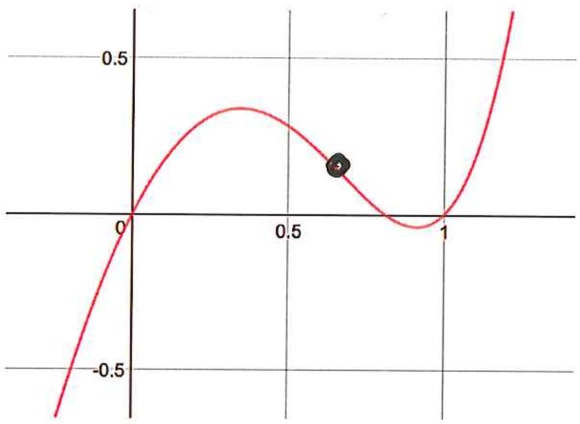
$$\iff 8 \cdot \frac{dB}{dt} = -32$$

$$\iff \frac{dB}{dt} = -4$$

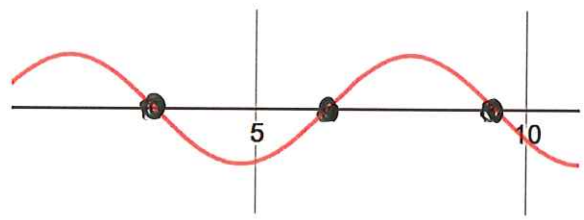
Exam version: 33

11. The following four images are graphs of twice differentiable functions. In which of the following graphs are exactly two distinct inflection points visible?

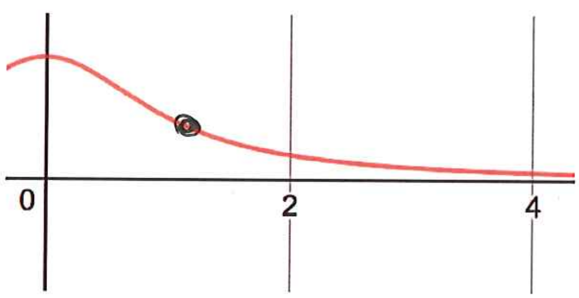
(a)



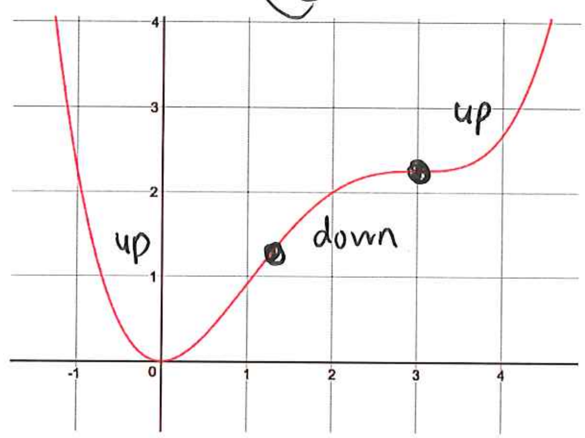
(b)



(c)



(d)



12. The limit $\lim_{x\to 0^+} \frac{\arcsin(x)}{\sqrt{x}}$ is equal to

(a) 0

(b) 1.

(c) $\frac{\pi}{2}$.

(d) ∞.

(e) This limit does not exist.

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{2\sqrt{x}}{\sqrt{1-x^2}} = \frac{0}{\sqrt{1-0}} = 0$$

Exam version: 55

Part A: Provide your full solution on these pages. Write your work in a neat and organized format, and take the time to fully justify your steps. Do not simplify your answer.

Question 1 (5 points) The radius of a sphere is increasing at a constant rate. Recall that the surface area of a sphere is given by $S = 4\pi r^2$.

(a) (2 points) Find an equation that relates $\frac{dS}{dt}$ and $\frac{dr}{dt}$.

$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

1 point for correct use of implicit diff.

1 point for answer

(b) (3 points) Assuming the radius of the sphere is initially 0m and is increasing at a rate of 3m/s, find the rate at which the surface area is expanding when $S = 16\pi$ m^2.

Have: $\frac{dr}{dt} = 3$m/s

$$\left.\begin{array}{l}\text{Need to know } r \text{ when } S = 16\pi: \\ 16\pi = 4\pi r^2 \\ \iff r^2 = 4 \\ \iff r = 2 \text{ (since } r > 0)\end{array}\right\} \text{1 point}$$

$$\underbrace{\text{Then } \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = 8\pi \cdot 2 \cdot 3 = 48\pi \text{ m}^2/\text{s}.}_{\text{2 points}}$$

Exam version: 55

Question 2 (5 points) Let $f(x) = \ln(x^2 - 2x + 3)$.

(a) (2 points) Calculate all singular points (if any) for f and all critical points (if any) for f.

$$f'(x) = \frac{2x-2}{x^2-2x+3} = 0 \iff 2x-2=0$$

$$\iff x = 1$$

(critical point)

1 point

Note: $x^2-2x+3>0$, so f' always exists and therefore no singular points.

0.5 points

0.5 points

(b) (3 points) Determine the absolute minimum for f on the closed interval $[0, 4]$.

Closed interval method:

endpoints $\begin{cases} f(0) = \ln 3 \\ f(4) = \ln(16-8+3) = \ln(11) \end{cases}$

crit/sing points in the desired interval $\left\{ f(1) = \ln(1-2+3) = \ln 2 \right.$

minimum.

2 points

1 point for identifying the minimum (either the x or y coordinate).

Exam version: 55

Part B: Choose the best answer to each of the following questions. Fill in the appropriate circle on the scantron sheet with a pencil AND circle your answer in the booklet. You may keep this booklet when the exam concludes. There are 12 problems.

1. If $f(x) = \arctan(4x)$, then

   (a) $f'(x) = \frac{1}{1+16x^2}$.
   (b) $f'(x) = \frac{4}{1+16x^2}$.
   (c) $f'(x) = \frac{1}{1+4x^2}$.
   (d) $f'(x) = \frac{4}{1+4x^2}$.
   (e) $f'(x) = \frac{32x}{1+16x^2}$.

$$f'(x) = \frac{(4x)'}{1+(4x)^2} = \frac{4}{1+16x^2}$$

2. If $f(x) = (x^2+1)^x$ then

   (a) $f'(x) = x(x^2+1)^{x-1}$.
   (b) $f'(x) = \ln(x^2+1)(x^2+1)^x$.
   (c) $f'(x) = (x^2+1)^x \left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)$. See VII
   (d) $f'(x) = (x^2+1)^x \left(\frac{2x^2}{x^2+1}\right)$.
   (e) $f'(x) = \ln(x^2+1) + \frac{2x^2}{x^2+1}$.

Exam version: 55

3. Suppose f is differentiable at every x in $\mathbb{R}$. Which of the following statements is true?

   (a) If $f'(c) = 0$, then f has a local maximum or local minimum at $x = c$.

   (b) If f has a local maximum at $x = c$ then $f'(c) = 0$.

   (c) If f has a local minimum at $x = c$ then $f'(c)$ does not exist.

   (d) If f has a local minimum at $x = c$, then $f(c)$ is the absolute maximum for f.

   (e) None of the above statements are true.

4. Let $f(x) = \frac{1}{x+3}$. The linear approximation for f at $a = -1$ is given by which of the following?

   (a) $L(x) = \frac{1}{2} + \frac{1}{4}(x + 1)$.

   (b) $L(x) = \frac{1}{2} - \frac{1}{4}(x + 1)$.

   (c) $L(x) = \frac{1}{2} - \frac{1}{2}(x + 1)$.

   (d) $L(x) = \frac{1}{2} + \frac{1}{2}(x + 1)$.

   (e) None of the above.

$$f(-1) = \frac{1}{2}$$

$$f'(x) = \frac{-1}{(x+3)^2}$$

$$f'(-1) = \frac{-1}{2^2} = -\frac{1}{4}$$

Exam version: 55

5. Let $T_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ denote the third degree Taylor polynomial for $f(x) = \sin x$ about the point $a = 0$ (also called the Maclaurin polynomial of degree 3).

   Which of the following is equal to a_3?

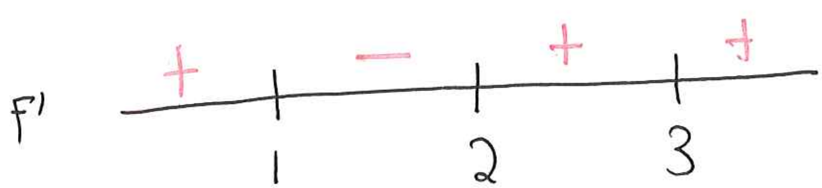
   (a) 0.
   (b) $-\frac{1}{6}$.   See VII
   (c) $-\frac{1}{3}$.
   (d) $\frac{1}{6}$.
   (e) $\frac{1}{3}$.

6. Suppose f is a differentiable function and

$$f'(x) = (x-1)(x-2)^3(x-3)^4.$$

   Which of the following statements is true?

   (a) f is increasing on the open interval $(1,2)$.
   (b) f is increasing on the open interval $(2,3)$.
   (c) f is not decreasing on any interval.
   (d) f is not increasing on any interval.
   (e) None of the above statements are true.



Exam version: 55

7. Suppose $f(x) = \sqrt[3]{x^3+1}$. Which of the following statements is true?

   (a) f has more than one critical point and no singular points.
   (b) f has more than one singular point and no critical points.
   (c) f has two distinct critical points and no singular points.
   (d) f has two distinct singular points and no critical points.
   (e) f has one critical point and one singular point.

See VII

8. The total cost required to build a large cylindrical tank is given by

$$C = \pi r^2 + 4\pi r h,$$

where r is the radius of the tank in metres and h is the height of the tank in metres. If the total volume required is 4π m^3, what radius will minimize the total cost? The volume of a cylinder is given by $V = \pi r^2 h$.

   (a) $r = 1$m.
   (b) $r = \frac{3}{2}$m.
   (c) $r = 2$m.
   (d) $r = 6$m.
   (e) $r = 8$m.

See VII

Exam version: 55

9. The limit

$$\lim_{x\to 0} \frac{x^2 + \sin(-4x)}{\sin(x) + x^3}$$

"$\frac{0}{0}$"

equals

(a) 0.
(b) 1.
(c) −1.
(d) −2.
(e) −4.

$$\overset{l'H}{=} \lim_{x\to 0} \frac{2x - 4\cos(4x)}{\cos(x) + 3x^2}$$

$2x \to 0$, $\cos(4x) \to 1$, $\cos(x) \to 1$, $3x^2 \to 0$

$$= \frac{-4}{1} = -4$$

10. Suppose A and B are two quantities depending on time t and that they are related by the equation

$$A^2 + 4AB = 12.$$

If $\frac{dA}{dt} = 1$, determine $\frac{dB}{dt}$ when $A = 2$.

(a) $\frac{dB}{dt} = 3$.
(b) $\frac{dB}{dt} = 2$.
(c) $\frac{dB}{dt} = -1$.
(d) $\frac{dB}{dt} = -2$.
(e) $\frac{dB}{dt} = -3$.

When $A = 2$: $4 + 8B = 12$

$\Leftrightarrow B = 1$

$$2A \cdot \frac{dA}{dt} + 4A\frac{dB}{dt} + 4\frac{dA}{dt}B = 0$$

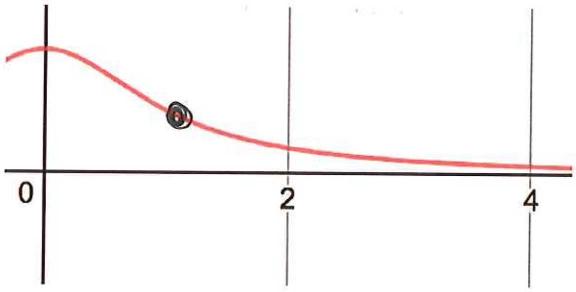
$$\Rightarrow 4 + 8 \cdot \frac{dB}{dt} + 4 = 0$$

$$\Leftrightarrow \frac{dB}{dt} = -1$$

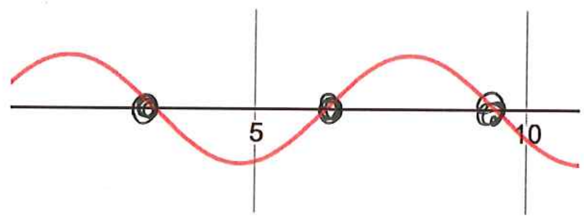
Exam version: 55

11. The following four images are graphs of twice differentiable functions. In which of the following graphs are exactly two distinct inflection points visible?

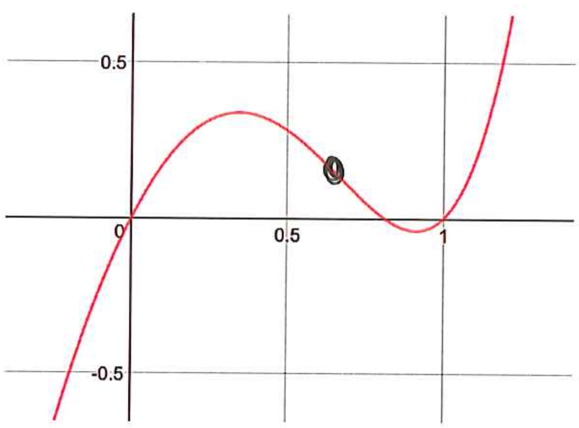
(a)



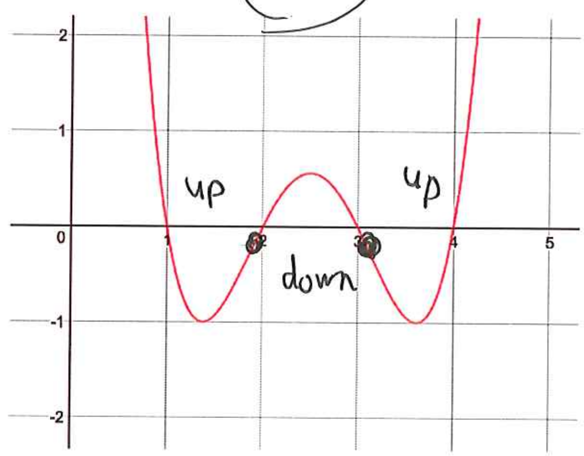
(b)



(c)



(d)



12. The limit $\lim_{x\to 0+} \frac{\arctan(x)}{\sqrt{x}}$ is equal to

(a) 0.
(b) 1.
(c) $\frac{\pi}{2}$.
(d) ∞.
(e) This limit does not exist.

$$\overset{l'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x^2}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0} \frac{2\sqrt{x}}{1+x^2} = \frac{0}{1+0} = 0$$

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Part A: Provide your full solution on these pages. Write your work in a neat and organized format, and take the time to fully justify your steps. Do not simplify your answer.

Question 1 (5 points) The radius of a sphere is increasing at a constant rate. Recall that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

(a) (2 points) Find an equation that relates $\frac{dV}{dt}$ and $\frac{dr}{dt}$.

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot (3r^2) \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(b) (3 points) Assuming the radius of the sphere is initially 0m and is increasing at a rate of 3m/s, find the rate at which the ~~surface area~~ volume is expanding when $V = 36\pi$ m^3.

Note: When $V = 36\pi$ we have

$$36\pi = \frac{4}{3}\pi r^3$$

$$\Leftrightarrow \quad 27 = r^3 \Leftrightarrow 3 = r.$$

Now $$\frac{dV}{dt} = 4\pi \underset{r}{(3)}^2 \cdot \underset{\frac{dr}{dt}}{3} = 108\pi \text{ m}^3/\text{s}$$

Exam version: 77

Question 2 (5 points) Consider the function $f(x) = \ln(x^2 + 1)$.

(a) (2 points) Calculate all singular points (if any) for f and all critical points (if any) for f.

The domain of f is $\mathbb{R}$.

$$f'(x) = \frac{2x}{x^2+1} \quad \text{(always exists)}$$

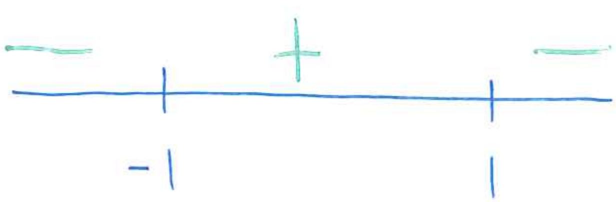
$$= 0 \iff x = 0.$$

$x = 0$ only critical point.
No singular points

(b) (3 points) Determine on which interval(s) f is *concave up*.

$$f''(x) = \frac{2(x^2+1) - 4x^2}{(x^2+1)^2} = \frac{2 - 2x^2}{(x^2+1)^2}$$

$$= \frac{2(1-x)(1+x)}{(x^2+1)^2}$$



So f is concave up on $(-1, 1)$ ($[-1,1]$ also acceptable)

Exam version: 77

Part B: Choose the best answer to each of the following questions. Fill in the appropriate circle on the scantron sheet with a pencil AND circle your answer in the booklet. You may keep this booklet when the exam concludes. There are 12 problems.

1. If $f(x) = \arcsin(4x)$, then

   (a) $f'(x) = \frac{4x}{\sqrt{1-16x^2}}$.
   (b) $f'(x) = \frac{4}{\sqrt{1-4x^2}}$.
   (c) $f'(x) = \frac{4}{\sqrt{1-16x^2}}$.
   (d) $f'(x) = \frac{8x}{\sqrt{1-16x^2}}$.
   (e) $f'(x) = \frac{32x}{\sqrt{1-16x^2}}$.

$$f'(x) = \frac{1}{\sqrt{1-(4x)^2}} \cdot (4x)'$$

$$= \frac{4}{\sqrt{1-16x^2}}$$

2. If $f(x) = (x^2+1)^{\cos x}$ then

   (a) $f'(x) = x(x^2+1)^{\cos x - 1}$.
   (b) $f'(x) = \ln(x^2+1)(x^2+1)^{\cos x}$.
   (c) $f'(x) = (x^2+1)^{\cos x}\left(-\sin x \ln(x^2+1) + \frac{2x\cos x}{x^2+1}\right)$.
   (d) $f'(x) = (x^2+1)^{\cos x}\left(\cos x \ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right)$.
   (e) $f'(x) = (x^2+1)^{\cos x}\left(\sin x \ln(x^2+1) + \frac{\cos x}{x^2+1}\right)$.

$$\ln(f(x)) = \cos x \ln(x^2+1)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{2x\cos x}{x^2+1} - \sin x \cdot \ln(x^2+1)$$

Exam version: 77

3. Suppose $f(x) = x^3 + 3x^{[redacted]} + [redacted]$ 3. Then $(f^{-1})'(7)$ is equal to which of the following?

   (a) [redacted] $\frac{1}{6}$.
   (b) [redacted] $\frac{1}{3}$.
   (c) [redacted] 1.
   (d) [redacted] 150
   (e) None of the above.

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))}$$

$f^{-1}(7) = x$ means $f(x) = 7$

$\Leftrightarrow x^3 + 3x^{[redacted]} + 3 = 7$

$\Rightarrow x = 1$ by guessing

$$\frac{1}{f'(1)} = \frac{1}{6}$$

4. The third degree Taylor approximation for the function $f(x) = \frac{1}{x+1}$ about the point $a = 0$ is given by

$$T_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

The coefficient a_3 is equal to

   (a) -1.
   (b) 1.
   (c) $\frac{1}{6}$.
   (d) $-\frac{1}{6}$.
   (e) 6.

$$f'(x) = \frac{-1}{(x+1)^2}$$

$$f''(x) = \frac{2}{(x+1)^3}$$

$$f'''(x) = \frac{-6}{(x+1)^4}$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{\frac{-6}{1^4}}{3!} = -\frac{6}{6} = -1$$

Exam version: 77

5. The linear approximation for the function $f(x) = e^x$ about the point $a = 0$ gives which of the following estimates for $e^{0.01}$?

   (a) 1.01.
   (b) 1.1.
   (c) 1.001.
   (d) $e^{0.01}$.
   (e) $1 + (0.01)e^{0.01}$.

$f'(x) = e^x \qquad f(0) = 1 = f'(0)$

$L(x) = 1 + x$

$L(0.01) = 1 + 0.01 = 1.01$

6. Suppose f is a twice differentiable function and

$$f''(x) = (x-1)(x-2)^3(x-3)^4.$$

   Which of the following statements is true?

   (a) f has no inflection points.
   (b) f has exactly one inflection point.
   (c) f has exactly two inflection points.
   (d) f has exactly three inflection points.
   (e) It is impossible to determine how many inflection points f has.

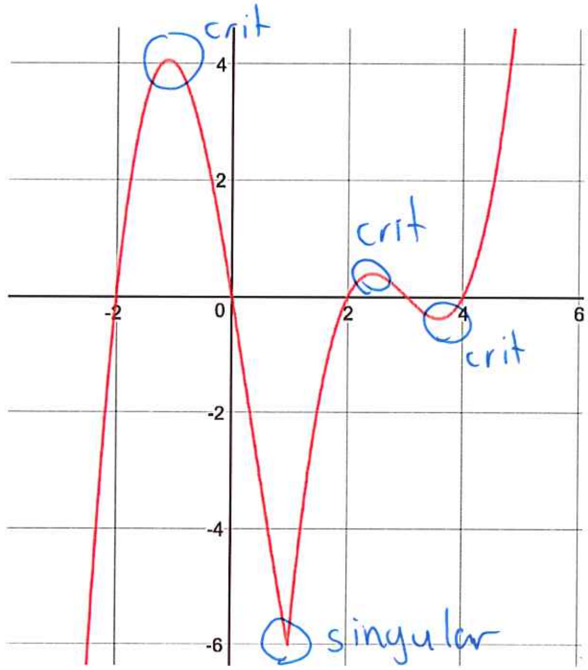
\+ − + +

1 2 3

both infl. pts.

Exam version: 77

7. Consider the following graph of a function f..



In the visible portion of the graph of f shown, which of the following is true?

(a) f has more than one critical point and no singular points.
(b) f has more than one singular point and no critical points.
(c) f has two distinct critical points and no singular points.
~~(d)~~ f has two distinct singular points and no critical points.
(e) f has three distinct critical points and one singular point.

8. The minimum value for the quantity $x^2 + 3y^2$ subject to the constraint $x - y = 2$ is which of the following?

(a) 0.
(b) $\frac{1}{2}$.
(c) $\frac{3}{2}$.
(d) 3.
(e) 4.

$\Rightarrow y = x - 2$.

So

$x^2 + 3y^2 = x^2 + 3(x-2)^2 = f(x)$

$f'(x) = 2x + 6(x-2)$
$= 8x - 12 = 0$
$\Leftrightarrow x = \frac{3}{2}$

f' $-$ $+$
$\searrow$ $\frac{3}{2}$ $\nearrow$

Global min at $x = \frac{3}{2}$
by general interval method

$f(\frac{3}{2}) = \frac{9}{4} + 3(-\frac{1}{2})^2 = \frac{9}{4} + \frac{3}{4} = \frac{12}{4} = 3$

Exam version: 77

9. The limit

$$\lim_{x\to 0} \frac{\sin(-4x+2x^2)}{\sin(2x+x^2)}$$

equals

(a) 0.
(b) −2.
(c) 2.
(d) −1.
(e) 1.

$$\overset{L'H}{=} \lim_{x\to 0} \frac{(-4+4x)\cos(-4x+2x^2)}{(2+2x)\cos(2x+x^2)}$$

$$= -\frac{4}{2} = -2$$

10. Suppose A and B are two quantities depending on time t and that they are related by the equation

$$A^2 + 4AB = 12.$$

If $\frac{dA}{dt} = 1$, determine $\frac{dB}{dt}$ when $A = 2$.

(a) $\frac{dB}{dt} = 3$.
(b) $\frac{dB}{dt} = 2$.
(c) $\frac{dB}{dt} = -1$.
(d) $\frac{dB}{dt} = -2$.
(e) $\frac{dB}{dt} = -3$.

$$2A\frac{dA}{dt} + 4A\frac{dB}{dt} + 4\frac{dA}{dt}B = 0$$

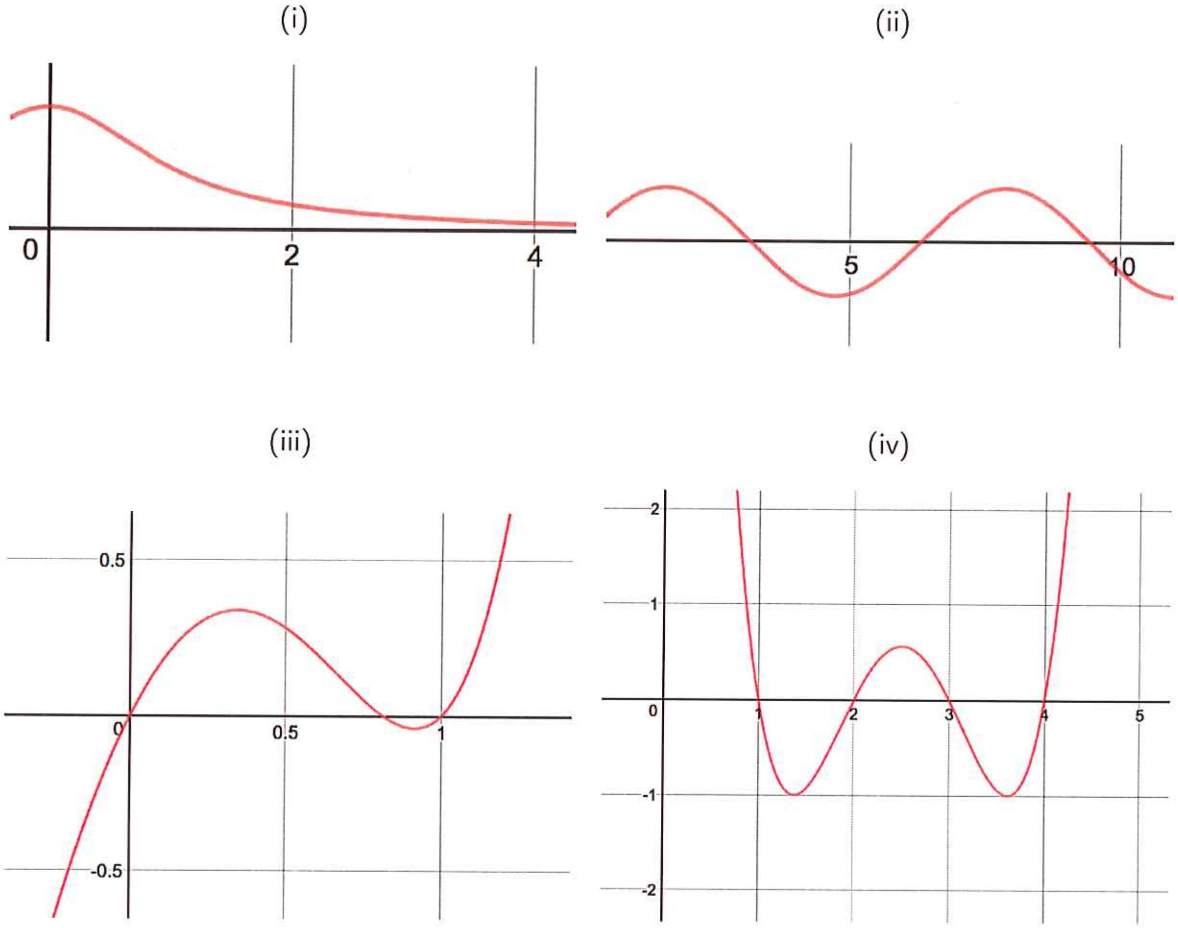
When $A=2$ we have. $4 + 8B = 12$
$\Rightarrow B = 1.$

So $4\cdot(1) + 8\cdot\frac{dB}{dt} + 4 = 0$

$\Rightarrow \frac{dB}{dt} = -1.$

Exam version: 77

11. The following four images are graphs of twice differentiable functions.



With respect to the visible portion of the graphs shown, which of the following statements is true?

(a) All of the above functions have at least two local extreme values.

(b) Exactly three of these functions have critical points.

(c) All of these functions have inflection points.

(d) One of the functions is increasing on $\mathbb{R}$.

(e) None of the above statements are true.

Exam version: 77

12. The limit $\lim_{x\to 0+} \frac{x^2-2x}{\arcsin(x)}$ is equal to

(a) 0.

(b) -1.

(c) -2.

(d) 1.

(e) This limit does not exist.

$$= \lim_{x\to 0^+} \frac{2x-2}{\frac{1}{\sqrt{1-x^2}}} = -2$$