Lesson 13: Limits and Definite Integrals
Textbook Notes

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The topics covered in this lesson are listed below and were extracted from the 12th edition of your textbook: CALCULUS for Business, Economics, Life Science, and Social Sciences, by Barnett, Ziegler, and Byleen.

Although the locations, reference numbers, and some of the illustrations of the topics may have changed in the newer versions of your textbook, the material is identical and you are only responsible for the topics as discussed below.

You can find the new locations of the lesson material in the current edition of your textbook by consulting its Table of Contents and Index.

The Study Notes were created with *Scientific Notebook*, a menu-driven computation system, at a time when *Mathematica* was unavailable at the Univerity. Today we use *Mathematica*. If need be, you can easily replace any illustrations and calculations referring to *Scientific Notebook* in the notes below by analogous *Mathematica* illustrations and calculations as illustrated in the *Mathematica Companion* and the *Mathematica Notes*.

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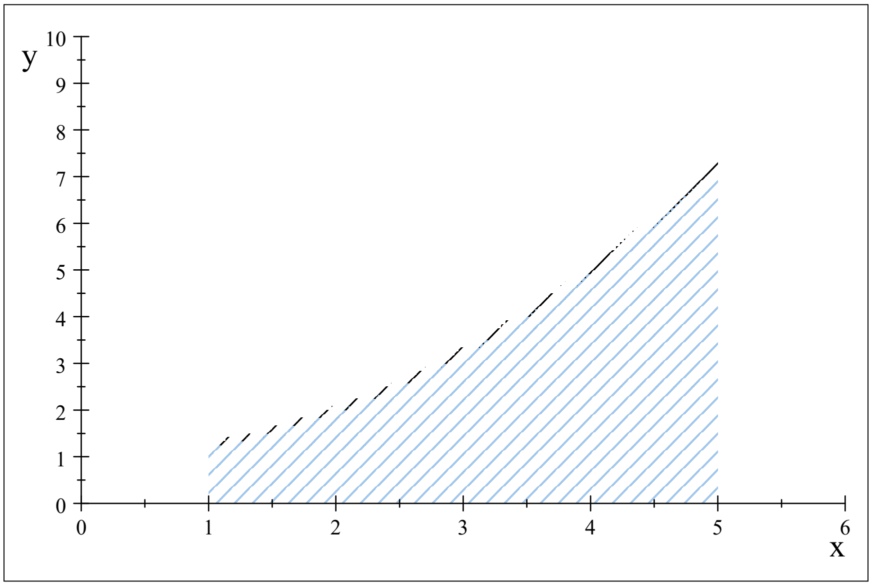
Section 6-4 (pp. 383-393)

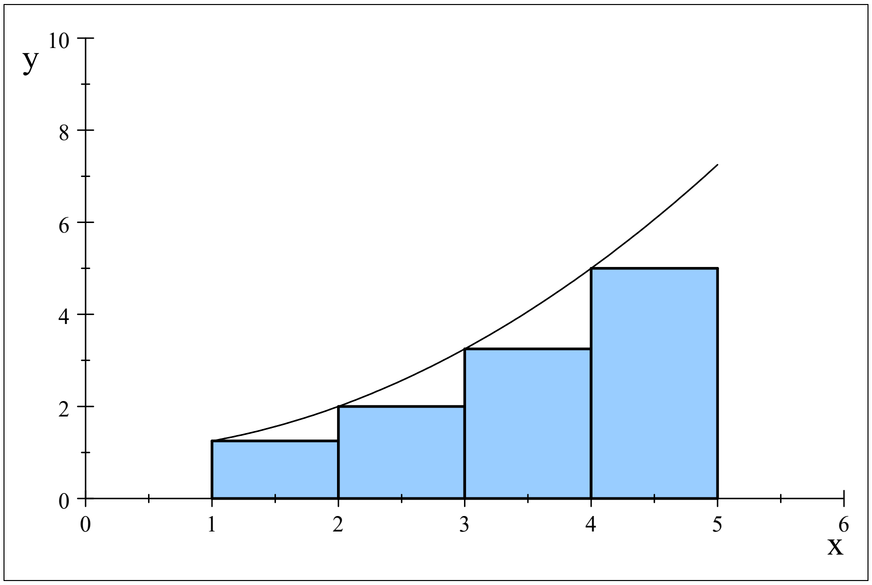
THE DEFINITE INTEGRAL

Approximating Areas by Left and Right Sums

The shaded area in the graph below is the area between the x axis, the curve $f(x) = 0.25x^2 + 1$ and the lines $x = 1$ and $x = 5$.



Is it possible to find the exact value of this area? We cannot apply our standard geometric formulas directly, but we can get an approximation of the area through a series of rectangles. To do so, divide the interval $[1,5]$ on the x axis into four equal parts, each of length $\Delta x = 1$. Then place a rectangle on each subinterval, with the height of the rectangle determined by the function evaluated at the left endpoint of the subinterval:



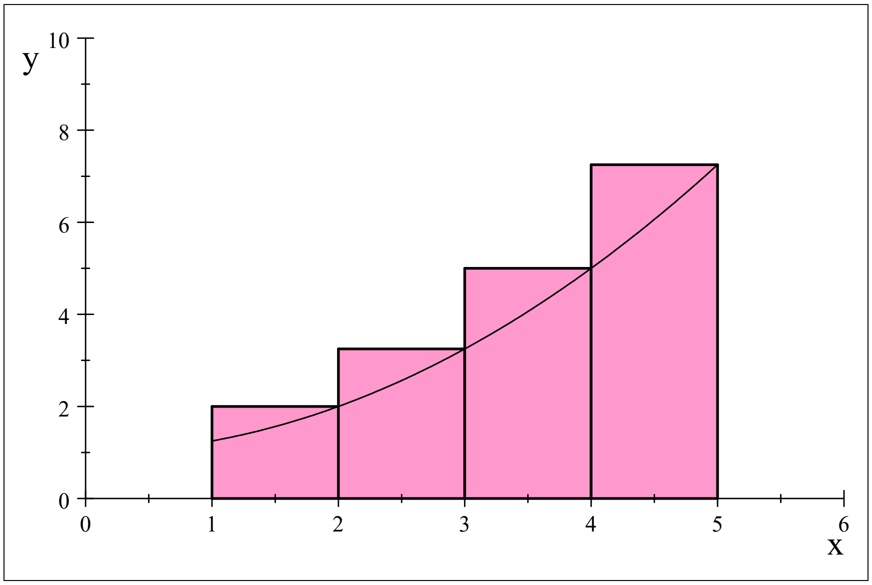
Summing the areas of the rectangles, we obtain a **left sum** of four rectangles, denoted by L_4, as follows:

$$\begin{aligned} L_4 &= f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 \\ &= 1.25 + 2.00 + 3.25 + 5 \\ &= 11.5 \end{aligned}$$

Since all rectangles are completely underneath the curve, it is clear that L_4 underestimates the area under the curve. We can therefore write

$$L_4 = 11.5 < Area$$

If we use the right endpoint instead, we obtain the following rectangles:



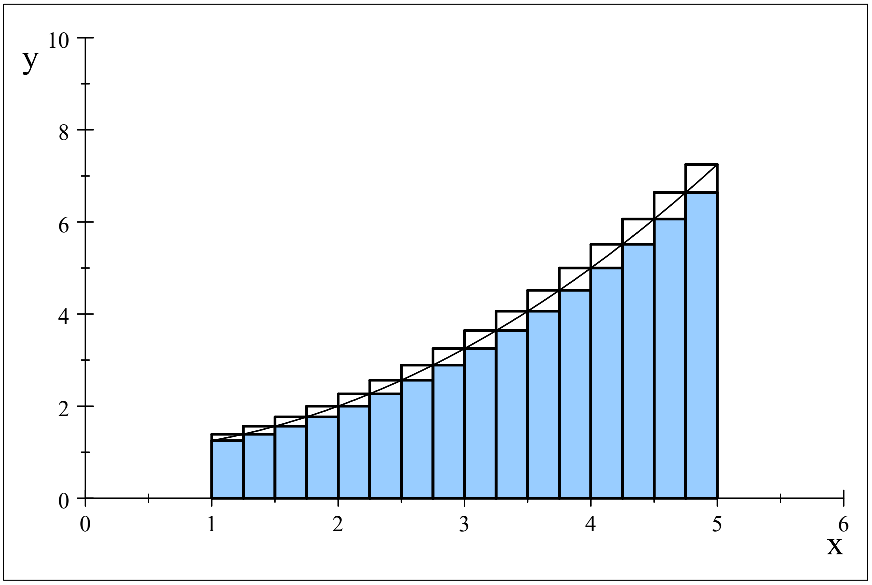
Summing the areas of these rectangles, we obtain the **right sum** of the four rectangles, denoted by R_4, as follows:

$$\begin{aligned} R_4 &= f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 \\ &= 2.00 + 3.25 + 5 + 7.25 \\ &= 17.5 \end{aligned}$$

From the graph, it is clear that R_4 overestimates the area. We can therefore write

$$11.5 = L_4 < Area < R_4 = 17.5$$

This approximation is fairly coarse, but the same method can be continued with increasingly accurate results by dividing the interval into more and more subintervals of equal length. Of course, this is not a job for hand calculation, but Scientific Notebook provides a tool for doing this easily. You can access it by going to **Compute Calculus** > **Plot Approximate Integral** in the menu. Here are the left and right rectangle approximations for the same function and 16 equal subdivisions:



For this case,

$$\begin{aligned} \Delta x &= \frac{5-1}{16} = 0.25 \\ L_{16} &= f(1)\cdot\Delta x + f(1.25)\cdot\Delta x + \ldots + f(4.75)\cdot\Delta x \\ &= 13.59 \\ R_{16} &= f(1.25)\cdot\Delta x + f(1.50)\cdot\Delta x + \ldots + f(5)\cdot\Delta x \\ &= 15.09 \end{aligned}$$

Thus, we know that the area under the curve is between 13.59 and 15.09. That is,

$$13.59 = L_{16} < Area < R_{16} = 15.09$$

For 100 equal subdivisions, computer calculations give us

$$14.214 = L_{100} < Area < R_{100} = 14.454$$

The **error in approximation** is the absolute value of the difference between the approximation and the actual value. In general, neither the actual value nor the error in approximation is known. However, it is often possible to calculate an **error bound**—a positive number which is guaranteed to be greater than the error in approximation.

The error in our approximation is the area between the rectangles and the curve. It is clear that this area is smaller than the area of the difference between the rectangles in the right and left sums: finding the sum of those small rectangles leads us to the next theorem.

Theorem (**1—Error Bounds for Approximations of Area by Left or Right Sums**) *If $f(x) > 0$ and is either increasing or decreasing on $[a,b]$, then*

$$|f(b) - f(a)| \cdot \frac{b-a}{n}$$

is an error bound for the approximation of the area between the graph of f and the x axis, from $x = a$ to $x = b$, by L_n or R_n.

Because the error bound of Theorem 1 approaches 0 as $n \to \infty$, it can be shown that left and right sums, for certain functions, approach one same limit as $n \to \infty$.

Theorem (**2—Limits of Left and Right Sums**) *If $f(x) > 0$ and is either increasing or decreasing on $[a,b]$, then its left and right sums approach the same real number as $n \to \infty$.*

The number approached as $n \to \infty$ by the left and right sums in Theorem 2 is the area between the graph of f and the x axis from $x = a$ to $x = b$.

Example (**1 in Section 6-4**) *Approximating Areas*

Given the function $f(x) = 9 - 0.25x^2$, we are interested in approximating the area under $y = f(x)$ from $x = 2$ to $x = 5$.

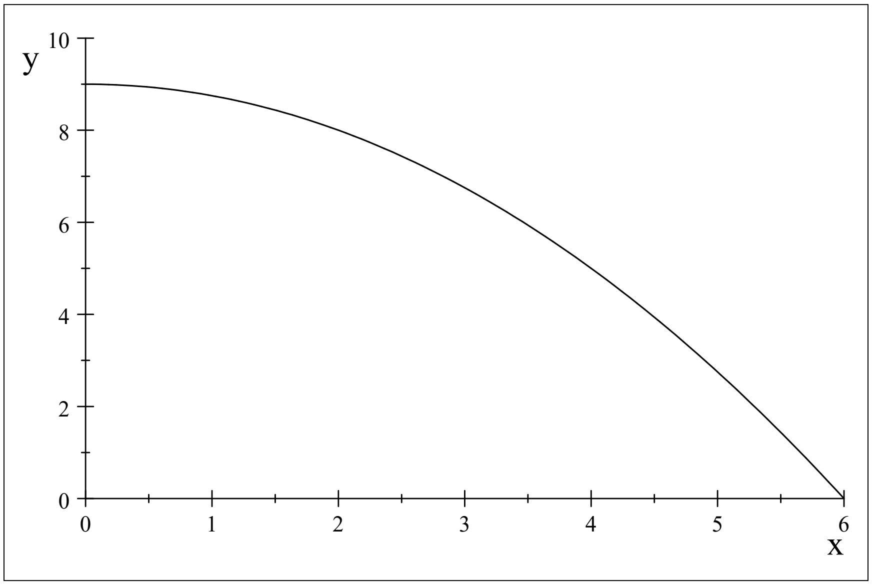
(A) Graph the function over the interval $[0,6]$; then draw left and right rectangles for the interval $[2,5]$ with $n = 6$.

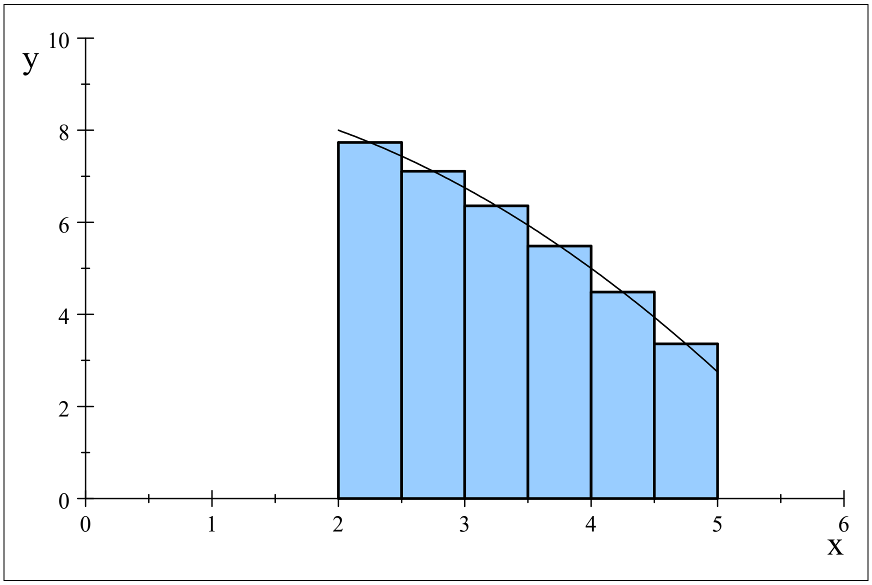
(B) Calculate L_6, R_6, and error bounds for each.

(C) How large should n be chosen for the approximation of the area by L_n or R_n to be within 0.05 of the true value?

Solution

(A) *Here is the graph of the function followed by the left and right rectangles. In this case, $\Delta x = 0.5$.*





(B)

$$L_6 = f(2)\cdot\Delta x + f(2.5)\cdot\Delta x + f(3)\cdot\Delta x + f(3.5)\cdot\Delta x + f(4)\cdot\Delta x + f(4.5)\cdot\Delta x$$
$$= 18.53$$
$$R_6 = f(2.5)\cdot\Delta x + f(3)\cdot\Delta x + f(3.5)\cdot\Delta x + f(4)\cdot\Delta x + f(4.5)\cdot\Delta x + f(5)\cdot\Delta x$$
$$= 15.91$$

The error bound for L_6 *and* R_6 *is*

$$\text{error} \le |f(5) - f(2)|\cdot\frac{5-2}{6} = |2.75 - 8|(0.5) = 2.625$$

(C) *For* L_n *and* R_n, *find* n *such that error* ≤ 0.05.

$$|f(b) - f(a)|\cdot\frac{b-a}{n} \le 0.05$$
$$|2.75 - 8|\frac{3}{n} \le 0.05$$
$$|-5.25|\frac{3}{n} \le 0.05$$
$$15.75 \le 0.05n$$
$$n \ge \frac{15.75}{0.05} = 315$$

Problem (Matched—1 in Section 6-4) *Given the function* $f(x) = 8 - 0.5x^2$, *we are interested in approximating the area under* $y = f(x)$ *from* $x = 1$ *to* $x = 3$.

(A) *On a sheet of paper, graph the function over the interval* $[0,4]$; *then draw left and right rectangles for the interval* $[1,3]$ *with* $n = 4$.

(B) *Calculate* L_4, R_4, *and error bounds for each.*

(C) *How large should* n *be chosen for the approximation of the area by* L_n *or* R_n *to be within* 0.5 *of the true value?*

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

The Definite Integral as a Limit of Sums

Left and right sums are special cases of more general sums, called **Riemann sums**, that are used to approximate areas by means of rectangles.

Let f be a function defined on the interval $[a,b]$. We partition $[a,b]$ into n subintervals of equal length $\Delta x = (b-a)/n$ with endpoints $(x_0, x_1, x_2, \ldots, x_n)$ such that

$$a = x_0 < x_1 < x_2 \ldots < x_n = b$$

Then we have

$$\text{Left sum: } L_n = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

$$\text{Right sum: } R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

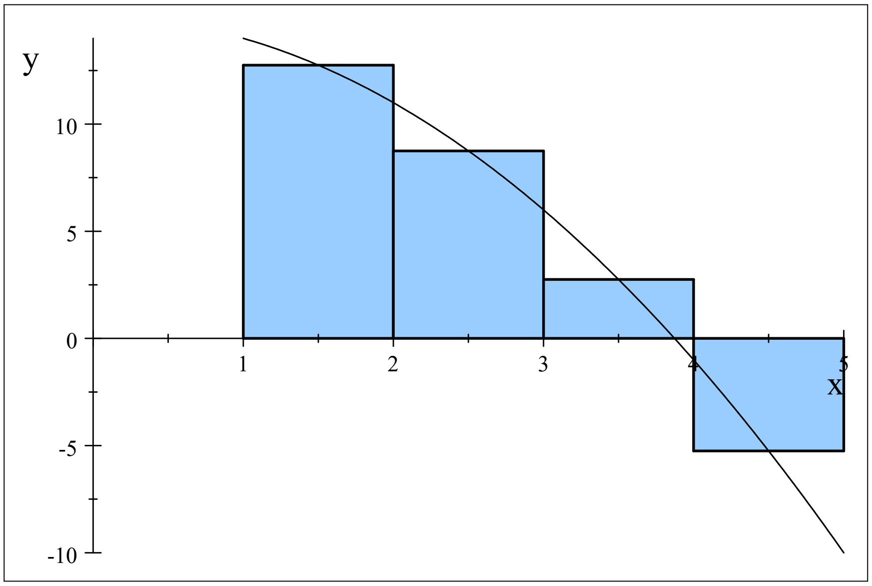
$$\text{Riemann sum:} S_n = f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x$$

In a Riemann sum, each c_k is only required to belong to the subinterval $[x_{k-1}, x_k]$. Left and right sums are the special cases of Riemann sums in which c_k is the left endpoint or right endpoint, respectively, of the subinterval. If $f(x) > 0$, then each term of a Riemann sum represents the area of a rectangle having height $f(c_k)$ and width Δx. If $f(c_k) < 0$, then the term of the Riemann sum associated with c_k is the negative of the area of the rectangle.

Example (**2 in Section 6-4**) *Riemann Sums*

Consider the function $f(x) = 15 - x^2$ on $[1,5]$. Partition the interval $[1,5]$ into four subintervals of equal length. For each subinterval $[x_{k-1}, x_k]$, let c_k be the midpoint. Calculate the corresponding Riemann sum S_4. (Riemann sums for which the c_k are the midpoints of the subintervals are called **midpoint sums**.)

Solution *Here is the graph of the rectangles:*



$$\Delta x = \frac{5-1}{4} = 1$$

$$\begin{aligned} S_4 &= f(c_1)\cdot\Delta x + f(c_2)\cdot\Delta x + f(c_3)\cdot\Delta x + f(c_4)\cdot\Delta x \\ &= f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 + f(4.5)\cdot 1 \\ &= 12.75 + 8.75 + 2.75 - 5.25 = 19 \end{aligned}$$

Problem (Matched—2 in Section 6-4) *Consider the function* $f(x) = x^2 - 2x - 10$ *on* $[2, 8]$. *Partition the interval* $[2, 8]$ *into three subintervals of equal length. For each subinterval* $[x_{k-1}, x_k]$, *let* c_k *be the midpoint. Calculate the corresponding Riemann sum* S_3.

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Theorem (3—Limit of Riemann Sums) *If* f *is a continuous function on* $[a, b]$, *then the Riemann sums for* f *on* $[a, b]$ *approach a real number limit* I *as* $n \to \infty$.

Definition (Definite Integral) *Let* f *be a continuous function on* $[a, b]$. *The limit* I *of Riemann sums for* f *on* $[a, b]$, *guaranteed to exist by Theorem 3, is called the* ***definite integral*** *of* f *from* a *to* b *and is denoted*

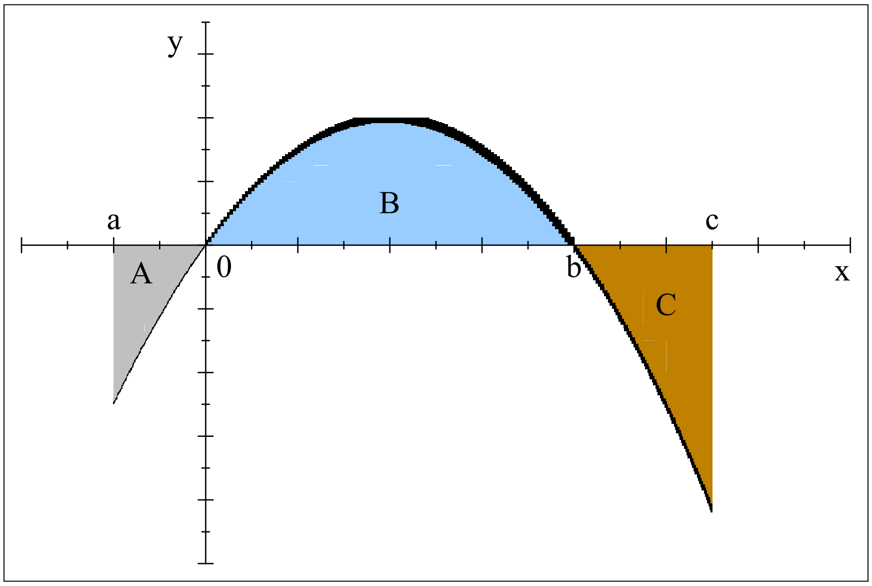
$$\int_a^b f(x)\,dx$$

The ***integrand*** *is* $f(x)$, *the* ***lower limit of integration*** *is* a, *and the* ***upper limit of integration*** *is* b.

Because area is a positive quantity, the definite integral does not truly represent the area between a curve and the x axis. Instead, it represents the cumulative sum of the signed areas between the graph of f and the axis from $x = a$ to $x = b$, where the areas above the x axis are counted as positive and the areas below the x axis are counted as negative.

Example (3 in Section 6-4) *Definite Integrals*

Calculate the definite integrals by referring to the following graph of $f(x)$ and the indicated areas.



Area $A = 2.33$

Area $B = 10.67$

Area $C = 5.63$

(A) $\int_a^b f(x)\,dx$

(B) $\int_a^c f(x)dx$

(C) $\int_b^c f(x)dx$

Solution

(A) $\int_a^b f(x)dx = -2.33 + 10.67 = 8.34$

(B) $\int_a^c f(x)dx = -2.33 + 10.67 - 5.62 = 2.71$

(C) $\int_b^c f(x)dx = -5.63$

Problem (Matched—3 in Section 6-4) *Referring to the figure in Example 3, calculate the definite integrals.*

(A) $\int_a^0 f(x)dx$

(B) $\int_0^c f(x)dx$

(C) $\int_0^b f(x)dx$

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Properties of the Definite Integral

Because the definite integral is defined as the limit of Riemann sums, many properties of sums are also properties of the definite integral.

Theorem (Properties of Definite Integrals)

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, k a constant

4. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Example (4 in Section 6-4) *Using Properties of the Definite Integral*

If

$$\int_0^2 xdx = 2,\ \int_0^2 x^2dx = \frac{8}{3},\ \int_2^3 x^2dx = \frac{19}{3}$$

then find

(A) $\int_0^2 12x^2dx$

(B) $\int_0^2 (2x - 6x^2)dx$

(C) $\int_3^2 x^2dx$

(D) $\int_5^5 3x^2dx$

(E) $\int_0^3 3x^2dx$

Solution

(A) $\int_0^2 12x^2dx = 12\int_0^2 x^2dx = 12\left(\frac{8}{3}\right) = 32$

(B) $\int_0^2 (2x - 6x^2)dx = 2\int_0^2 xdx - 6\int_0^2 x^2dx = 2(2) - 6\left(\frac{8}{3}\right) = -12$

(C) $\int_3^2 x^2dx = -\int_2^3 x^2dx = -\frac{19}{3}$

(D) $\int_5^5 3x^2dx = 0$

(E) $\int_0^3 3x^2dx = 3\int_0^2 x^2dx + 3\int_2^3 x^2dx = 3\left(\frac{8}{3}\right) + 3\left(\frac{19}{3}\right) = 27$

Problem (Matched—4 in Section 6-4) *Using the same integral values as in Example 4, find*

(A) $\int_2^3 6x^2dx$

(B) $\int_0^2 (9x^2 - 4x)dx$

(C) $\int_2^0 3xdx$

(D) $\int_{-2}^{-2} 3xdx$

(E) $\int_0^3 12x^2dx$

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Section 6-5 (pp. 393-405)

THE FUNDAMENTAL THEOREM OF CALCULUS

Introduction to the Fundamental Theorem

Suppose that the daily cost function for a small manufacturing firm (in dollars) is given by

$$C(x) = 180x + 200,\ 0 \le x \le 20$$

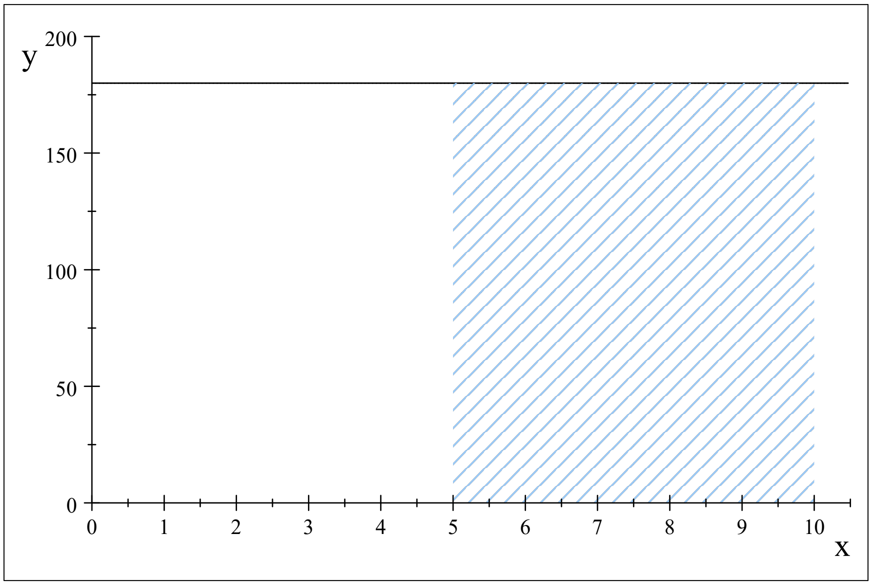
Then the marginal cost function is given (in dollars per unit) by

$$C'(x) = 180$$

The change in cost when production is increased from $x = 5$ units to $x = 10$ units is equal to

$$\begin{aligned} C(10) - C(5) &= (180 \cdot 10 + 200) - (180 \cdot 5 + 200) \\ &= 180(10 - 5) \\ &= \$900 \end{aligned}$$

Notice that $180(10 - 5)$ is equal to the area between the graph of $C'(x)$ and the x axis from $x = 5$ to $x = 10$.



Therefore,

$$C(10) - C(5) = \int_{5}^{10} 180dx$$

In other words, the change in cost from $x = 5$ to $x = 10$ is equal to the area between the marginal cost function and the x axis from $x = 5$ to $x = 10$.

Example (1 in Section 6-5) *Change in Cost vs. Area under Marginal Cost*

The daily cost function for a company (in dollars) is given by

$$C(x) = -5x^2 + 210x + 400,\ 0 \le x \le 20$$

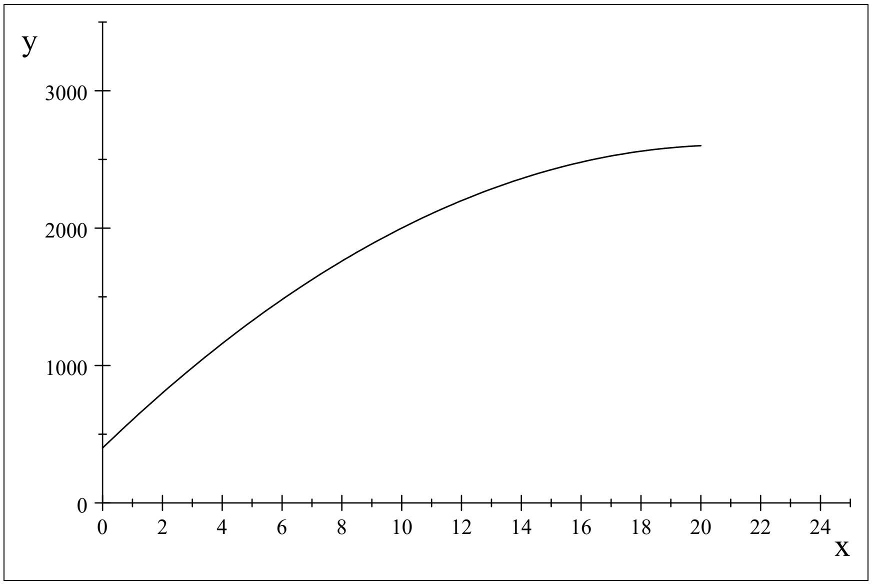
(A) Graph $C(x)$ for $0 \le x \le 20$ and calculate the change in cost from $x = 5$ to $x = 10$.

(B) Graph the marginal cost function $C'(x)$ for $0 \le x \le 20$, and use geometric formulas to calculate the area between $C'(x)$ and the x axis from $x = 5$ to $x = 10$.

(C) Compare the results of the calculations in parts (A) and (B).

Solution

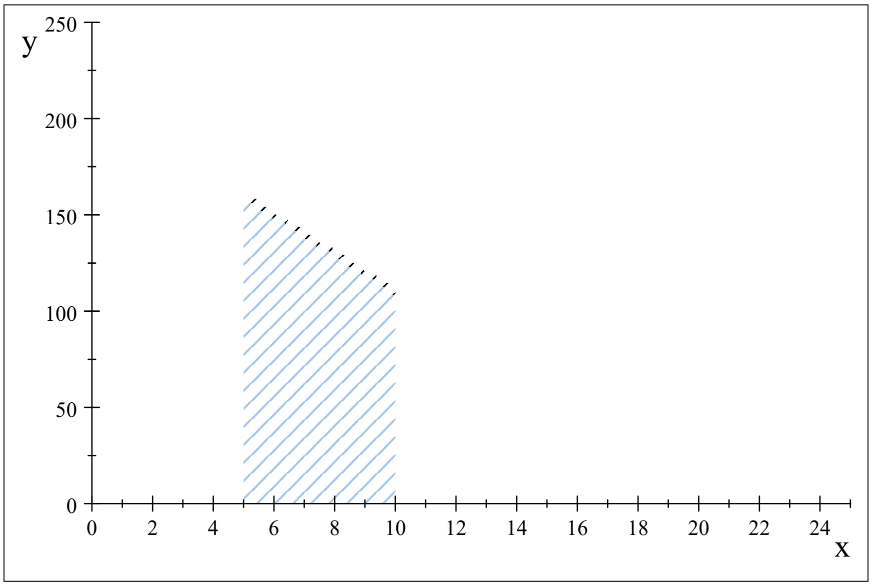
(A) $C(x)$



$$C(10) - C(5) = 2000 - 1325 = 675$$

(B) $C'(x) = -10x + 210$, *so the area between* $C'(x)$ *and the* x *axis from* $x = 5$ *to* $x = 10$ *is the area of a trapezoid (geometric formulas are given in Appendix C of the textbook):*

$$Area = \frac{C(5) + C(10)}{2}(10 - 5) = \frac{160 + 110}{2}(5) = 675$$



(C) *The change in cost from* $x = 5$ *to* $x = 10$ *is equal to the area between the marginal cost function and the* x *axis from* $x = 5$ *to* $x = 10$.

Problem (Matched—1 in Section 6-5) *Repeat Example 1 for the daily cost function*

$$C(x) = -7.5x^2 + 305x + 625$$

▸ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

The connection illustrated in the previous example, between the change of a function from $x = a$ to $x = b$ and the area under the derivative of the function, provides the link between antiderivatives (or indefinite integrals) and the definite integral. This link is known as the Fundamental Theorem of Calculus.

Theorem (**1–Fundamental Theorem of Calculus**) *If f is a continuous function on $[a,b]$, and F is any antiderivative of f, then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

Evaluating Definite Integrals

By the fundamental theorem of calculus, we can evaluate $\int_a^b f(x)dx$ easily and exactly whenever we can find an antiderivative $F(x)$ of $f(x)$. Since the constant of integration C will cancel out in the difference $F(b) - F(a)$, we usually leave it out altogether when evaluating definite integrals. Furthermore, we use the notation $F(x)|_a^b$ (the dashed line at the beginning of the symbol is added by Scientific Notebook, you should not write it down when calculating by hand) as an intermediate step in the calculation to represent the change in $F(x)$ from $x = a$ to $x = b$. That is, $F(x)|_a^b = F(b) - F(a)$.

Example (**2 in Section 6-5**) *Evaluating Definite Integrals*

Evaluate $\int_1^2 \left(2x + 3e^x - \frac{4}{x}\right)dx$.

Solution

$$\begin{aligned}\int_1^2 \left(2x + 3e^x - \frac{4}{x}\right)dx &= 2\int_1^2 xdx + 3\int_1^2 e^x dx - 4\int_1^2 \frac{1}{x}dx \\ &= 2\frac{x^2}{2}\Big|_1^2 + 3e^x|_1^2 - 4\ln|x||_1^2 \\ &= (2^2 - 1^2) + (3e^2 - 3e^1) + (4\ln 2 - 4\ln 1) \\ &= 3 + 3e^2 - 3e - 4\ln 2 \approx 14.24\end{aligned}$$

Problem (**Matched—2 in Section 6-5**) *Evaluate* $\int_1^3 \left(4x - 2e^x + \frac{5}{x}\right)dx$.

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Example (**3 in Section 6-5**) *Definite Integrals and Substitution Techniques*

Evaluate $\int_0^5 \frac{x}{x^2 + 10}dx$.

Solution *We will solve this problem by using substitution in two different ways.*

Method 1*. Use substitution in an indefinite integral to find an antiderivative as a function of x; then evaluate the definite integral:*

$$\int \frac{x}{x^2 + 10}dx = \frac{1}{2}\int \frac{1}{x^2 + 10}2xdx$$

Let $u = x^2 + 10$, $du = 2xdx$,

$$\frac{1}{2}\int \frac{1}{x^2+10}2xdx = \frac{1}{2}\int \frac{1}{u}du$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln(x^2+10) + C$$

We choose $C = 0$ *and use the antiderivative* $\frac{1}{2}\ln(x^2+10)$ *to evaluate the definite integral:*

$$\int_0^5 \frac{x}{x^2+10}dx = \frac{1}{2}\ln(x^2+10)\Big|_0^5$$
$$= \frac{1}{2}\ln 35 - \frac{1}{2}\ln 10 \approx 0.626$$

Method 2. *Substitute directly into the definite integral, changing both the variable of integration and the limits of integration: In the definite integral*

$$\int_0^5 \frac{x}{x^2+10}dx$$

the upper limit is $x = 5$ *and the lower limit is* $x = 0$. *When we make the substitution* $u = x^2 + 10$ *in this definite integral, we must change the limits of integration to the corresponding values of* u*:*

$$x = 5 \text{ implies } u = 5^2 + 10 = 35$$
$$x = 0 \text{ implies } u = 0^2 + 10 = 10$$

Thus, we have

$$\int_0^5 \frac{x}{x^2+10}dx = \frac{1}{2}\int_0^5 \frac{1}{x^2+10}2xdx$$
$$= \frac{1}{2}\int_{10}^{35} \frac{1}{u}du$$
$$= \frac{1}{2}\left(\ln|u|\,\big|_{10}^{35}\right)$$
$$= \frac{1}{2}(\ln 35 - \ln 10) \approx 0.626$$

Problem (Matched—3 in Section 6-5) *Use both methods described in Example 3 to evaluate* $\int_0^1 \frac{1}{2x+4}dx$.

▸ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Example (4 in Section 6-5) *Definite Integrals and Substitution*

Use method 2 described in Example 3 to evaluate

$$\int_{-4}^{1} \sqrt{5-t}\,dt$$

Solution *If* $u = 5 - t$, *then* $du = -dt$, *and*

$$t = 1 \text{ implies } u = 5 - 1 = 4$$
$$t = -4 \text{ implies } u = 5 - (-4) = 9$$

Notice that the lower limit for u *is larger than the upper limit. Be careful not to reverse these two values when substituting into the definite integral:*

$$
\begin{aligned}
\int_{-4}^{1} \sqrt{5-t}\, dt &= -\int_{-4}^{1} \sqrt{5-t}\,(-dt) \\
&= -\int_{9}^{4} \sqrt{u}\, du \\
&= -\int_{9}^{4} u^{1/2} du \\
&= -\left.\frac{u^{3/2}}{\frac{3}{2}}\right|_{9}^{4} \\
&= -\left(\frac{2}{3}(4^{3/2}) - \frac{2}{3}(9^{3/2})\right) \\
&= -\left(\frac{16}{3} - \frac{54}{3}\right) = \frac{38}{3} \approx 12.667
\end{aligned}
$$

Problem (Matched—4 in Section 6-5) *Use method 2 described in Example 3 to evaluate*

$$\int_{2}^{5} \frac{1}{\sqrt{6-t}} dt$$

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Example (5 in Section 6-5) *Change in Profit*

A company manufactures x television sets per month. The monthly marginal profit (in dollars) is given by

$$P'(x) = 165 - 0.1x,\ 0 \le x \le 4000$$

The company is currently manufacturing 1500 sets per month, but is planning to increase production. Find the change in the monthly profit if monthly production is increased to 1600 sets.

Solution

$$
\begin{aligned}
P(1600) - P(1500) &= \int_{1500}^{1600} (165 - 0.1x) dx \\
&= (165x - 0.05x^2)|_{1500}^{1600} \\
&= (165(1600) - 0.05(1600^2)) - (165(1500) - 0.05(1500^2)) \\
&= 136000 - 135000 \\
&= 1000
\end{aligned}
$$

Thus, increasing monthly production from 1500 *units to* 1600 *units will increase the monthly profit by* \$1000.

Problem (Matched—5 in Section 6-5) *Repeat Example 5 if*

$$P'(x) = 300 - 0.2x,\ 0 \le x \le 3000$$

and monthly production is increased from 1400 *sets to* 1500 *sets.*

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Example (6 in Section 6-5) *Useful Life*

An amusement company maintains records for each video game it installs in an arcade. Suppose that $C(t)$ and $R(t)$ represent the total accumulated costs and revenues (in thousands of dollars), respectively, t years after a particular game has been installed. Suppose also that

$$C'(t) = 2$$

$$R'(t) = 9e^{-0.5t}$$

The value for which $C'(t) = R'(t)$ is called the **useful life** of the game.

(A) Find the useful life of the game, to the nearest year.

(B) Find the total profit accumulated during the useful life of the game.

Solution

(A)

$$\begin{aligned} R'(t) &= C'(t) \\ 9e^{-0.5t} &= 2 \\ e^{-0.5t} &= \frac{2}{9} \\ -0.5t &= \ln\frac{2}{9} \\ t &= -2\ln\frac{2}{9} \approx 3 \text{ years} \end{aligned}$$

Thus, the game has a useful life of 3 *years.*

(B) *The total profit accumulated during the useful life of the game is*

$$\begin{aligned} P(3) - P(0) &= \int_0^3 P'(t)dt \\ &= \int_0^3 (R'(t) - C'(t))dt \\ &= \int_0^3 (9e^{-0.5t} - 2)dt \\ &= \left(\frac{9}{-0.5}e^{-0.5t} - 2t\right)\Big|_0^3 \\ &= (-18e^{-0.5t} - 2t)|_0^3 \\ &= (-18e^{-1.5} - 6) - (-18e^0 - 0) \\ &= 12 - 18e^{-1.5} \approx 7.984 \text{ or } \$7984 \end{aligned}$$

Problem (Matched—6 in Section 6-5) *Repeat Example 6 if* $C'(t) = 1$ *and* $R'(t) = 7.5e^{-0.5t}$.

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Example (7 in Section 6-5) *Numerical Integration with Scientific Notebook*

Evaluate $\int_{-1}^{2} e^{-x^2} dx$ to four decimal places.

Solution *The integrand e^{-x^2} does not have an elementary antiderivative, so we are unable to use the fundamental theorem to evaluate the definite integral. Instead, we use a numerical integration routine that has been programmed into the Scientific Notebook software.*

To do so, we first write the integral by going to the menu **Insert** > **Operator...** *and selecting the single integral sign. We then write the lower and upper limits of integration as a subscript and a superscript, respectively, of the integral sign. We write the expression $e^{-x^2} dx$ by using the superscript button (or pressing crtl+H) to write the exponents. The result is $\int_{-1}^{2} e^{-x^2} dx$. We then place our cursor anywhere in the expression and go to* **Compute** > **Evaluate Numerically** *in the menu. The result appears next to the expression.*

$$\int_{-1}^{2} e^{-x^2} dx : 1.6289$$

(Note that you can adjust the number of decimal places shown by going to the menu **Tools** > **Computation Setup...** *and the* **General** *tab.)*

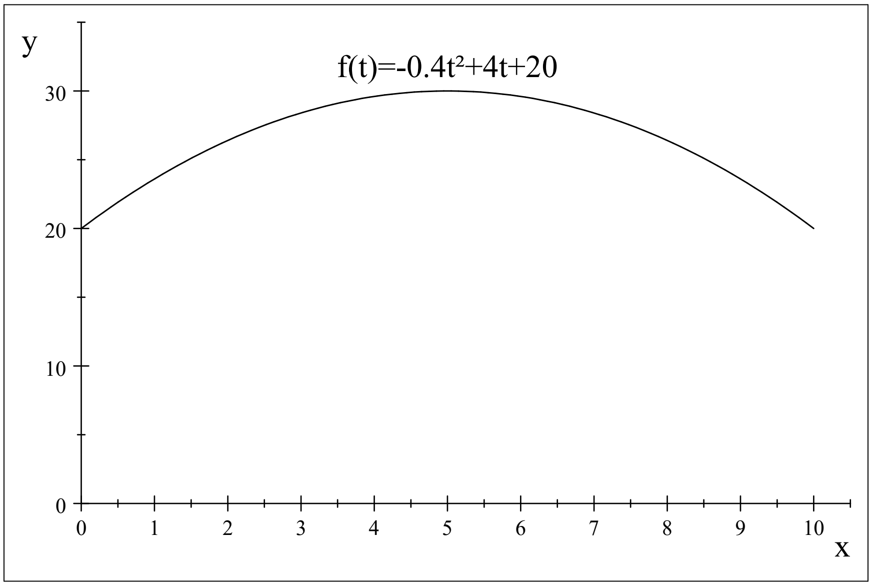
The routine used by Scientific Notebook is an approximation algorithm, more powerful than the left-sum and right-sum methods discussed in Section 6-4.

Problem (Matched—7 in Section 6-5) *Use Scientific Notebook to evaluate $\int_{1.5}^{4.3} \frac{x}{\ln x} dx$ to four decimal places.*

▸ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Recognizing a Definite Integral: Average Value

The definite integral can be used to find the average value of a continuous function over a range. For example, say the following is a graph of the temperature, in Celsius, over a 10-hour period.



If we had a finite number n of measurements, we would simply add them up and divide by n to find the average value. But how do we proceed with a continuous function? It would seem reasonable to take the value of the function at n regular intervals and compute the average of those n measurements. We would thus get an increasingly precise average with greater n. This is analogous to taking Riemann sums. Because of the definition of the definite integral as the limit of the Riemann sum of n rectangles as $n \to \infty$, we can use it to find an exact average value.

$$\begin{aligned}\text{Average} &= \frac{1}{10}\int_0^{10}(-0.4t^2+4t+20)dt \\ &= \frac{1}{10}\left(-0.4\frac{t^3}{3}+2t^2+20t\right)\Big|_0^{10} \\ &= \frac{1}{10}\left(-\frac{2}{15}(10^3)+2(10^2)+20(10)-0\right) \\ &= \frac{1}{10}\left(-\frac{2000}{15}+400\right) \approx 26.7°\text{C}\end{aligned}$$

In general, the formula for the average value of an arbitrary continuous function f over an interval $[a,b]$ is

Definition (Average Value of a Continuous Function f over $[a,b]$)

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Example (8 in Section 6-5) *Average Value of a Function*

Find the average value of $f(x) = x - 3x^2$ over the interval $[-1,2]$.

Solution

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &= \frac{1}{2-(-1)}\int_{-1}^{2}(x-3x^2)dx \\ &= \frac{1}{3}\left(\frac{x^2}{2}-x^3\right)\Big|_{-1}^{2} = -\frac{5}{2}\end{aligned}$$

Problem (Matched—8 in Section 6-5) *Find the average value of $g(t) = 6t^2 - 2t$ over the interval $[-2,3]$.*

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Example (9 in Section 6-5) *Average Price*

Given the demand function

$$p = D(x) = 100e^{-0.05x}$$

find the average price (in dollars) over the demand interval $[40,60]$.

Solution

$$\begin{aligned} \textit{Average price} &= \frac{1}{b-a}\int_a^b D(x)dx \\ &= \frac{1}{60-40}\int_{40}^{60} 100e^{-0.05x}dx \\ &= \frac{100}{20}\int_{40}^{60} e^{-0.05x}dx \\ &= -\frac{5}{0.05}e^{-0.05x}\Big|_{40}^{60} \\ &= -100(e^{-3}-e^{-2}) \approx \$8.55 \end{aligned}$$

Problem (Matched—9 in Section 6-5) *Given the supply equation*

$$p = S(x) = 10e^{0.05x}$$

find the average price (in dollars) over the supply interval $[20, 30]$.

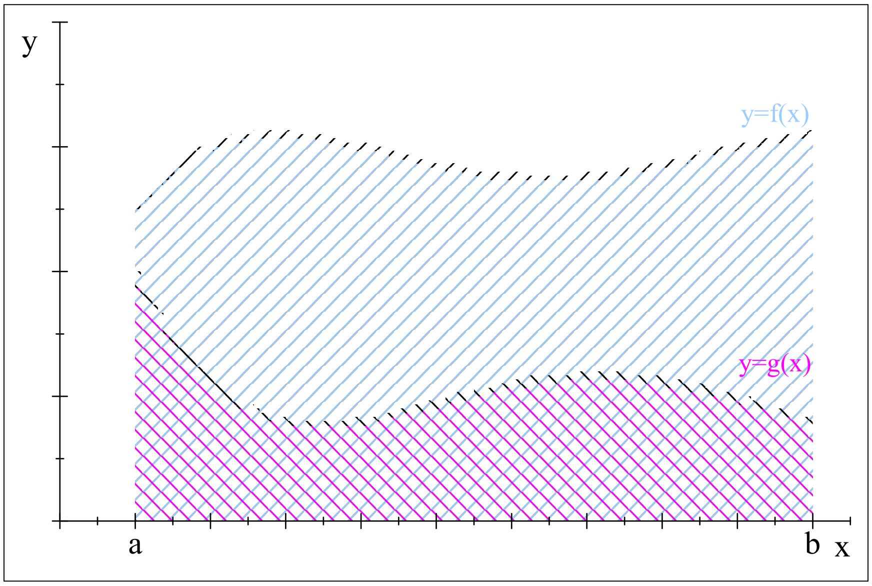
▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Section 7-1 (pp. 411-420)

AREA BETWEEN CURVES

Area between Two Curves

Consider the area bounded by $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x) \geq 0$, for $a \leq x \leq b$, as indicated in the following graph:



It is clear that

$$\begin{aligned}\text{Area between } f(x) \text{ and } g(x) &= \left(\text{Area under } f(x)\right) - \left(\text{Area under } g(x)\right)\\ &= \int_a^b f(x)dx - \int_a^b g(x)dx\\ &= \int_a^b (f(x) - g(x))dx\end{aligned}$$

It can be shown that the preceding result does not require $f(x)$ or $g(x)$ to remain positive over the interval $[a,b]$.

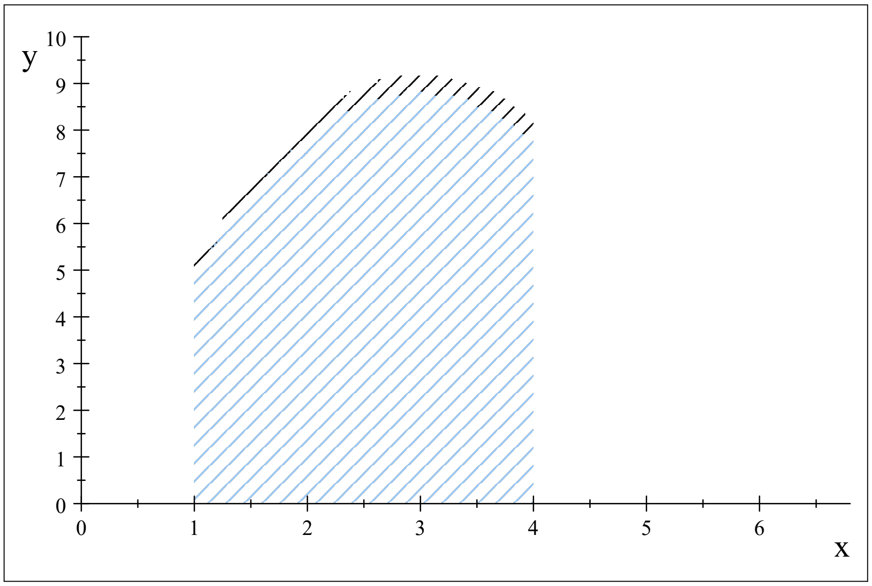
Theorem (**1–Area between Two Curves**) *If* f *and* g *are continuous and* $f(x) \geq g(x)$ *over the interval* $[a,b]$, *then the area bounded by* $y = f(x)$ *and* $y = g(x)$ *for* $a \leq x \leq b$ *is given exactly by*

$$A = \int_a^b (f(x) - g(x))dx$$

Example (**1 in Section 7-1**) *Area between a Curve and the* x *Axis*

Find the area bounded by $f(x) = 6x - x^2$ and $y = 0$ for $1 \leq x \leq 4$.

Solution *We sketch a graph of the region first:*



(The solution of every area problem should begin with a sketch. If you are not comfortable with Scientific Notebook's graphing tools, you can sketch the area by hand instead.)

Since $f(x) \geq 0$ *on* $[1,4]$,

$$\begin{aligned} A &= \int_1^4 (6x - x^2)dx \\ &= \left(3x^2 - \frac{x^3}{3}\right)\Big|_1^4 \\ &= \left(3(4^2) - \frac{4^3}{3}\right) - \left(3(1^2) - \frac{1^3}{3}\right) \\ &= 48 - \frac{64}{3} - 3 + \frac{1}{3} \\ &= 48 - 21 - 3 \\ &= 24 \end{aligned}$$

Problem (Matched—1 in Section 7-1) *Find the area bounded by* $f(x) = x^2 + 1$ *and* $y = 0$ *for* $-1 \le x \le 3$.

▸ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

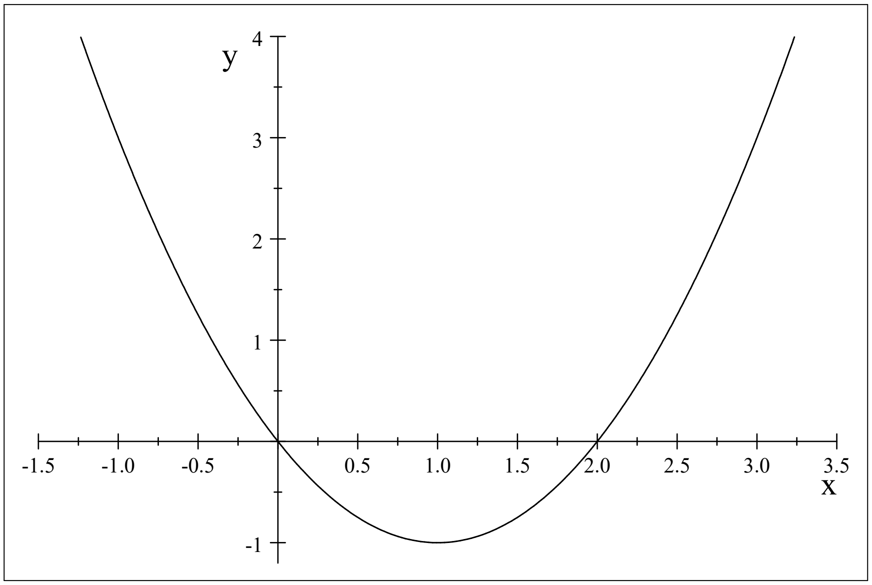
Example (2 in Section 7-1) *Area between a Curve and the* x *Axis*

Find the area between the graph of $f(x) = x^2 - 2x$ and the x axis over the indicated intervals:

(A) $[1, 2]$

(B) $[-1, 1]$

Solution *We begin by sketching the graph of* f:



(A) *From the graph, we see that* $f(x) \le 0$ *for* $1 \le x \le 2$*, so we integrate* $-f(x)$ *to get the area.*

$$\begin{aligned} A &= \int_1^2 (-f(x))dx \\ &= \int_1^2 (2x - x^2)dx \\ &= \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2 \\ &= \left(2^2 - \frac{2^3}{3}\right) - \left(1^2 - \frac{1^3}{3}\right) \\ &= 4 - \frac{8}{3} - 1 + \frac{1}{3} = \frac{2}{3} \approx 0.667 \end{aligned}$$

(B) *Since the graph shows that* $f(x) \geq 0$ *on* $[-1, 0]$ *and* $f(x) \leq 0$ *on* $[0, 1]$*, the computation of this area will require two integrals:*

$$\begin{aligned} A &= \int_{-1}^0 f(x)dx + \int_0^1 (-f(x))dx \\ &= \int_{-1}^0 (x^2 - 2x)dx + \int_0^1 (2x - x^2)dx \\ &= \left(\frac{x^3}{3} - x^2\right)\Big|_{-1}^0 + \left(x^2 - \frac{x^3}{3}\right)\Big|_0^1 \\ &= \frac{4}{3} + \frac{2}{3} = 2 \end{aligned}$$

Problem (Matched—2 in Section 7-1) *Find the area between the graph of* $f(x) = x^2 - 9$ *and the* x *axis over the indicated intervals:*

(A) $[0, 2]$

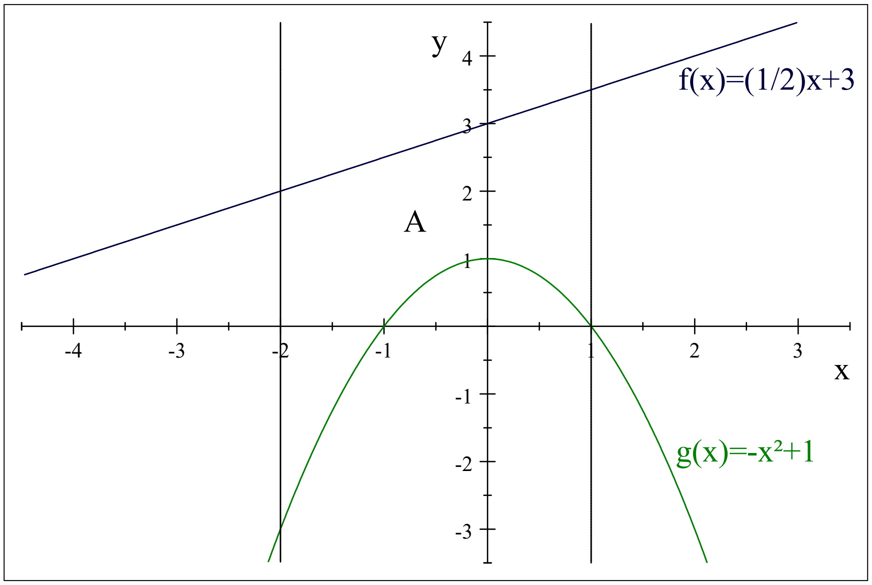
(B) $[2, 4]$

▸ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Example (3 in Section 7-1) *Area between Two Curves*

Find the area bounded by the graphs of $f(x) = \frac{1}{2}x + 3$, $g(x) = -x^2 + 1$, $x = -2$, and $x = 1$.

Solution *We first sketch the area by going to the menu* **Compute** > **Plot 2D** > **Implicit** *and adding more* **Items Plotted**.



We observe from the graph that $f(x) \geq g(x)$ *for* $-2 \leq x \leq 1$, *so*

$$\begin{aligned}
A &= \int_{-2}^{1} (f(x) - g(x))dx \\
&= \int_{-2}^{1} \left(\left(\frac{x}{2} + 3\right) - (-x^2 + 1)\right)dx \\
&= \int_{-2}^{1} \left(x^2 + \frac{x}{2} + 2\right)dx \\
&= \left(\frac{x^3}{3} + \frac{x^2}{4} + 2x\right)\Big|_{-2}^{1} \\
&= \left(\frac{1}{3} + \frac{1}{4} + 2\right) - \left(\frac{-8}{3} + \frac{4}{4} - 4\right) = \frac{33}{4} = 8.25
\end{aligned}$$

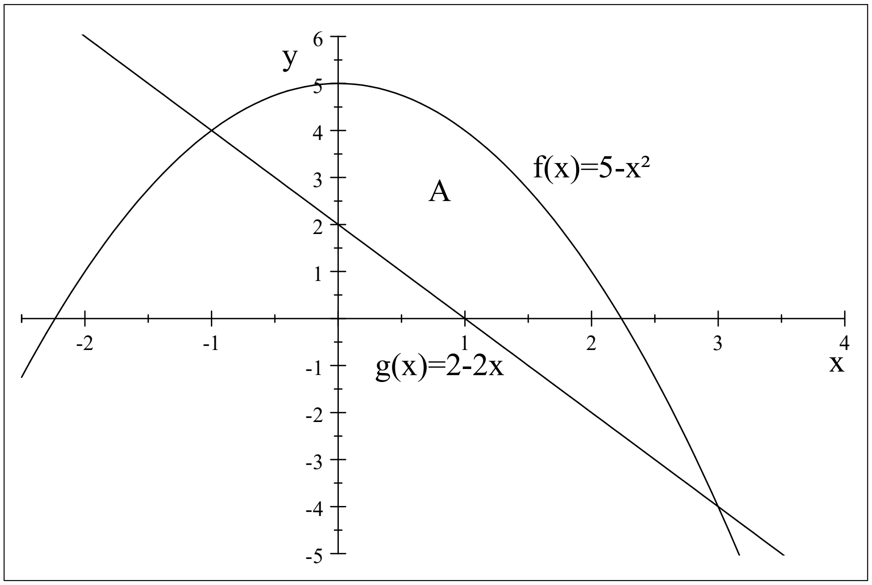
Problem (Matched—3 in Section 7-1) *Find the area bounded by* $f(x) = x^2 - 1$, $g(x) = -\frac{1}{2}x - 3$, $x = -1$, *and* $x = 2$.

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Example (4 in Section 7-1) *Area between Two Curves*

Find the area bounded by $f(x) = 5 - x^2$ and $g(x) = 2 - 2x$.

Solution *First, graph* f *and* g *on the same coordinate system.*



Since the statement of the problem does not include any limits on the values of x*, we must determine the appropriate values from the graph. The graph of* f *is a parabola and the graph of* g *is a line, as shown. The area bounded by these two graphs extends from the intersection point on the left to the intersection point on the right. To find these intersection points, we solve the equation* $f(x) = g(x)$ *for* x*:*

$$\begin{aligned} f(x) &= g(x) \\ 5 - x^2 &= 2 - 2x \\ x^2 - 2x - 3 &= 0 \\ (x+1)(x-3) &= 0 \\ x &= -1,\ 3 \end{aligned}$$

You should check these values in the original equations. (Note that the area between the graphs for $x < -1$ *is unbounded on the left, and the area between the graphs for* $x > 3$ *is unbounded on the right.) The graph shows that* $f(x) \geq g(x)$ *over the interval* $[-1, 3]$*, so we have*

$$\begin{aligned} A &= \int_{-1}^{3} (f(x) - g(x))dx \\ &= \int_{-1}^{3} (5 - x^2 - (2 - 2x))dx \\ &= \int_{-1}^{3} (3 + 2x - x^2)dx \\ &= \left(3x + x^2 - \frac{x^3}{3}\right)\Big|_{-1}^{3} \\ &= \left(3(3) + 3^2 - \frac{3^3}{3}\right) - \left(3(-1) + (-1)^2 - \frac{(-1)^3}{3}\right) = \frac{32}{3} \approx 10.667 \end{aligned}$$

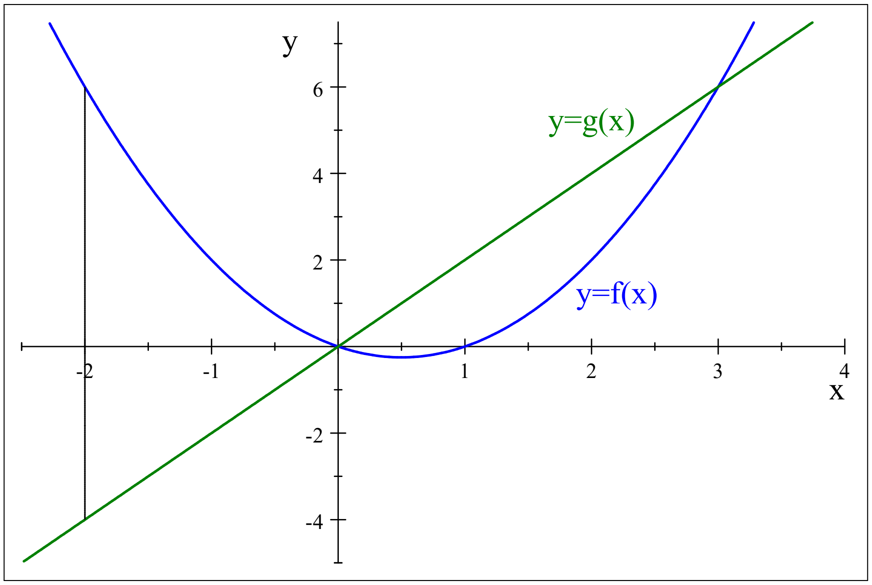
Problem (Matched—4 in Section 7-1) *Find the area bounded by* $f(x) = 6 - x^2$ *and* $g(x) = x$*.*

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Example (**5 in Section 7-1**) *Area between Two Curves*

Find the area bound by $f(x) = x^2 - x$ and $g(x) = 2x$ for $-2 \le x \le 3$.

Solution *Here are the graphs of* f *and* g*:*



Examining the graph, we see that $f(x) \ge g(x)$ *on the interval* $[-2, 0]$*, but* $g(x) \ge f(x)$ *on the interval* $[0, 3]$*. Thus, two integrals are required to compute this area:*

$$\begin{aligned} A &= \int_{-2}^{0} (f(x) - g(x))dx + \int_{0}^{3} (g(x) - f(x))dx \\ &= \int_{-2}^{0} (x^2 - x - 2x)dx + \int_{0}^{3} (2x - (x^2 - x))dx \\ &= \int_{-2}^{0} (x^2 - 3x)dx + \int_{0}^{3} (3x - x^2)dx \\ &= \left(\frac{x^3}{3} - \frac{3}{2}x^2\right)\Big|_{-2}^{0} + \left(\frac{3}{2}x^2 - \frac{x^3}{3}\right)\Big|_{0}^{3} \\ &= \left(0 - \left(\frac{(-2)^3}{3} - \frac{3}{2}(-2)^2\right)\right) + \left(\frac{3}{2}(3^2) - \frac{3^3}{3} - 0\right) \\ &= \frac{26}{3} + \frac{9}{2} = \frac{79}{6} \approx 13.167 \end{aligned}$$

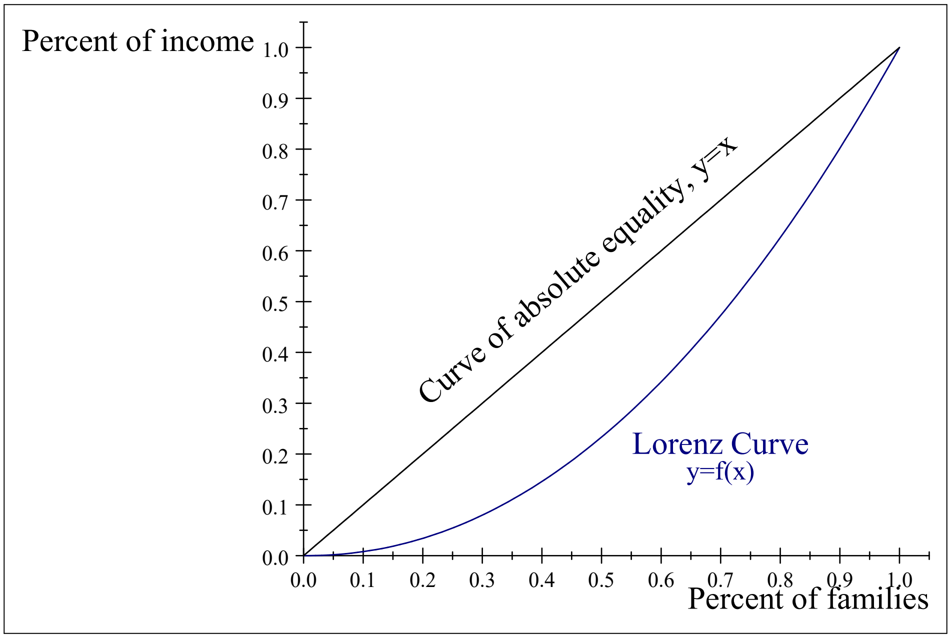
Problem (**Matched—5 in Section 7-1**) *Find the area bounded by* $f(x) = 2x^2$ *and* $g(x) = 4 - 2x$ *for* $-2 \le x \le 2$.

▸ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Application: Income Distribution

The U.S. Census Bureau compiles and analyzes a great deal of data having to do with the distribution of income among families in the United States. For 2003, the Bureau reported that the lowest 20% of families received 4% of all family income and the top 20% received 47%. The following table and graph give a detailed picture of the distribution of family income in 2003.

Income level	x	y
Under $\$24000$	0.20	0.04
Under $\$42000$	0.40	0.14
Under $\$65000$	0.60	0.30
Under $\$98000$	0.80	0.53



The graph of $y = f(x)$ in the preceding graph is called a **Lorenz Curve**. The variable x represents the cumulative percentage of families at or below a given income level, and y represents the cumulative percentage of total family income received. For example, data point $(0.40, 0.14)$ in the table indicates that the bottom 40% of families (those with incomes under $\$42000$) received 14% of the total income for all families in 2003, data point $(0.60, 0.30)$ indicates that the bottom 60% of families received 30% of the total income for all families that year, and so on.

Absolute equality of income would occur if the area between the Lorenz curve and $y = x$ were 0. In this case, the Lorenz curve would be $y = x$ and all families would receive equal shares of the total income. That is, 5% of the families would receive 5% of the income, 20% of the families would receive 20% of the income, 65% of the families would receive 65% of the income, and so on. The maximum possible area between a Lorenz curve and $y = x$ is $\frac{1}{2}$, the area of the triangle below $y = x$. In this case, we would have **absolute inequality**: all the income would be in the hands of one family and the rest would have none. In actuality, Lorenz curves lie between these two extremes. But as the area between the curves increases, the greater the inequality of income distribution.

We use a single number, the **Gini index** (named after Italian sociologist Corrado Gini), to measure income concentration. The Gini index is the ratio of two areas: the areas between $y = x$ and the Lorenz curve, and the area between $y = x$ and the x axis, from $x = 0$ to $x = 1$.

The first area equals $\int_0^1 (x - f(x))\,dx$ and the second (triangular) area equals $\frac{1}{2}$, giving the following definition:

Definition (**Gini Index of Income Concentration**) *If $y = f(x)$ is the equation of a Lorenz curve, then*

$$\textit{Gini index} = 2\int_0^1 (x - f(x))\,dx$$

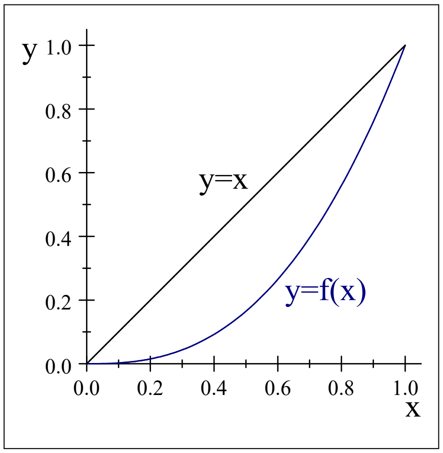
The Gini index is always a number between 0 and 1. A measure of 0 indicates absolute equality—all people share equally in the income. A measure of 1 indicates absolute inequality—one person has all the income and the rest have none. The closer the index is to 0, the closer the income is to being equally distributed. The closer the index is to 1, the closer the income is to being concentrated in a few hands. The Gini index of income concentration is used to compare income distributions at various points in time, between different groups of people, before and after taxes are paid, between different countries, and so on.

Example (**7 in Section 7-1**) *Distribution of Income*

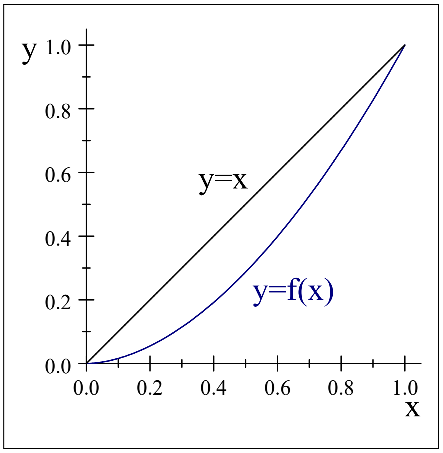
The Lorenz curve for the distribution of income in a certain country in 2005 is given by $f(x) = x^{2.6}$. Economists predict that the Lorenz curve for the country in the year 2020 will given by $g(x) = x^{1.8}$. Find the Gini index of income concentration for each curve, and interpret the results.

Solution *Here are the Lorenz curves for both years:*

2005



2020



The Gini index in 2005 is

$$2\int_0^1 (x - f(x))\,dx = 2\int_0^1 (x - x^{2.6})\,dx = 2\left(\frac{1}{2}x^2 - \frac{1}{3.6}x^{3.6}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3.6}\right) \approx 0.444$$

The projected Gini index in 2020 is

$$2\int_0^1 (x - f(x))\,dx = 2\int_0^1 (x - x^{1.8})\,dx = 2\left(\frac{1}{2}x^2 - \frac{1}{2.8}x^{2.8}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{2.8}\right) \approx 0.286$$

If this projection is correct, the Gini index will decrease, and income will be more equally distributed in the year 2020 than in 2005.

Problem (Matched—7 in Section 7-1) *Repeat Example 7 if the projected Lorenz curve in the year 2020 is given by* $g(x) = x^{3.8}$.

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Section 7-2 (pp. 421-432)

APPLICATIONS IN BUSINESS AND ECONOMICS

Probability Density Functions

We will now take a look at the use of the definite integral to determine probabilities. Suppose that an experiment is designed in such a way that any real number x on the interval $[c, d]$ is a possible outcome. For example, x may represent an IQ score, the height of a person, or the life of a lightbulb in hours. Technically, we refer to x as a **continuous random variable**.

In certain situations, it is possible to find a function f with x as an independent variable such that the function f can be used to determine the probability that the outcome x of an experiment will be in the interval $[c, d]$. Such a function, called a **probability density function**, must satisfy three conditions:

1. $f(x) \geq 0$ for all real x.
2. The area under the graph of $f(x)$ over the interval $(-\infty, \infty)$ is exactly 1.
3. If $[c, d]$ is a subinterval of $(-\infty, \infty)$, then

$$\text{Probability}(c \leq x \leq d) = \int_c^d f(x)\,dx$$

Example (1 in Section 7-2) *Duration of Telephone Calls*

Suppose that the length of telephone calls (in minutes) in a public telephone booth is a continuous random variable with probability density function

$$f(t) = \begin{cases} \frac{1}{4}e^{-t/4}, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(A) Determine the probability that a call selected at random will last between 2 and 3 minutes.

(B) Find b (to two decimal places) such that the probability of a call selected at random lasting between 2 and b minutes is 0.5.

Solution

(A)

$$\text{Probability}(2 \le t \le 3) = \int_2^3 \frac{1}{4} e^{-t/4} dt$$
$$= (-e^{-t/4})|_2^3$$
$$= -e^{-3/4} + e^{-1/2} \approx 0.13$$

(B) *We want to find* b *such that Probability*$(2 \le t \le b) = 0.5$*. Therefore,*

$$\int_2^b \frac{1}{4} e^{-t/4} dt = 0.5$$
$$-e^{-b/4} + e^{-1/2} = 0.5$$
$$e^{-b/4} = e^{-1/2} - 0.5$$
$$-\frac{b}{4} = \ln(e^{-0.5} - 0.5)$$
$$b = 8.96 \text{ minutes}$$

Thus, the probability of a call selected at random lasting between 2 *and* 8.96 *minutes is* $.5$*.*

Problem (Matched—1 in Section 7-2)

(A) *In Example 1, find the probability that a call selected at random will last* 4 *minutes or less.*

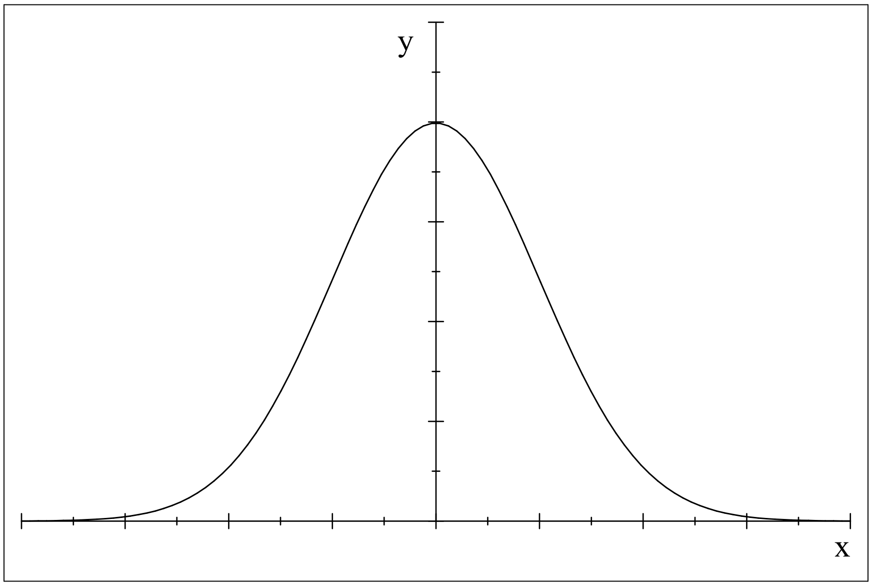
(B) *Find* b *(to two decimal places) so that the probability of a call selected at random lasting* b *minutes or less is* $.9$*.*

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

One of the most important probability distribution functions, the normal probability distribution function, is defined as follows:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

where μ is the mean and σ is the standard deviation. Here is its familiar bell-shaped graph:



Since $\int e^{-x^2} dx$ is nonintegrable in terms of elementary functions (that is, the antiderivative cannot be expressed as a finite combination of simple functions), probabilities such as

$$\text{Probability}(c \leq x \leq d) = \frac{1}{\sqrt{2\pi}} \int_c^d e^{-(x-\mu)^2/2\sigma^2} dx$$

are generally determined by using tables of areas under the normal curve or computers that employ refined techniques of approximation.

Continuous Income Stream

Suppose that your aunt has established a trust that pays you $\$2000$ a year for 10 years. What is the total amount you will receive from the trust by the end of the 10th year? Since there are 10 payments of $\$2000$ each, you will receive

$$10 \times 2000 = \$20000$$

Let us now assume that the income stream is continuous at the rate $\$2000$ per year. That is, after $\frac{1}{4}$ year you will have received $2000\left(\frac{1}{4}\right) = \$500$, after 2 years you will have received $2000(2) = \$4000$, after 5.3 years you will have received $2000(5.3) = \$10600$, and so on. The area under the graph of $f(t) = 2000$ from 0 to t represents the amount accumulated t years after the start. The total income over a 10 year period is thus also given by the definite integral

$$\int_0^{10} 2000 dt = 2000t|_0^{10} = 2000(10) = \$20000$$

Let us now apply the idea of a continuous income stream to a problem where the income rate is not constant.

Example (**2 in Section 7-2**) *Continuous Income Stream*

The rate of change of the income produced by a vending machine located at an airport is given by

$$f(t) = 5000e^{0.04t}$$

where t is time in years since the installation of the machine. Find the total income produced by the machine during the first 5 years of operation.

Solution *The area under the graph of the rate-of-change function from* 0 *to* 5 *represents the total change in income over the first* 5 *years, and hence is given by a definite integral:*

$$\begin{aligned} \textit{Total income} &= \int_0^5 5000e^{0.04t} dt \\ &= 125000e^{0.04t}|_0^5 \\ &= 125000e^{0.04(5)} - 125000e^{0.04(0)} \\ &= 152675 - 125000 \\ &= \$27675 \end{aligned}$$

Thus, the vending machine produces a total income of $\$27675$ *during the first* 5 *years of operation.*

Problem (**Matched—2 in Section 7-2**) *Referring to Example 2, find the total income produced (to the nearest dollar) during the second* 5 *years of operation.*

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

In reality, income produced from a vending machine is not usually received as a single payment at the end of the year, even though the rate is given as a yearly rate. In this type of problem, it is convenient to assume that income is actually received as a **continuous stream**; that is, we assume that the rate at which income is received is a continuous function of time. The rate of change is called the **rate of flow** of the continuous income stream.

Definition (**Total Income for a Continuous Income Stream**) *If* $f(t)$ *is the rate of flow of a continuous income stream, the total income produced during the period from* $t = a$ *to* $t = b$ *is*

$$\text{Total income} = \int_a^b f(t)dt$$

Future Value of a Continuous Income Stream

Recall the continuous compound interest formula

$$A = Pe^{rt}$$

where P is the principal (or present value), A is the amount (or future value), r is the annual rate of continuous compounding (expressed as a decimal), and t is time in years.

Now suppose that the income produced by a continuous income stream is invested as soon as it is received at a rate r, compounded continuously. How can we find the total income produced and the interest earned by this income? Since the income does not arrive as one lump sum at the beginning but rather as a continuous stream, we cannot simply use the continuous compound interest formula directly. Instead, by using a Riemann-sum approach, we are able to devise another formula:

Definition (**Future Value of a Continuous Income Stream**) *If* $f(t)$ *is the rate of flow of a continuous income stream,* $0 \le t \le T$*, and if the income is continuously invested at a rate* r*, compounded continuously, then the future value* FV *at the end of* T *years is given by*

$$FV = \int_0^T f(t)e^{r(T-t)dt} = e^{rT}\int_0^T f(t)e^{-rt}dt$$

For example, returning to the trust set up by your aunt, suppose that you invest the income as soon as you receive it at a rate of 8% compounded continuously. At the end of the 10th year, the future value will be

$$\begin{aligned} FV &= e^{rT}\int_0^T f(t)e^{-rt}dt \\ &= e^{0.08(10)}\int_0^{10} 2000e^{-0.08t}dt \\ &= 2000e^{0.8}\int_0^{10} e^{-0.08t}dt \\ &= 2000e^{0.8}\left(\frac{e^{-0.08t}}{-0.08}\right)\Bigg|_0^{10} \\ &= 2000e^{0.8}(-12.5e^{-0.8} + 12.5) = \$30639 \end{aligned}$$

Since the money coming from the trust is $\$20000$, you will have earned

$30639 - 20000 = \$10639$ of interest during the 10 years.

Example (3 in Section 7-2) *Future Value of a Continuous Income Stream*

Using the continuous income rate of flow for the vending machine in Example 2, namely,

$$f(t) = 5000e^{0.04t}$$

find the future value of this income stream at 12%, compounded continuously for 5 years, and find the total interest earned. Compute answers to the nearest dollar.

Solution *Using the formula*

$$FV = e^{rT}\int_0^T f(t)e^{-rt}dt$$

with $r = 0.12$, $T = 5$, *and* $f(t) = 5000e^{0.04t}$, *we have*

$$\begin{aligned} FV &= e^{0.12(5)}\int_0^5 5000e^{0.04t}e^{-0.12t}dt \\ &= 5000e^{0.6}\int_0^5 e^{-0.08t}dt \\ &= 5000e^{0.6}\left(\frac{e^{-0.08t}}{-0.08}\right)\Big|_0^5 \\ &= 5000e^{0.6}(-12.5e^{-0.4} + 12.5) \\ &= \$37545 \end{aligned}$$

Thus, the future value of the income stream compounded continuously at the end of 5 *years is* $\$37545$.

In Example 2, we saw that the total income produced by this vending machine over a 5*-year period was* $\$27675$. *The difference between future value and income is interest. Thus,*

$$37545 - 27675 = \$9870$$

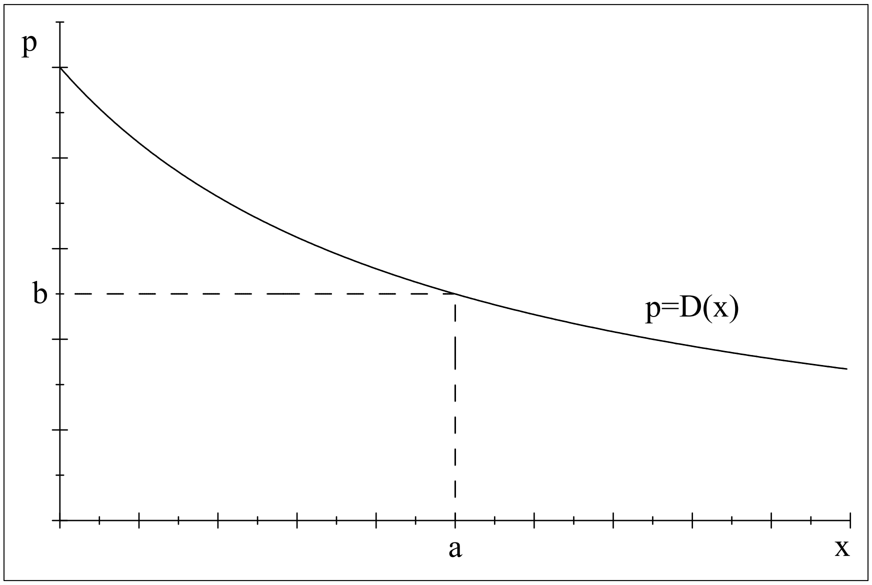
is the interest earned by the income produced by the vending machine during a 5-year period.

Problem (Matched—3 in Section 7-2) *Repeat Example 3 if the interest rate is* 9%, *compounded continuously.*

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

Consumers' and Producers' Surplus

Let $p = D(x)$ be the price-demand equation for a product, where x is the number of units of the product that consumers will purchase at a price of $p\$$ per unit. Suppose that $\bar{p}$ represents the current price and $\bar{x}$ is the number of units that can be sold at that price. As you can see in the following graph, if the price is higher than $\bar{p}$, the demand x is less than $\bar{x}$, but some consumers are still willing to pay the higher price.



$$a = \bar{x}$$

$$b = \bar{p}$$

Consumers who are willing to pay more than $\bar{p}$ but are buying the product at $\bar{p}$ are saving money. We call **consumers' surplus** the total amount of money saved by all consumers willing to pay more than $\bar{p}$. Through a series of Riemann sums, it is possible to show that this amount is equal to the area between the graph of $p = D(x)$ the line $p = \bar{p}$.

Definition (Consumers' Surplus) *If* $(\bar{x}, \bar{p})$ *is a point on the graph of the price-demand equation* $p = D(x)$ *for a particular product, then the consumers' surplus* CS *at a price level of* $\bar{p}$ *is*

$$\int_0^{\bar{x}} (D(x) - \bar{p})dx$$

which is the area between $p = \bar{p}$ *and* $p = D(x)$ *from* $x = 0$ *to* $x = \bar{x}$.
The consumers' surplus represents the total savings to consumers who are willing to pay more than $\bar{p}$ *for the product, but are still able to buy the product for* $\bar{p}$.

Example (4 in Section 7-2) *Consumers' Surplus*

Find the consumers' surplus at a price level of $\$8$ for the price-demand equation

$$p = D(x) = 20 - 0.05x$$

Solution

Step 1. *Find* $\bar{x}$*, the demand when the price is* $\bar{p} = 8$*:*

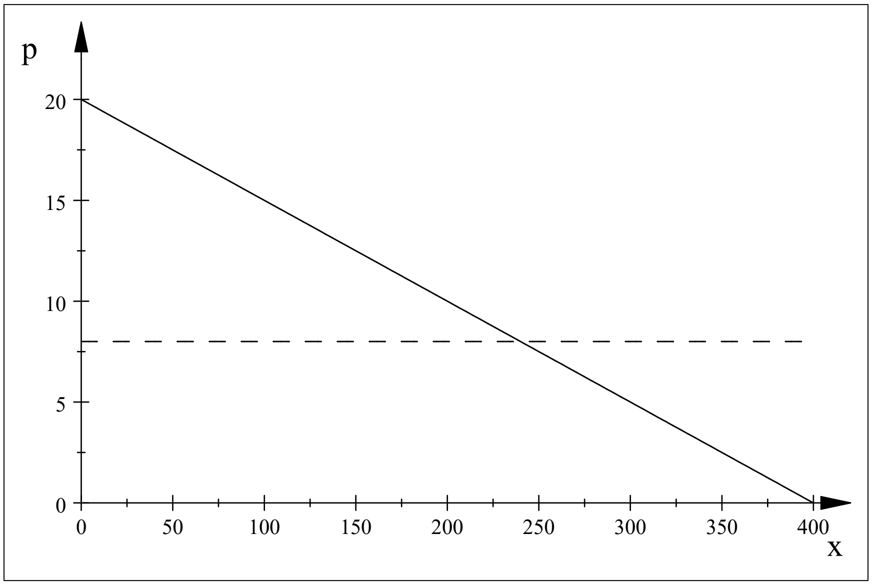
$$\bar{p} = 20 - 0.05\bar{x}$$

$$8 = 20 - 0.05\bar{x}$$

$$0.05\bar{x} = 12$$

$$\bar{x} = 240$$

Step 2. *Sketch a graph:* $20 - 0.05x$



Step 3. *Find the consumers' surplus:*

$$\begin{aligned} CS &= \int_0^{\bar{x}} (D(x) - \bar{p})dx \\ &= \int_0^{240} (20 - 0.05x - 8)dx \\ &= \int_0^{240} (12 - 0.05x)dx \\ &= (12x - 0.025x^2)\Big|_0^{240} \\ &= 2800 - 1440 = \$1440 \end{aligned}$$

Thus, the total savings to consumers who are willing to pay a higher price for the product is $1440.

Problem (Matched—4 in Section 7-2) *Repeat Example 4 for a price level of* $4.

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

The analogous amount for $p = S(x)$ a price-supply equation is the producers' surplus.

Definition (Producers' Surplus) *If* $(\bar{x}, \bar{p})$ *is a point on the graph of a price-supply equation* $p = S(x)$, *then the* ***producers' surplus*** PS *at a price level of* $\bar{p}$ *is*

$$PS = \int_0^{\bar{x}} (\bar{p} - S(x))dx$$

which is the area between $p = \bar{p}$ *and* $p = S(x)$ *from* $x = 0$ *to* $x = \bar{x}$.
The producers' surplus represents the total gain to producers who are willing to supply units at a lower price than $\bar{p}$, *but are still able to supply units at* $\bar{p}$.

Example (5 in Section 7-2) *Producers' Surplus*

Find the producers' surplus at a price level of $20 for the price-supply equation

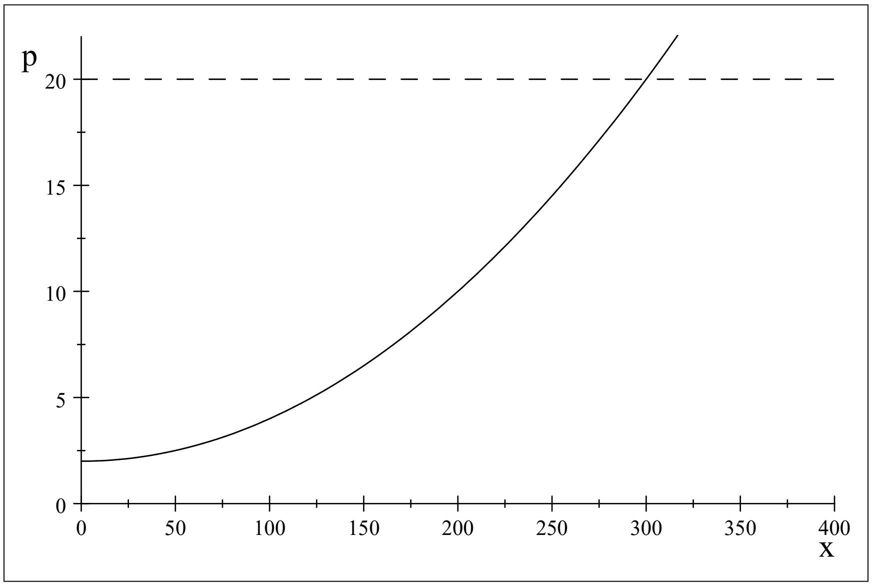
$$p = S(x) = 2 + 0.0002x^2$$

Solution

Step 1. *Find* $\bar{x}$*, the supply when price is* $\bar{p} = 20$*:*

$$\begin{aligned} \bar{p} &= 2 + 0.0002\bar{x}^2 \\ 20 &= 2 + 0.0002\bar{x}^2 \\ 0.0002\bar{x}^2 &= 18 \\ \bar{x}^2 &= 90000 \\ \bar{x} &= 300 \end{aligned}$$

Step 2, *Sketch a graph:* $2 + 0.0002x^2$



Step 3. *Find the producers' surplus:*

$$\begin{aligned} PS &= \int_0^{\bar{x}} (\bar{p} - S(x))dx = \int_0^{300} (20 - (2 + 0.0002x^2))dx \\ &= \int_0^{300} (18 - 0.0002x^2)dx = \left(18x - 0.0002\frac{x^3}{3}\right)\Big|_0^{300} \\ &= 5400 - 1800 = \$3600 \end{aligned}$$

Thus, the total gain to producers who are willing to supply units at a lower price is $3600*.*

Problem (Matched—5 in Section 7-2) *Repeat Example 5 for a price level of* $4.

▶ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

In a free competitive market, the price of a product is determined by the relationship between supply and demand. If $p = D(x)$ and $p = S(x)$ are the price-demand and price-supply equations, respectively, for a product, and if $(\bar{x}, \bar{p})$ is the point of intersection of these equations, $\bar{p}$ is called the **equilibrium price** and $\bar{x}$ is called the **equilibrium quantity**.

Example (6 in Section 7-2) *Equilibrium Price and Consumers' and Producers' Surplus*

Find the equilibrium price, and then find the consumers' surplus and producers' surplus at the equilibrium price level if

$$p = D(x) = 20 - 0.05x$$

and

$$p = S(x) = 2 + 0.0002x^2$$

Solution

Step 1. *Find the equilibrium point. Set $D(x)$ equal to $S(x)$ and solve:*

$$D(x) = S(x)$$

$$20 - 0.05x = 2 + 0.0002x^2$$

$$0.0002x^2 + 0.05x - 18 = 0$$

$$x^2 + 250x - 90000 = 0$$

$$x = 200, -450$$

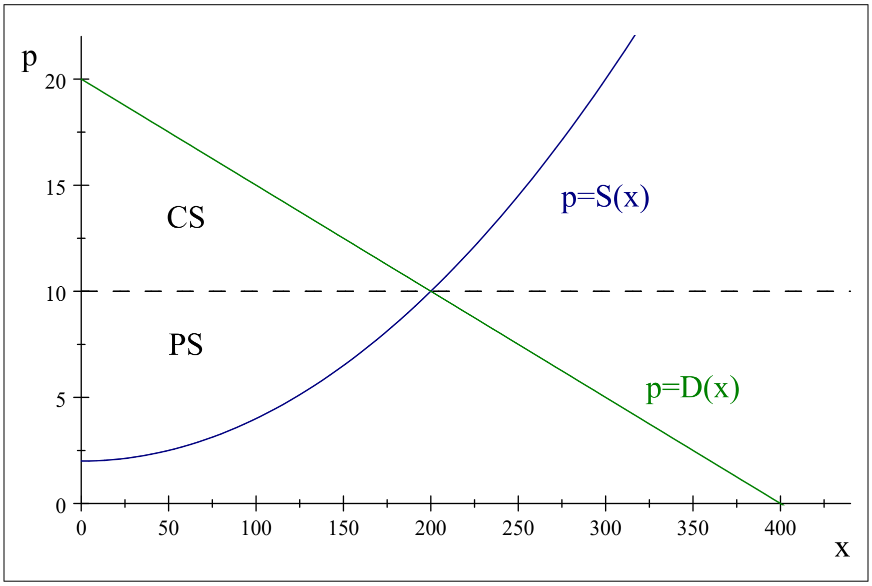
Since x cannot be negative, the only solution is $x = 200$. The equilibrium price can be determined by using $D(x)$ or $S(x)$. We will use both to check our work:

$$\bar{p} = D(200) = 20 - 0.05(200) = 10$$

$$\bar{p} = S(200) = 2 + 0.0002(200^2) = 10$$

Thus, the equilibrium price is $\bar{p} = 10$, and the equilibrium quantity is $\bar{x} = 200$.

Step 2. *Sketch a graph of the two functions:*



Step 3. *Find the consumers' surplus:*

$$
\begin{aligned}
CS &= \int_0^{\bar{x}} (D(x) - \bar{p})dx = \int_0^{200} (20 - 0.05x - 10)dx \\
&= \int_0^{200} (10 - 0.05x)dx \\
&= (10x - 0.025x^2)|_0^{200} \\
&= 2000 - 1000 = \$1000
\end{aligned}
$$

Step 4. *Find the producers' surplus:*

$$
\begin{aligned}
PS &= \int_0^{\bar{x}} (\bar{p} - S(x))dx \\
&= \int_0^{200} (10 - (2 + 0.0002x^2))dx \\
&= \int_0^{200} (8 - 0.0002x^2)dx \\
&= \left(8x - 0.0002\frac{x^3}{3}\right)\Big|_0^{200} \\
&= 1600 - \frac{1600}{3} \approx \$1067
\end{aligned}
$$

Problem (Matched—6 in Section 7-2) *Repeat Example 6 for*

$$p = D(x) = 25 - 0.001x^2$$

and

$$p = S(x) = 5 + 0.1x$$

▸ Solve this problem in Mathematica. This problem is designed to reinforce the ideas and techniques presented in the previous example.

End of Textbook Notes for Lesson 13