1. [1 point] If we use the linear approximation of $f(x) = \arctan(2x)$ at $a = 0$ to approximate $\arctan(0.05)$, it will give?

Solution: We want $\arctan(0.05) = f(0.05/2)$.

We will use the linearization around $x = 0$ since at $x = 0$ we can compute $f(x)$ easily (and also because the questions tells us to use it!).

First $f'(x) = \dfrac{2}{1+x^2}$.

This means that $f(0) = arctan(2 \times 0) = arctan(0) = 0$, and $f'(0) = 2$.

So the equation of the tangent line to $f(x)$ at $x = 0$ is $L(x) = f(a) + f'(a)(x - a) = f(0) + f'(0)(x - 0) = 2x$.

Therefore $\arctan(0.05) = f(0.05/2) \approx L(0.05/2) = 2(0.05/2) = 0.05$.

2. [1 point] What is $\dfrac{d}{dx}\left(\displaystyle\int_x^{x^2} f(t)\,dt\right)$?

Solution:

$$\frac{d}{dx}\left(\int_x^{x^2} f(t)\,dt\right) f(x^2)(x^2)' - f(x)(x)' = 2xf(x^2) - f(x)$$

3. [1 point] If we wish to integrate $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$ with a trigonometric substitution, we would let $x =$?

Solution: We want $\sqrt{9-x^2}$ to simplify due to a trig identity. If we set $x = 3\sin\theta$ then $\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = 3\sqrt{1-\sin^2\theta} = 2\cos\theta$. So this seems promising.

Moreover we would have $dx = 3\cos\theta d\theta$, so we would have the following.

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{27\sin^3\theta}{\sqrt{9-(3\sin\theta)^2}} 3\cos\theta d\theta = \int \frac{27\sin^3\theta}{3\cos\theta} 3\cos\theta d\theta = 27\int(1-\cos^2\theta)\sin\theta d\theta$$

Setting $u = \cos\theta$ we have $du = -\sin\theta d\theta$.

$$27\int(1-\cos^2\theta)\sin\theta d\theta = -27\int(1-u^2)du = -27u + \frac{27u^3}{3} + C$$

Now $x = 3\sin\theta$, which mean we might imagine a right triangle where the side opposite θ is of length x and the hypoteneuse has length 3 in order to get $x = 3\sin\theta$. Then we seen by Pythagoras that the third side is $\sqrt{9-x^2}$, so $\cos\theta = \sqrt{9-x^2}/3$. So we have $u = \sqrt{9-x^2}/3$, and we get as our final answer

$$-9\sqrt{9-x^2} + \frac{(9-x^2)^{3/2}}{3} + C$$

4. [1 point] The length of the sides of a cube is increasing at a rate of 2 cm/s. At the moment when the volume is 64 cm^3, at what rate is the volume increasing (in cm^3/s) ?

Solution: We are given the rate of change of the side-length and we want to know the rate of change of the volume. So we need an equation that relates the volume and the side-length. If we let the side-length be x and the volume be v then we have $v = x^3$. Now we think of both v and x

as being functions of time t, and differentiate $v = x^3$ with respect to t: remember to use the chain rule since v and x are both functions.

$$\begin{aligned}\frac{d}{dt}v &= \frac{d}{dt}x^3\\ \frac{dv}{dt} &= 3x^2\frac{dx}{dt}\end{aligned}$$

At the moment that the volume is 64, we have that $x = (64)^{1/3} = 4$, and we know that $dx/dt = 2$. So we have

$$\frac{dv}{dt} = 3(4)^2(2) = 96$$

5. [1 point] The derivative of $f(x) = x^{\cos x}$ has the form $f'(x) = f(x)\,p(x)$. What is $p(x)$?

Solution: We compute the derivative. We can write the function as $f(x) = e^{\ln(f(x))}$, use a property of logarithms to bring the exponent down, and then the chain and product rules.

$$\begin{aligned}\frac{d}{dx}x^{\cos(x)} = \frac{d}{dx}e^{\ln(x^{\cos(x)})} &= \frac{d}{dx}e^{\cos(x)\ln(x)}\\ &= e^{\cos(x)\ln(x)}\frac{d}{dx}\left(\cos(x)\ln(x)\right)\\ &= e^{\cos(x)\ln(x)}\left(-\sin(x)\ln(x) + \cos(x)\frac{1}{x}\right)\\ &= x^{\cos(x)}\left(-\sin(x)\ln(x) + \cos(x)\frac{1}{x}\right)\end{aligned}$$

6. [2 points] Find $\frac{dy}{dx}$ if $5x^3y^4 + 3xy^3 = 7x$.

Solution: We differentiate implicitly.

$$\begin{aligned}\frac{d}{dx}\left(5x^3y^4 + 3xy^3\right) &= \frac{d}{dx}(7x)\\ 15x^2y^4 + 20x^3y^3y' + 3xy^3 + 9xy^2y' &= 7\\ y'\left(20x^3y^3 + 9xy^2\right) &= 7 - 15x^2y^4 - 3xy^3\\ y' &= \frac{7 - 15x^2y^4 - 3xy^3}{20x^3y^3 + 9xy^2}\end{aligned}$$

7. [2 points] Set up the approximation of the definite integral $\displaystyle\int_0^1 e^x\,dx$ with $n = 4$ rectangles using the right-hand endpoints. You do not need to evaluate the sum.

Solution: We have $a = 0$ and $b = 1$, so $\triangle x = (b-1)/4 = 1/4$. Since we want the right hand endpoints, we have $x_1 = 1/4$, $x_2 = 2/4$, $x_3 = 3/4$, $x_4 = 4/4$. This gives the following approximation.

$$\left(e^{1/4} + e^{2/4} + e^{3/4} + e^{4/4}\right)(1/4)$$

Of course this can be simplified somewhat...

We can also use $\sum$-notation. For the right-hand endpoints we have $x_i = a + i\triangle x = 0 + i(1-0)/4 = i/4$.

$$\triangle x \sum_{i=1}^{n} e^{x_i} = \frac{1}{4}\sum_{i=1}^{4} e^{1/4}$$

8. [6 points] Evaluate the following integrals:

(a) $\displaystyle\int \sin^7 x \cos^3 x \, dx$

Solution: We have an odd power, so we can substitute. Since both are odd we have our choice, so we choose the larger one, and set $u = \sin(x)$ and $du = \cos(x)\,dx$.

$$\int \sin^7 x \cos^3 x \, dx = \int \sin^7 x \cos^2 x \cos x \, dx = \int \sin^7 x(1-\sin^2 x)\cos x \, dx$$

Now our substitution gives a polynomial.

$$= \int u^7(1-u^2)\,du = \int u^7 - u^9 \, du = \frac{u^8}{8} - \frac{u^{10}}{10} + C = \frac{\sin^8 x}{8} - \frac{\sin^{10} x}{10} + C$$

(b) $\displaystyle\int_1^e \left(\frac{4(\ln x)^3}{x} - \frac{2(\ln x)^{1/3}}{x}\right) dx$

Solution: We identify that the derivative of $\ln x$ is $1/x$, so the substitution $u = \ln x$ lkooks promising, with $du = (1/x)\,dx$.

$$\begin{aligned}
\int_1^e \left(\frac{4(\ln x)^3}{x} - \frac{2(\ln x)^{1/3}}{x}\right) dx &= \int_1^e \left(4(\ln x)^3 - 2(\ln x)^{1/3}\right)\frac{1}{x}\,dx \\
&= \int_{x=1}^{x=e} \left(4u^3 - 2u^{1/3}\right) du \\
&= \left(u^4 - \frac{3}{2}u^{4/3}\right)\Big|_{x=1}^{x=e} \\
&= \left((\ln x)^4 - \frac{3}{2}(\ln x)^{4/3}\right)\Big|_{x=1}^{x=e} \\
&= \left(1^4 - \frac{3}{2}1^{4/3}\right) - \left(0^4 - \frac{3}{2}0^{4/3}\right) \\
&= -\frac{1}{2}
\end{aligned}$$

(c) $\int x^2\, e^{-x}\, dx$

Solution: Integration by parts. We first choose $u = x^2$ and $dv = e^{-x} dx$. This gives $du = 2x\, dx$ and $v = -e^{-x}$.

$$\int x^2\, e^{-x}\, dx = -x^2 e^{-x} - \int (-e^{-x} 2x)\, dx = -x^2 e^{-x} + 2 \int x e^{-x}\, dx$$

Again, integration by parts with $u = x$ and $dv = e^{-x}$, so we have $du = dx$ and $v = -e^{-x}$.

$$\begin{aligned} -x^2 e^{-x} + 2 \int x e^{-x}\, dx = -x^2 e^{-x} + 2 \left(-x e^{-x} - \int (-e^{-x})\, dx \right) &= -x^2 e^{-x} + 2 \left(-x e^{-x} + \int e^{-x}\, dx \right) \\ &= -x^2 e^{-x} + 2 \left(-x e^{-x} - e^{-x} \right) \\ &= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \end{aligned}$$

9. [0 points] What is the sum of the angles of that rectangle?