CONCORDIA UNIVERSITY **Department of Mathematics and Statistics**

Course	Number		Section(s)
Mathematics	**MATH 364**		**A**
Examination	Date	Time	Pages
Midterm	**October 2015**	**$1\frac{1}{4}$ hours**	**4**
Instructor			Course Examiner
Paweł Góra			**Paweł Góra**

Special Instructions: Approved calculators permitted. Lined paper booklets.

READ THE QUESTIONS CAREFULLY !!! SHOW ALL WORK !!! JUSTIFY ALL STEPS !!! GOOD LUCK !!!

MARKS: marks for each problem are shown in front of the problems. Maximum total is 100.

↓MARKS

15 **Problem 1 :** **(a)** Is the following sentence always true ? Give the proof or a counterexample.

$$[(p \Rightarrow q) \wedge (p \Rightarrow \neg q)] \Rightarrow (\neg p) \ .$$

(b) Write the negation of the sentence:

$$\forall_{\ \varepsilon>0}\ \exists_{\ \delta>0}\ \forall_{\ x\in\mathbb{R}}\ \forall_{\ y\in\mathbb{R}}\ \ |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon \ .$$

15 **Problem 2 :** Let $\mathcal{T}$ be the set of all triangles in the plane which have all vertices at rational points. Show that $\mathrm{card}(\mathcal{T}) = \aleph_0$. What would be the cardinality if we allow one non-rational vertex?

15 **Problem 3 :** Find the limit

$$\lim_{n\to+\infty} \frac{1}{n} \ln(5^n + \pi\cdot(-1)^n + \sin^2(n^n + (10!)^n)) \ .$$

15 **Problem 4 :** Find the supremum and the infimum of the set $A = \left\{ \left(\frac{n^2+1}{n^2}\right) : n = 1, 2, \dots \right\}$.

25 **Problem 4a : This is a replacement problem for Problem 4.** You can be awarded points for Problem 4 or Problem 4a but not for both. Your choice.

Find the supremum and the infimum of the set

$$A = \left\{ \sqrt[n]{n} : n = 2, 3, \dots \right\} \ .$$

15 **Problem 5 :** Let $a_1 = 2$ and $a_{n+1} = a_n - \frac{a_n^2 - a_n}{2a_n - 1}$ for $n = 1, 2, \dots$ Prove that the sequence $\{a_n\}$ is decreasing and bounded below by 1. Conclude that it is convergent (refer to a theorem) and find its limit. What would happen if $a_1 = 0.7$?

15 **Problem 6 :** Prove that for any $n \in \mathbb{N}$

$$12 | 3^n + 7^{n-1} + 8 \ ,$$

i.e., $3^n + 7^{n-1} + 8$ is divisible by 12.

10 **Problem 7 :** **(a)** Prove that $\sqrt{\sqrt{2}+\sqrt{3}}$ is not a rational number.

(b) Assume that $a \cdot b$ is rational. Must both a and b be rational numbers?

SOLUTIONS:

Problem 1:

(a)

$$[(p \Rightarrow q) \wedge (p \Rightarrow \neg q)] \Rightarrow (\neg p) \ .$$

First method, checking all possibilities:

p	q	$p \implies q$	$p \implies \neg q$	$\wedge$	$\neg p$	full
0	1	1	1	1	1	1
1	0	0	1	0	0	1
0	0	1	1	1	1	1
1	1	1	0	0	0	1

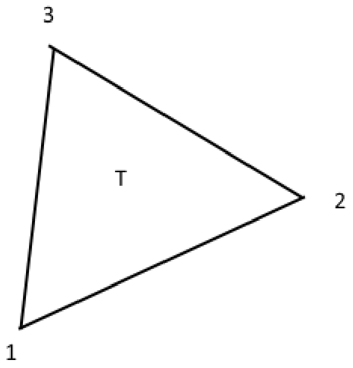
Since for all possible values of p and q the sentence is true it is always true.

Second method: Assume that the sentence is sometimes false. Then, since the only false implications is $1 \implies 0$, we have $p = 1$ and $(p \implies q) = 1$ and $(p \implies \neg q) = 1$. Then, $q = 1$ and $\neg q = 1$, which is a contradiction. Thus, the sentence is always true.

(b)

$$\exists_{\ \varepsilon>0} \ \forall_{\ \delta>0} \ \exists_{\ x\in\mathbb{R}} \ \exists_{\ y\in\mathbb{R}} \ \ |x-y| < \delta \wedge |f(x) - f(y)| \geq \varepsilon \ .$$

Problem 2: If two triangles are different, then they cannot have identical vertices, i.e., the map $f(T) = (x_1, y_1, x_2, y_2, x_3, y_3)$ is injective, where the vertices of T are (x_1, y_1), $(x_2, y_2$, (x_3, y_3) and we agree for some order of vertices for all triangles. For example, we can agree that the vertex 1 is the one with the smallest x-coordinate and if there are two with the same the smallest x-coordinate, the one of them with smaller y-coordinate. Then, the vertex 2 is the next vertex when we move from vertex 1 along the perimeter of the triangle in counter-clock direction. The remaining vertex is vertex 3.



The map f defines an injection of $\mathcal{T}$ into $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$. Since $\mathbb{Q}$ is countable and a finite Cartesian product of countable sets is countable, $\mathcal{T}$ is countable. Since it is obviously infinite $\text{card}(\mathcal{T}) = \aleph_0$.

If we allow one non-rational vertex, there is a continuum choices for this vertex so $\text{card}(\mathcal{T}) \geq \mathfrak{c}$. The map f above defines injection of of $\mathcal{T}$ into $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The cardinality of this set is again $\mathfrak{c}$ (finite Cartesian product of sets of cardinality $\mathfrak{c}$). Thus, $\text{card}(\mathcal{T}) = \mathfrak{c}$.

Problem 3: We have

$$\frac{1}{n}\ln(5^5 - (1/2)\cdot 5^n) \leq \frac{1}{n}\ln(5^n + \pi\cdot(-1)^n + \sin^2(n^n + (10!)^n)) \leq \frac{1}{n}\ln(5^n + 5^n + 5^n),$$

or

$$\frac{1}{n}\ln((1/2)\cdot 5^n) \leq \frac{1}{n}\ln(5^n + \pi\cdot(-1)^n + \sin^2(n^n + (10!)^n)) \leq \frac{1}{n}\ln(3\cdot 5^n),$$

or

$$\frac{1}{n}\ln(1/2) + \frac{1}{n}\ln(5^n) \le \frac{1}{n}\ln(5^n + \pi \cdot (-1)^n + \sin^2(n^n + (10!)^n)) \le \frac{1}{n}\ln(3) + \frac{1}{n}\ln(5^n).$$

The limit on the LHS is $\ln 5$ and the limit on the RHS is also $\ln 5$. Thus, by Squeeze Th., the limit of our sequence is $\ln 5$.

Problem 4: We can write $\frac{n^2+1}{n^2} = 1 + \frac{1}{n^2}$ which clearly shows that elements of A form a decreasing sequence. Thus,

$$\sup A = \max A = 1 + 1/1^2 = 2 \ .$$

We will prove that $\inf A = 1$. We have $1 < 1 + 1/n^2$ for all $n \ge 1$, so 1 is a lower bound for A. We will show that for arbitrary $\varepsilon >$ we can find element $1 + 1/n^2 \in A$ such that $1 + 1/n^2 < 1 + \varepsilon$. We need $1 + 1/n^2 < 1 + \varepsilon$, or $1/n^2 < \varepsilon$, or $1/\varepsilon < n^2$, or $\sqrt{1/\varepsilon} < n$. Since natural numbers are unbounded, for arbitrary $\varepsilon > 0$ we can find n satisfying this inequality. This proves

$$\inf A = 1 \ .$$

Problem 4a: We have

$$\sqrt[2]{2} \sim 1.414213562,$$
$$\sqrt[3]{3} \sim 1.442249570,$$
$$\sqrt[4]{4} = \sqrt[2]{2} \sim 1.414213562,$$
$$\sqrt[5]{5} \sim 1.379729661.$$

It seems that starting from $n = 3$ the sequence is decreasing. We will prove this. We want:

$$\sqrt[n]{n} \ge \sqrt[n+1]{n+1},$$

or

$$n^{n+1} \ge (n+1)^n,$$

or

$$n \ge \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n .$$

Since $(1 + 1/n)^n < e \sim 2.718281828$ for all n, the above inequality holds for $n = 3, 4, 5, \ldots$ as we guessed. This shows that

$$\sup A = \max A = \sqrt[3]{3}.$$

Obviously $1 < \sqrt[n]{n}$, for all $n = 2, 3, 4, \ldots$. We know from class that $\sqrt[n]{n} \to 1$, as $n \to \infty$. We also proved a theorem saying that a decreasing bounded below sequence converges to its infimum. Thus,

$$\inf A = 1.$$

Problem 5: We will prove that $a_{n+1} \le a_n$. We need

$$a_n - \frac{a_n^2 - a_n}{2a_n - 1} \le a_n,$$

or

$$2a_n^2 - a_n - a_n^2 + a_n \le 2a_n^2 - a_n,$$

or

$$0 \le a_n(a_n - 1).$$

We will prove that $a_n \ge 1$ for all $n = 1, 2 \ldots$ by induction.

$n = 1$ $a_1 = 2$ so yes, it holds.

Assume for n: $a_n \ge 1$.

Prove for $n+1$: We have

$$a_n - \frac{a_n^2 - a_n}{2a_n - 1} = \frac{a_n^2}{2a_n - 1}.$$

Since $a_n \geq 1$ the denominator is positive and the inequality

$$\frac{a_n^2}{2a_n - 1} \geq 1$$

is equivalent to $a_n^2 \geq 2a_n - 1$ or $a_n^2 - 2a_n + 1 \geq 0$ which is always true ($t^2 - 2t + 1 = (t-1)^2 \geq 0$). Proof by induction is completed.

We proved both facts: $\{a_n\}$ is decreasing and bounded below by 1. Note, that in the proof, the only information about a_1 we used was that $a_1 \geq 1$. By theorem about monotone sequences the sequence $\{a_n\}$ is convergent.

Let $a_n \to L$. We can go to the limit in the definition

$$a_{n+1} = a_n - \frac{a_n^2 - a_n}{2a_n - 1},$$

and obtain equation

$$L = L - \frac{L^2 - L}{2L - 1},$$

with two solutions : $L = 0$ or $L = 1$. Since $a_n \geq 1$ it cannot converge to 0. Thus, $a_n \to 1$.

If $a_1 = 0.7$, then $a_2 = 1.225$ and starting from element number two the sequence is decreasing and bounded below by 1 as before. The limit again is 1.

Problem 6: Prove

$$21 | 4^{n+1} + 5^{2n-1}\ .$$

$n = 1$: $4^{1+1} + 5^{2-1} = 21$ is divisible by 21.

Assume for n: $4^{n+1} + 5^{2n-1} = 21k$.

Prove for $n+1$: We have

$$4^{n+1+1} + 5^{2n+2-1} = 4 \cdot 4^{n+1} + 25 \cdot 5^{2n-1} = 4(4^{n+1} + 5^{2n-1}) + 21 \cdot 5^{2n-1} = 4 \cdot 21k + 21 \cdot 5^{2n-1},$$

and again is divisible by 21.

This completes the proof by induction.

Problem 7: (a) Assume that $\sqrt{\sqrt{2} + \sqrt{3}} = \frac{p}{q}$ where p, q are natural numbers (it is obviously positive). Then,

$$\sqrt{2} + \sqrt{3} = \frac{p^2}{q^2},$$

and

$$2 + 2\sqrt{6} + 3 = \frac{p^4}{q^4},$$

so $\sqrt{6}$ is a rational number. Let $\sqrt{6} = \frac{r}{s}$, where r, s are natural numbers without common divisors. Then,

$$6 = \frac{r^2}{s^2},$$

or $6s^2 = r^2$ and r^2 is even. Thus, r is even and $r = 2k$ where k is a natural number. We have $6s^2 = 4k^2$ or $3s^2 = 2k^2$. Thus, s^2 is even and then s is also even. Thus, we proved that both r and s are even which contradicts the assumption that they have no common divisors. This contradiction proves that $\sqrt{\sqrt{2} + \sqrt{3}}$ is not a rational number.

(b) Simple example

$$\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1.$$

Concordia University	**Course**: Math	**Number**: 364

Examination	**Instructor**	**Date**	**Time**	**Pages**
Midterm	M. Bertola	October 2015	1.25 hours	1

Special Instructions: Explain carefully but not verbosely your steps!

Problem 1. **(a)** *Write the negation of the sentence*

$$\forall \epsilon > 0 \ \exists N_0 \in \mathbb{N} : \quad \forall n, m \geq N_0 \quad (|x_n - x_m| < \epsilon) \vee (x_n x_m = 0)$$

(b) *Prove that for any sets* A *and* B *we have* $A = (A \cap B) \cup (A \setminus B)$.

Problem 2. *Use the "squeeze" theorem to show*

$$\lim_{n \to \infty} \sqrt[n]{5^n + 2^n - \sin(n^2)} = 5$$

You can use without proof the fact that, for every $n \in \mathbb{N}$, *the function* $f(x) = \sqrt[n]{x}$ *is increasing.*

Problem 3. *The Fibonacci sequence is defined by induction as follows*

$$x_1 := 1 \ , \quad x_2 := 1 \quad x_n = x_{n-1} + x_{n-2}$$

The first few ones are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$ *Prove that* $\forall k \in \mathbb{N}$, *the element* x_{3k} *is* **even**.

Problem 4. *Prove that* $\sqrt{3} + \sqrt{2}$ *is not rational. Prove that for any* $a, b \in \mathbb{Q}$ *the number* $x = a\sqrt{3} + b\sqrt{2}$ *is not rational, except for the case* $a = b = 0$. *You do not need to prove that* $\sqrt{2}$ *and* $\sqrt{3}$ *(separately) are not rational. Explain carefully all your logic.*

Problem 5. *Let* f *and* g *be two functions from a set* X *to the interval* $[1, \infty)$. *Let* $h(x) = f(x)g(x)$ *be the product function and* $r(x) = \frac{f(x)}{g(x)}$ *be the quotient function. Recall that the* sup *or* inf *of a function means the* sup *or* inf *of its image. Prove that;* **(1)** $\sup h \leq (\sup f)(\sup g)$; **(2)** $\sup r \leq \frac{\sup f}{\inf g}$.

Problem 6. *Show that the set of the roots of polynomials with integer coefficients is countable.*

Problem 7 (Bonus)**.** *Suppose that* $(x_n)_{n \in \mathbb{N}}$ *is a convergent sequence of real numbers. Prove that it satisfies the following property (***Cauchy***)*

$$\forall \epsilon > 0 \ \exists N_0 \in \mathbb{N} : \forall n, m \geq N_0 \quad |x_n - x_m| < \epsilon.$$

Solution of Problem 1

$$\exists \epsilon : \ \forall \delta > 0 \ \exists x, \ \exists y : \ (|x-y| < \delta) \wedge (|f(x) - f(y)| \geq \epsilon)$$

Solution of Problem ??

If $x \in A \cap B$ then $x \in A$; if $x \in A \setminus B$ then $x \in A$. Thus the RHS is contained in A. Viceversa, if $x \in A$ then it either belongs to B or not. If it does then $x \in A \cap B$, if it doesn't then $x \in A \setminus B$. This proves the reverse inclusion and concludes the proof.

Solution of Problem 7

1. By induction: $x_1 = 3 > 2$. Suppose $x_k > 2$, then

$$x_k > 2 \ \Leftrightarrow \ \frac{2}{x_k} < 1 \ \Leftrightarrow \ -\frac{2}{x_k} > -1$$

So

$$x_{k+1} = 3 - \frac{2}{x_k} > 3 - 1 = 2$$

Hence the sequence is bounded below by 2.

2. We have

$$x_{k+1} - x_k = 3 - \frac{2}{x_k} - x_k = \frac{3x_k - 2 - {x_k}^2}{x_k} = -\frac{(x_k - 2)(x_k - 1)}{x_k}$$

Since we know already that $x_k > 2$ the numerator is strictly positive and hence (because of the minus sign)

$$x_{k+1} - x_k < 0 \ \Leftrightarrow \ x_{k+1} < x_k$$

so the sequence is strictly decreasing.

3. By the monotone convergence theorem, the limit exists. Say $\lim_{n\to\infty} x_n = L \geq 2$ (since 2 is a lower bound for the sequence) Then also $\lim_{n\to\infty} x_{n+1} = L$ because this is a subsequence. Thus

$$\begin{aligned} L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} 3 - \frac{2}{x_n} = 3 - \frac{2}{\lim x_n} &= 3 - \frac{2}{L} \\ L &= 3 - \frac{2}{L} \\ L^2 - 3L + 2 = 0 &= (L-2)(L-1) \end{aligned}$$

Thus $L = 2$ ($L = 1$ is not possible since we already know $L \geq 2$).

Solution of Problem 4

1. Strictly speaking it is False: take $x = \sqrt{2}$ and $y = 0$. However, I counted also the answer True (under the untold assumption that the rational number is not zero). In this case one needs to prove something.

   So, let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $y \in \mathbb{Q} \setminus \{0\}$; write $y = \frac{p}{q}$ with $p, q \in \mathbb{Z}$. Suppose by contradiction that $xy \in \mathbb{Q}$. Then $xy = \frac{r}{s}$ for some $r, s \in \mathbb{Z}$. Then

$$x = \frac{1}{y}\frac{r}{s} = \frac{qr}{sp}$$

   which is in $\mathbb{Q}$ hence a contradiction.

2. False: $x = \sqrt{2}$ and $y = \sqrt{2}$. Then $\frac{x}{y} = 1 \in \mathbb{Q}$.

Solution of Problem ??
Let $0 < x < y$; then $x - y < 0$ and

$$\sqrt{x} - \sqrt{y} = \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}{\sqrt{x} + \sqrt{y}} = \frac{x - y}{\sqrt{x} + \sqrt{y}} < 0$$

where the last inequality holds because the denominator is positive. Thus $\sqrt{x} < \sqrt{y}$ and the function is increasing.

Solution of Problem ??

$$\left| \frac{4n^2 - 1}{2n^2 + 3} - 2 \right| = \left| \frac{-7}{2n^2 + 3} \right| = \frac{7}{2n^2 + 3} < \frac{7}{2n}$$

Now, for any $\epsilon > 0$ let $\mathbb{N} \ni n_0 > \frac{2}{7\epsilon}$. Then, forall $n \geq n_0$ we have

$$\left| \frac{4n^2 - 1}{2n^2 + 3} - 2 \right| < \frac{7}{2n} \leq \frac{7}{2n_0} < \epsilon$$

Solution of Problem 5 Since f, g take values in $[1, \infty)$ then $\sup f \geq 1 \leq \sup g$ and also $\inf f \geq 1 \leq \inf g$. In particular all of these numbers are **positive**.

1. Now

$$f(x)g(x) \leq (\sup f)g(x) \leq (\sup f)(\sup g)$$

where the first is valid because $g(x) > 0$ and the last inequality is valid only because $\sup f > 0$. In particular $\sup f \sup g$ is an upper bound for the range of $h(x) = f(x)g(x)$. Thus $\sup h \leq \sup f \sup g$.

2.

$$\frac{f(x)}{g(x)} \leq \frac{f(x)}{\inf g} \leq \frac{\sup f}{\inf g}$$

The inequalities are valid as stated because all the quantities are strictly positive. Taking the sup of the LHS yields $\sup h \leq \frac{\sup f}{\inf g}$.

Solution of Problem 3
By induction: $x_3 = 1 + 1 = 2$ is even. Suppose that x_{3k} is even: then

$$x_{3(k+1)} = x_{3k+3} = x_{3k+2} + x_{3k+1} = x_{3k+1} + x_{3k} + x_{3k+1} = x_{3k} + 2x_{3k+1}$$

Thus $x_{3(k+1)}$ is the sum of two even numbers, hence even. (It should be noted that the sequence is constituted of integers, but I take this as self-evident).

Solution of Problem ??
I claim the limit is 1 and I prove it directly:

$$\left| \frac{n}{\sqrt{n^2 + 1}} - 1 \right| = \frac{|n - \sqrt{n^2 + 1}|}{\sqrt{n^2 + 1}} = \frac{|n^2 - n^2 - 1|}{(n + \sqrt{n^2 + 1})\sqrt{n^2 + 1}} =$$

$$\frac{1}{(n + \sqrt{n^2 + 1})\sqrt{n^2 + 1}} < \frac{1}{n}$$

For any $\epsilon > 0$ take $\mathbb{N} \ni n_0 > \frac{1}{\epsilon}$: then for any $n \geq n_0$ we have

$$\left| \frac{n}{\sqrt{n^2+1}} - 1 \right| < \frac{1}{n} \leq \frac{1}{n_0} < \epsilon$$

which proves the result.

Solution of Problem ??

Suppose by contradiction that $\sqrt[5]{3} \in \mathbb{Q}$. Then it can be written as

$$\sqrt[5]{3} = \frac{p}{q}$$

with p, q two relatively prime integers (i.e. without common divisors (except 1)). Elevating to the fifth power we get

$$3q^5 = p^5$$

Thus p^5 is divisible by 3 and since 3 is a prime number, p itself must be divisible by 3 and can be written $p = 3k$ for some other integer k. Then

$$3q^5 = 3^5 k^5 \quad \Rightarrow \quad q^5 = 3^4 k^5.$$

By the same argument also q must be divisible by 3, a contradiction with the assumption that p, q were relatively prime.

Sample questions for MATH 364

October 2012

Section A only

This is a *representative* list of problems. The actual midterm will consist in 6 (or less) questions.

Problem 1. *Rewrite the following statements using only the symbols* $\vee, \wedge, \neg$ *and brackets*

$$\textbf{a}.\ (\neg A \Rightarrow A) \Rightarrow A \qquad \textbf{b}.\ \neg A \Rightarrow (A \Rightarrow B)$$

Problem 2. *Prove that*

$$[(A \vee B) \Rightarrow C] \Rightarrow [(A \Rightarrow C) \vee (B \Rightarrow C)]$$

Problem 3. *Write the negation of the sentence:*

$$\forall a\ \forall \epsilon\ \exists \delta > 0 :\ \forall x \forall y \quad [|f(x) - f(y)| < \epsilon] \Rightarrow [(|x-a| < \delta) \vee (|y-a| < \delta)]$$

Problem 4. *Consider the following statements: if they are true prove them, if they are false give a counterexample*

1. *The product of two rational numbers is rational;*
2. *The product of two irrational numbers is irrational;*
3. *the sum of two irrational numbers is irrational;*
4. *the sum of a rational number and an irrational number is irrational.*

Problem 5. *Let* $A \subset \mathbb{R}$ *be a non-empty set, bounded above. Prove that*

$$\sup(A) = \inf\{L :\ \forall a \in A, a \le L\}$$

Problem 6. *Let* A *be a nonempty subset of* $\mathbb{R}$ *bounded above and below. Let* $c \neq 0$ *be some number. Define*

$$cA := \{c\,x : \quad x \in A\}$$

Prove that

1. *if* $c > 0$ *then* $\sup cA = c \sup A$*;*
2. *if* $c < 0$ *then* $\sup cA = c \inf A$*.*

Problem 7. *Let* f *and* g *be two functions from the interval* $[0,1]$ *to* $\mathbb{R}$*. Suppose that both functions are bounded above and below. Let* $h = f - g$ *be the difference function, namely*

$$h(x) = f(x) - g(x)\ .$$

1. *Prove that*

   $$\sup h \le \sup f - \inf g$$

   where by sup *or* inf *of a function we mean the* sup *or* inf *of its image.*
2. *Show, by a* **simple** *example that the equality sign instead of* $\le$ *would be false in general*

Problem 8. *Prove by induction that*

1. $$1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$
2. *for any* $n = 1, 2, \ldots$ *the following expression is divisible by* 25

   $$2^{n+2} \cdot 3^n + 5n - 4$$

Problem 9. *Prove that the limits of the following sequences are correct*

$$\lim_{n\to\infty} n - \sqrt{1+n^2} = 0 \qquad\qquad \lim_{n\to\infty} \sqrt[n]{3^n - 2^n} = 3\ .$$

Hint: **rationalize** *in the first, use* **"sandwich"** *in the second.*

Problem 10. *Prove that* $\sqrt{3}+\sqrt{4}$ *is not a rational number. Find an irrational number within the interval* $(100, 101)$.

Problem 11. *Find the* sup *and* inf *of*

(1) $A := \{\frac{n+5}{n} : \ n = 1, 2, \ldots\}$
(2) $B := \left\{\frac{t^2}{t^2+5} : \ t \in \mathbb{R}\right\}$

Problem 12. *Prove that the set*

$$B := \left\{a + b\sqrt{2} + c\sqrt{3} : \quad a, b, c \in \mathbb{Z}\right\}$$

is **countable**.

Problem 13. *Prove that a convergent sequence is bounded. Is every bounded sequence convergent?*

Problem 14. *Let a non-empty set* A *be bounded below and let* $m = \inf A$.

1. *Show that for any* $\varepsilon > 0$ *there is an* $a \in A$ *such that* $a < m + \varepsilon$.
2. *Use part 1. to show that* A *contains a sequence convergent to* m.

Proof of Problem 1.

(1) Recall that $P \Rightarrow Q$ is the same as

$$\neg(P \wedge \neg Q)$$

Thus the following statements are equivalent

$$\begin{gathered}(\neg A \Rightarrow A) \Rightarrow A \\ \neg\left((\neg A \Rightarrow A) \wedge \neg A\right) \\ \neg\left(\neg(\neg A \wedge \neg A) \wedge \neg A\right)\end{gathered}$$

This is enough, but we can simplify further

$$\begin{aligned}\neg\left(\neg(\neg A \wedge \neg A) \wedge \neg A\right) = \\ \neg\left((A \vee A) \wedge \neg A\right) = \\ \neg\left(A \wedge \neg A\right) = \\ \neg A \vee A\end{aligned}$$

In particular the statement is true, because whatever A is (True or False) the statement is True.

(2) For the second

$$\begin{aligned}\neg A \Rightarrow (A \Rightarrow B) = \\ \neg\left(\neg A \wedge \neg(A \Rightarrow B)\right) = \\ \neg\left(\neg A \wedge \neg\neg(A \wedge \neg B)\right) = \\ \neg\left(\neg A \wedge (A \wedge \neg B)\right) = \\ \neg\left((\neg A \wedge A) \wedge \neg B\right) = \\ A \vee \neg A \vee B\end{aligned}$$

□

Proof of Problem 2. We start by simplifying the hypothesis and the thesis using that

$$P \Rightarrow Q = \neg(P \wedge \neg Q) = \neg P \vee Q$$

The hypothesis P reads

$$(A \vee B) \Rightarrow C = \neg(A \vee B \wedge \neg C) = \neg A \wedge \neg B \vee C$$

The thesis Q reads

$$(A \Rightarrow C) \vee (B \Rightarrow C) = \neg(A \wedge \neg C) \vee \neg(B \wedge \neg C) = \neg A \vee C \vee \neg B \vee C = \neg Q \vee \neg B \vee C$$

Thus we need to check that $\neg P \vee Q$ is true:

$$\begin{aligned}\neg\left(\neg A \wedge \neg B \vee C\right) \vee \neg A \vee C \vee \neg B \vee C = \\ A \vee B \wedge \neg C \vee \neg A \vee C \vee \neg B = \\ (A \vee \neg A) \wedge (C \vee \neg C) \vee (B \vee \neg B)\end{aligned}$$

Since all the sentences $X \vee \neg X$ are always true, then the above is always true. □

Proof. **Proof of Problem 3** In negating, ∀ becomes ∃ and viceversa, and the last statement is negated. The last statement in this case is

$$\overbrace{[|f(x) - f(y)| < \epsilon]}^{P} \Rightarrow \overbrace{[((|x - a| < \delta) \vee (|y - a| < \delta)]}^{Q}$$

Now $P \Rightarrow Q = \neg(P \wedge \neg Q)$ thus $\neg(P \to Q) = P \wedge \neg Q$ and the negation of the above is

$$|x - a| \geq \delta \wedge |y - a| \geq \delta \wedge |f(x) - f(y)| < \epsilon$$

Thus

$$\exists a: \ \exists \epsilon: \ \forall \delta > 0 \ \exists x: \ \exists y: \ |x-a| \geq \delta \wedge |y-a| \geq \delta \wedge |f(x)-f(y)| < \epsilon$$

$\square$

Proof. **Proof of Problem 4** The first is true; indeed two rational numbers are of the form $\frac{p}{q}$ and $\frac{r}{s}$ with $p, q, r, s \in \mathbb{Z}$ Thus

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

is the ratio of two integers and hence a rational number by definition.

The second is false: counterexample is $x = \sqrt{2}$ and $y = \sqrt{2}$ for which $xy = 2 \in \mathbb{Q}$.

The third is false: counterexample is $x = \sqrt{2}$ and $y = -\sqrt{2}$ for which $x + y = 0 \in \mathbb{Q}$.

The fourth is true: indeed if $q \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus Q$ then $y := q + x$ is irrational, for, otherwise, we would have

$$\mathbb{Q} \ni y = q + x \Rightarrow \ x = y - q \in \mathbb{Q}$$

a contradiction. $\square$

Proof. **Proof of Problem 5** Let

$$S = \sup A \ , \quad M := \inf \overbrace{\{L: \ \forall a \in A, \ a \leq L\}}^{B}$$

The set B is the set of upper bounds of A. Thus, M is the least upper bound. By definition M is the sup of A. $\square$

Proof. **Proof of Problem 6** Let $c > 0$: let $S = \sup A$ and $K = \sup(cA)$. This means that

$$\forall a \ , \quad K \geq ca \quad \Rightarrow \frac{K}{c} \geq a$$

Thus $\frac{K}{c}$ is an upper bound of A and thus $S \leq \frac{K}{c}$ or equivalently $cS \leq K$. On the other hand

$$\forall a \in A \quad ca \leq cS$$

thus cS is an upper bound for cA, and hence $K \leq cS$. Combining the two we have $cS = K$.

Suppose now $c < 0$. Let $J := \inf cA$. We have

$$\forall a \ , \quad J \leq ca \quad \Rightarrow \frac{J}{c} \geq a$$

where the inequality has flipped because c is negative. Thus $\frac{J}{c}$ is an upper bound of A and thus $S \leq \frac{J}{c}$ or equivalently $cS \geq J$. On the other hand

$$\forall a \in A \quad ca \geq cS$$

(again because S is an upper bound for A and c is negative). thus cS is a **lower** bound for cA, and hence $J \geq cS$ because $J = \inf cA$ is the **greatest lower bound**. Combining the two we have $cS = J$. $\square$

Proof. **Proof of Problem 7** We have, $\forall x, y \in [0, 1]$

$$f(x) - g(y) \leq \sup f - g(y) \leq \sup f - \inf g$$

In particular, if we take $y = x$ we have, $\forall x \in [0, 1]$

$$h(x) = f(x) - g(v) \leq \sup f - \inf g.$$

Thus the right hand side is an **upper bound** for the range of $h(x)$: by definition of $\sup h$, this cannot exceed any other upper bound and hence

$$\sup h \leq \sup f - \inf g$$

A counterexample that the equality is wrong is as follows

$$f(x) = x, \ g(x) = x \quad , \quad x \in [-1, 1] \ .$$

Then $\sup f = \sup[-1, 1] = 1$ and $\inf g = \inf[-1, 1] = -1$ so $\sup f - \inf g = 2$. But $h = x - x = 0$ and $\sup h = 0$. $\square$

Proof. **Proof of Problem 8** We verify it for $n = 1$:

$$1 \cdot 2 = \frac{1(1+1)(1+2)}{3} = 2$$

Suppose that the formula is true for n, namely that

$$\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$$

then

$$\sum_{j=1}^{n+1} j(j+1) = \left(\sum_{j=1}^{n} j(j+1)\right) + (n+1)(n+2) \overset{\star}{=} \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) =$$
$$(n+1)(n+2)\left(\frac{n}{3}+1\right) = (n+1)(n+2)\frac{n+3}{3} = \frac{(n+1)(n+1+1)(n+1+2)}{3}$$

which proves the case with $n+1$. On the step marked with $\star$ we have used the induction hypothesis.

For the second part: for $n = 1$ we have $8 \cdot 3 + 5 - 4 = 25$, so it is verified.

Suppose we know that the statement holds for n; then

$$2^{n+1+2}3^{n+1} + 5(n+1) - 4 = 6 \cdot 2^{n+2}3^{n} + 5n + 5 - 4 = 6\overbrace{\left(2^{n+2}3^{n} + 5n - 4\right)}^{\text{divisible by 25}} - 25n + 25$$

Since each number is divisible by 25 the proof follows.

□

Proof. **Proof of Problem 9** We estimate the distance of the sequence from the alleged limit

$$\left|n - \sqrt{n^2+1} - 0\right| = \left|n - \sqrt{n^2+1}\right| = \left|\frac{(n-\sqrt{n^2+1})(n+\sqrt{n^2+1}}{n+\sqrt{n^2+1}}\right| =$$
$$= \left|\frac{n^2-n^2-1}{n+\sqrt{n^2+1}}\right| = \left|\frac{-1}{n+\sqrt{n^2+1}}\right| = \frac{1}{n+\sqrt{n^2+1}} \le \frac{1}{n}$$

Therefore, for any $\epsilon > 0$ the definition is verified by choosing $n_0 > \frac{1}{\epsilon}$ any integer. Then forall $n > n_0$ we have $|x_n - 0| < \epsilon$. □

Proof. We use "sandwich" theorem

$$\sqrt[n]{3^n - \frac{1}{2}3^n} < \sqrt[n]{3^n - 2^n} < \sqrt[n]{3^n} = 3\ .$$

The left hand side estimate is valid for large n as $(3/2)^n > 2$ for large n. Since we want to find a limit considering only large n is correct. The left hand side estimate

$$\sqrt[n]{3^n - \frac{1}{2}3^n} = \sqrt[n]{\frac{1}{2}3^n} = \sqrt[n]{\frac{1}{2}}\sqrt[n]{3^n} \underset{n\to+\infty}{\longrightarrow} 1 \cdot 3 = 3\ .$$

□

Proof. **Proof of Problem 10** Well, $\sqrt{4} = 2$ so if $\sqrt{3} + 2$ were rational than so would be $\sqrt{3}$. But we have seen that the square root of any prime is not rational.

Now, $1 < \sqrt{3} < 2$ and thus $99 + \sqrt{3}$ is the desired example (one of many). □

Proof. **Proof of Problem 11**

(1) For the first part: we note that the sequence is **positive** and **decreasing**, namely if $k > n$

$$\frac{n+5}{n} > \frac{k+5}{k}$$

Indeed

$$\frac{n+5}{n} - \frac{k+5}{k} = \frac{nk+5k-nk-5n}{nk} = \frac{5}{nk}(k-n) < 0$$

Thus the sup is achieved for $n = 1$ and it is 6. The inf is the limit (by the monotone convergence theorem) which is 1.

(2) The function $\frac{t^2}{t^2+5}$ is obviously positive and for $t = 0$ it is zero. So the inf is actually a min and it is 0.
The sup is the limit at infinity which is 1.

$\square$

Proof. **Proof of Problem 12** We set-up a surjection from $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ onto B as follows

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} & \longrightarrow & B \\ (a,b,c) & \mapsto & a + b\sqrt{2} + c\sqrt{3} \end{array}$$

Since $\mathbb{Z}^3$ is the cartesian product of $\mathbb{Z}^2$ and $\mathbb{Z}$ (both of which are countable), then it is countable. By the theorem we proved that says: "if X is countable and $F : X \to Y$ is surjective, then Y is countable"
we can assert that B is countable $\square$

Proof. **Proof of Problem 13** Let $x_n \to x$. We will prove that the sequence (x_n) is bounded. The convergence means

$$\forall_{\varepsilon>0} \exists_{N\geq 1} \forall_{n\geq N} \quad x - \varepsilon < x_n < x + \varepsilon \ .$$

Let us fix an epsilon, say $\varepsilon = 2012 > 0$ (in principle we think about epsilon as a small number, but it does not have to be small). For this ε we can find an N such that

$$x - 2012 < x_n < x + 2012 \ ,$$

for $n = N, N+1, N+2, \ldots$. So the "tail" of the sequence is bounded. It is enough to bound the terms $x_1, x_2, \ldots, x_{N-1}$. Since there is only finitely many of them, there is a number $M > 0$ such that

$$-M \leq x_1, x_2, \ldots, x_{N-1} \leq M \ .$$

Thus, all terms of the of the sequence satisfy

$$\min\{x - 2012, -M\} < x_n < \max\{x + 2012, M\} \ .$$

No, a bounded sequence does not have to converge.

$$(x_n) = (1, 0, 1, 0, 1, 0, \ldots) \ ,$$

is an example.

$\square$

Proof. **Proof of Problem 14**

1. Fix an $\varepsilon > 0$. If no element of A satisfies $a < m + \varepsilon$, then $m + \varepsilon > m$ is a lower bound for A. This contradicts the definition of the infimum m. This proves the claim.

2. In 1. we proved that for any $\varepsilon > 0$ we can find an element $a_\varepsilon < m + \varepsilon$. Thus, for $\varepsilon_n = 1/n$, there are elements $a_n \in A$ such that $a_n < m + 1/n$, $n = 1, 2, 3, \ldots$ We have

$$m \leq a_n < m + 1/n \quad , \quad n = 1, 2, 3, \ldots$$

By "sandwich" theorem, $a_n \to m$.

$\square$

Proof. **Proof of Problem 15** Obviously the numbers are not written in base 10. Let the base be a. Then, we have

$$2(4 \cdot a + 0) = 1 \cdot a^2 + 2 \cdot a + 0 \ ,$$

or $a = 6$ as $a = 0$ is not a valid solution. Then, $(4 \cdot 6 + 0)/3 = 8_{(10)} = 12_{(6)}$. The answer is 12.

□

Proof. **Proof of Problem 16** Consider the sequence $a_0 = 1$ and $a_n = a_0 + \cdots + a_{n-1}$. I claim that

$$a_n = 2^n \ \ , n \geq 1$$

Indeed $a_1 = a_0 = 1$ verifies the assertion. If the assertion is true for n then

$$a_{n+1} = a_n + a_{n-1} + \cdots + a_0 = a_n + a_n = 2a_n = 2 \cdot 2^n = 2^{n+1}.$$

Now since $(A + B) mod\ 9 = (A mod\ 9 + B) mod\ 9$ we have that

$$x_n = a_{n-1} mod\ 9 = 2^{n-1} mod\ 9 \ , \quad n \geq 1$$

We see that $64 mod\ 9 = 2^6 mod\ 9 = 1$. Thus $x_{n+6} = 2^{n+6} mod\ 9 = (2^6 mod\ 9)(2^n mod\ 9) mod\ 9 = 2^n mod\ 9 = x_n$. Thus the sequence is periodic and hence the number is rational.

□