

# MATH1004 – Notes — By Eric Hua

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# Precalculus Review

## 1. Real numbers and intervals

Interval Notation	Set Notation
$[a, b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$
$(a, b)$	$\{x \in \mathbb{R} : a < x < b\}$
$[a, b)$	$\{x \in \mathbb{R} : a \leq x < b\}$
$(a, b]$	$\{x \in \mathbb{R} : a < x \leq b\}$
$(a, +\infty)$	$\{x \in \mathbb{R} : x > a\}$
$[a, +\infty)$	$\{x \in \mathbb{R} : x \geq a\}$
$(-\infty, b)$	$\{x \in \mathbb{R} : x < b\}$
$(-\infty, b]$	$\{x \in \mathbb{R} : x \leq b\}$
$(-\infty, +\infty)$	$\mathbb{R}$

## 2. Exponents and radicals

- $x^0 = 1, \quad x \neq 0.$
- $x^{-n} = \frac{1}{x^n}, \quad x \neq 0.$
- $x^{1/n} = \sqrt[n]{x}, \quad x^{m/n} = \sqrt[n]{x^m}.$
- $x^m x^n = x^{m+n}, \quad x^m / x^n = x^{m-n}.$
- $(x^m)^n = x^{mn}.$
- $x^n y^n = (xy)^n.$

### Example 1

$$\frac{x^{3/2} + 5x^2}{x^{1/2}} = x(1 + 5x^{1/2}).$$

## 3. Factoring Polynomials

- $a^2 - b^2 = (a - b)(a + b).$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2) \quad \text{and} \quad a^3 + b^3 = (a + b)(a^2 - ab + b^2).$
- $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1}).$
- $(a \pm b)^2 = a^2 \pm 2ab + b^2.$

- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$     and     $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ .

**Example 2**  $x^2 - 8x - 9 = (x - 9)(x + 1)$ .

**Example 3**

$$\begin{aligned}x^4 + x^2 + 1 &= x^4 + 2x^2 + 1 - x^2 \\ &= (x^2 + 1)^2 - x^2 \\ &= (x^2 + 1 - x)(x^2 + 1 + x).\end{aligned}$$

#### 4. Rationalizing denominator or numerator

- If the denominator is  $\sqrt{a}$ , then multiply both top and bottom by  $\sqrt{a}$ .
- If the denominator is  $\sqrt{a} \pm \sqrt{b}$ , then multiply both top and bottom by  $\sqrt{a} \mp \sqrt{b}$ .

**Example 4**

$$\frac{x}{\sqrt{x+4}-2} = \frac{x(\sqrt{x+4}+2)}{(\sqrt{x+4}-2)(\sqrt{x+4}+2)} = \frac{x(\sqrt{x+4}+2)}{x} = \sqrt{x+4}+2.$$

# Chapter 1 – Functions and Their Properties

## Functions, domain and range

Function: A function  $y = f(x)$  from a set  $D$  to a set  $R$  is a rule that assigns a unique element  $f(x) \in R$  to each element  $x \in D$ . ( $x$  is called independent variable,  $y$  is called dependent variable).

- (Natural) Domain  $D$  = The set of all values of  $x$  for which the function is defined.
- Range  $R$  = The set of all values of  $f(x)$  as  $x$  varies throughout  $D$ .

**Example 5** For MATH1004 class, we define  $D$ =the set of all students,  $R$ =the set of final grades. Define two relations  $f$  and  $g$  such that  $f$  is a function,  $g$  is not.

**Solution:**

$f(\text{student})$ =the grade of the student;  $g(\text{grade})$ =the student(s) who receive the grade.

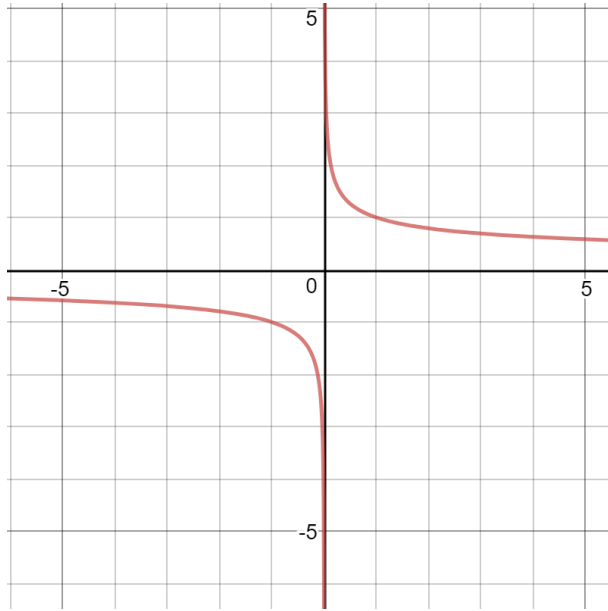
Some special functions:

- Linear function:  $y = f(x) = mx + b$ .
- Piecewise defined functions:  $f(x) = \begin{cases} 2x, & \text{if } x \leq 0; \\ 3x, & \text{if } x > 0. \end{cases}$
- Odd functions:  $f(-x) = -f(x)$ ; even functions:  $f(-x) = f(x)$ .
- Power function:  $f(x) = kx^p$ , where  $k \neq 0$  and  $p$  are constants.

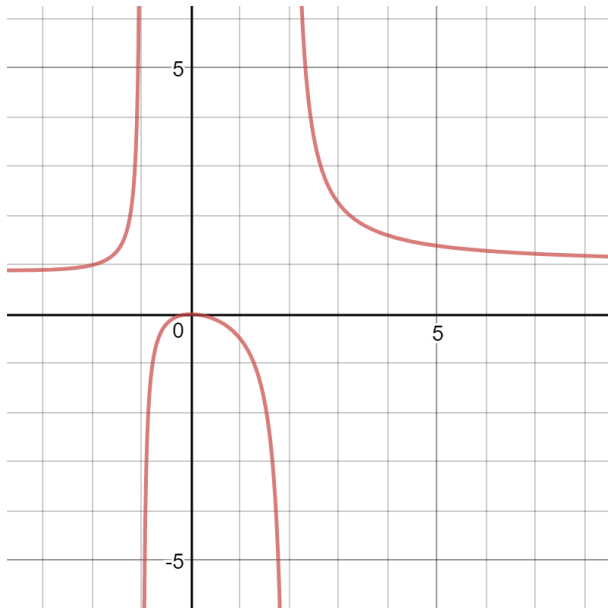
Domain of  $f(x) = x^{1/2}$ :  $\{x \in \mathbb{R} : x \geq 0\}$ ;

Domain of  $f(x) = x^{-1/2}$ :  $\{x \in \mathbb{R} : x > 0\}$ ;

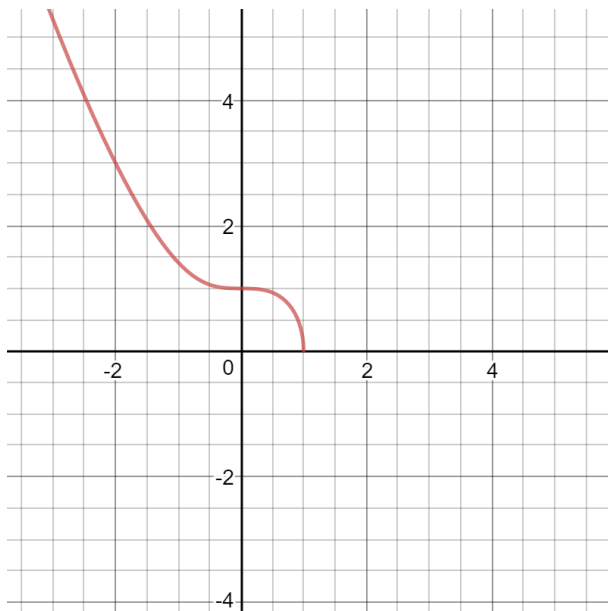
Domain of  $f(x) = x^{-1/3}$ :  $\{x \in \mathbb{R} : x \neq 0\}$ .



- Polynomials  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ , where  $n$  is a positive integer (which is called the degree of  $P(x)$ ).  $D = \{x \in \mathbb{R}\}$ .
- Rational function:  $f(x) = \frac{p(x)}{q(x)}$ . Domain =  $\{x : q(x) \neq 0\}$ . For example,  $f(x) = \frac{x^2}{(x-2)(x+1)}$ ,  $D = \{x \in \mathbb{R} : x \neq 2, -1\}$ .



- Root functions: e.g.,  $\sqrt{1-x^3}$ :  $D = \{x : x \leq 1\}$ .



- Composite function  $f(g(x))$  or  $f \circ g(x)$ . The domain of  $f \circ g(x)$  is the set of all  $x \in D(g)$  such that  $g(x) \in D(f)$ .

**Example 6** Let  $f(x) = \sqrt{2x-4}$ ,  $g(x) = \sqrt{3-2x}$ . Find the domain of  $f(g(x))$  and  $g(f(x))$ .

**Solution:** Then  $f(g(x)) = \sqrt{2\sqrt{3-2x}-4}$  with  $D(f \circ g) = \{x \leq -0.5\}$  (the solution of  $3-2x \geq 0$  and  $2\sqrt{3-2x}-4 \geq 0$ ).  $g(f(x)) = \sqrt{3-2\sqrt{2x-4}}$ ,  $D(g \circ f) = \{2 \leq x \leq 3.125\}$  (the solution of  $2x-4 \geq 0$  and  $3-2\sqrt{2x-4} \geq 0$ ).

## Absolute value functions

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

**Example 7** Let  $x > 10$ . Then  $|3 - |5 - x|| = |3 - (x - 5)| = |8 - x|$ .

**Example 8** Remove absolute value from  $f(x) = |2x - 4| + |2x + 4|$ .

**Solution:** Let  $2x - 4 = 0$ , then  $x = 2$ ; from  $2x + 4 = 0$ ,  $x = -2$ .

When  $x < -2$ ,  $f(x) = -(2x - 4) - (2x + 4) = -4x$ ;

When  $-2 < x < 2$ ,  $f(x) = -(2x - 4) + (2x + 4) = 8$ ;

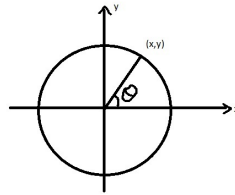
When  $x > 2$ ,  $f(x) = (2x - 4) + (2x + 4) = 4x$ .

$$f(x) = \begin{cases} -4x & \text{if } x < -2, \\ 8 & \text{if } -2 \leq x \leq 2, \\ 4x & \text{if } x > 2. \end{cases}$$

## Trigonometric functions

Radian  $\Leftrightarrow$  Degree:  $t$  degree  $= \frac{t}{180}\pi$ .

For any point  $(x, y)$ , let  $r = \sqrt{x^2 + y^2}$ .



$$\begin{aligned} \sin \theta &= \frac{y}{r}, & \cos \theta &= \frac{x}{r}, & \tan \theta &= \frac{\sin \theta}{\cos \theta}, \\ \sec \theta &= \frac{1}{\cos \theta}, & \csc \theta &= \frac{1}{\sin \theta}, & \cot \theta &= \frac{1}{\tan \theta}. \end{aligned}$$

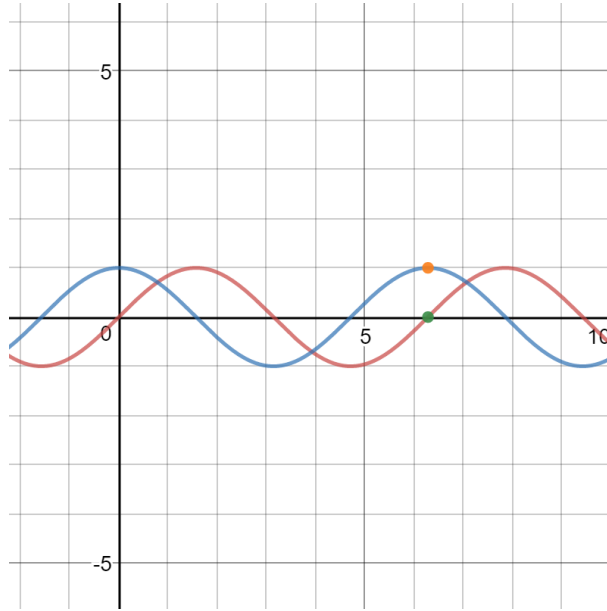


Figure 1: *blue* =  $\cos x$ , *red* =  $\sin x$

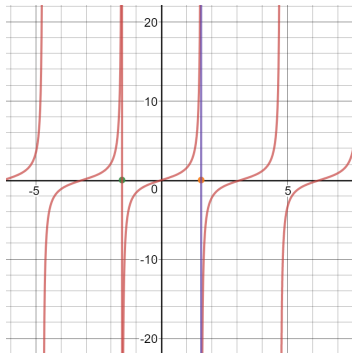


Figure 2:  $y = \tan x$

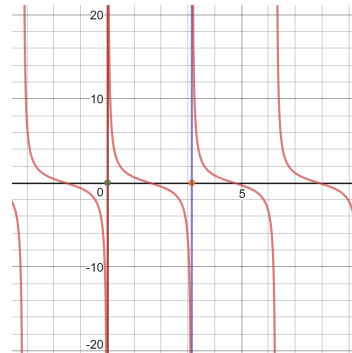


Figure 3:  $y = \cot x$

Basic relations:

$$\begin{array}{l}
 \sin(\theta + \frac{\pi}{2}) = +\cos \theta \\
 \cos(\theta + \frac{\pi}{2}) = -\sin \theta \\
 \tan(\theta + \frac{\pi}{2}) = -\cot \theta \\
 \cot(\theta + \frac{\pi}{2}) = -\tan \theta \\
 \sec(\theta + \frac{\pi}{2}) = -\csc \theta \\
 \csc(\theta + \frac{\pi}{2}) = +\sec \theta
 \end{array}
 \left|
 \begin{array}{l}
 \sin(\theta + \pi) = -\sin \theta \\
 \cos(\theta + \pi) = -\cos \theta \\
 \tan(\theta + \pi) = +\tan \theta \\
 \cot(\theta + \pi) = +\cot \theta \\
 \sec(\theta + \pi) = -\sec \theta \\
 \csc(\theta + \pi) = -\csc \theta
 \end{array}
 \right|
 \begin{array}{l}
 \sin(\pi - \theta) = \sin \theta \\
 \cos(\pi - \theta) = -\cos \theta \\
 \tan(\pi - \theta) = -\tan \theta \\
 \cot(\pi - \theta) = -\cot \theta \\
 \sec(\pi - \theta) = -\sec \theta \\
 \csc(\pi - \theta) = \csc \theta
 \end{array}
 \left|
 \begin{array}{l}
 \sin(2\pi - \theta) = -\sin \theta \\
 \cos(2\pi - \theta) = \cos \theta \\
 \tan(2\pi - \theta) = -\tan \theta \\
 \cot(2\pi - \theta) = -\cot \theta \\
 \sec(2\pi - \theta) = \sec \theta \\
 \csc(2\pi - \theta) = -\csc \theta
 \end{array}
 \right.$$

**Pythagorean trigonometric identity:**  $\sin^2 x + \cos^2 x = 1$ .

	t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Special values:	$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
	$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Addition formulas:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

Double-angle formulas:

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x.$$

Half-angle formula.

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

**Periods:**  $\sin x$  and  $\cos x$  have period  $2\pi$ ,  $\tan x$  and  $\cot x$  have period  $\pi$ .

**Graphs.**

**Example 9** Find all values of  $x$  in the interval  $[0, 2\pi]$  such that  $\sin^2 x - 3 \cos^2 x = 0$ .

Solution:  $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ .

**Example 10** Find  $\cos x$  where  $x \in [\frac{\pi}{2}, 2\pi]$  such that  $\sin x = 0.8$ .

Solution:  $\cos x = -0.6$

**Example 11** Find  $\sin x$  and  $\cos x$  where  $x \in [\frac{3\pi}{2}, 2\pi]$  such that  $\tan x = -\frac{1}{5}$ .

Solution:  $\tan x = \frac{y}{x} = \frac{-1}{5}$ .

**Example 12** Evaluate  $\sin \frac{\pi}{12}$ .

Solution:  $\frac{\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12}$ .

Special inequalities:

$$-|x| \leq \sin x \leq |x|, \quad -|x| \leq 1 - \cos x \leq |x|.$$

## Inequalities

- $|x| \leq t \iff -t \leq x \leq t$ .
- $|x| \geq t \iff x \geq t \quad \text{or} \quad x \leq -t$ .
- Triangle Inequality: Let  $a, b$  be real numbers. Then  $|a + b| \leq |a| + |b|$ .

**Example 13** *Solve the inequality*

$$-2x - 3 \leq -13.$$

Solution: We have

$$-2x - 3 \leq -13 \Rightarrow -2x \leq -13 + 3 \Rightarrow -2x \leq -10.$$

The next step would be to divide both sides by  $-2$ . Since  $-2 < 0$ , the sense of the inequality is inverted, and so

$$-2x \leq -10 \Rightarrow x \geq \frac{-10}{-2} \Rightarrow x \geq -5.$$

**Example 14** *Solve the inequality*

$$x^2 + 2x - 35 < 0.$$

**Solution:** Observe that  $x^2 + 2x - 35 = (x - 5)(x + 7)$ , which vanishes when  $x = -7$  or when  $x = 5$ . Now we construct the table:

$x \in$	$(-\infty, -7)$	$(-7, 5)$	$(5, +\infty)$
$x + 7$	-	+	+
$x - 5$	-	-	+
$(x + 7)(x - 5)$	+	-	+

On the last row, the sign of the product  $(x + 7)(x - 5)$  is determined by the sign of each of the factors  $x + 7$  and  $x - 5$ .

From the sign diagram above we see that

$$\{x \in \mathbb{R} : x^2 + 2x - 35 < 0\} = (-7, 5).$$

Notice that we exclude both  $x = -7$  and  $x = 5$  in the set, as  $(x + 7)(x - 5)$  vanishes there.

**Example 15** *Solve the inequality  $|2x - 1| \leq 1$ .*

Solution:

$$|2x - 1| \leq 1 \iff -1 \leq 2x - 1 \leq 1 \iff 0 \leq 2x \leq 2 \iff 0 \leq x \leq 1 \iff x \in [0, 1].$$

The solution set is the interval  $[0, 1]$ .

**Example 16** Solve the inequality  $|2x - 5| < |1 - x|$ .

**Solution:**  $|2x - 5|^2 < |1 - x|^2 \Rightarrow$

$$4x^2 - 20x + 25 < 1 - 2x + x^2 \Rightarrow$$

$$3x^2 - 18x + 24 < 0 \Rightarrow$$

$$3(x - 2)(x - 4) < 0 \Rightarrow$$

$$2 < x < 4.$$

## Chapter 2 – Limits and Continuity

### Rates of change and tangents to curves

The average rate of change of  $y = f(x)$  with respect to  $x$  over the interval  $[x_1, x_2]$  is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, h \neq 0.$$

Geometrically, it is the slope of the secant through two points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$ .

**Instantaneous rates of change and tangent lines:** What is a tangent line at point P on a curve? We chose another point Q on the curve. The line PQ is called a secant line. When Q tends to P, the secant PQ will tends to a line, which is called a the tangent line of the curve at P.

**Example 17** Estimate the slope of the tangent line to the parabola  $y = x^2$  at the point  $(2, 4)$ .

Solution:

$$m = \frac{x^2 - 4}{x - 2}.$$

x	m
2.1	4.1
2.01	4.01
2.001	4.001
1.9	3.9
1.99	3.99
1.999	3.999

## 2.1 One-sided limits

**Definition 1** We write

$$f(a - 0) = \lim_{x \rightarrow a^-} f(x) = L$$

and say that the limit of  $f(x)$  is  $L$  as  $x$  approaches  $a$  from the left. Similarly, We write

$$f(a + 0) = \lim_{x \rightarrow a^+} f(x) = L$$

and say that the limit of  $f(x)$  is  $L$  as  $x$  approaches  $a$  from the right.

**Example 18** Study one-sided limits at  $x = 0$  of the Heaviside function

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

**Solution:**

$$\begin{aligned} \lim_{t \rightarrow 2^-} H(t) &= 1, \\ \lim_{t \rightarrow 0^+} H(t) &= 1, \quad \lim_{t \rightarrow 0^-} H(t) = 0. \end{aligned}$$

**Example 19** Let

$$f(x) = \begin{cases} x - 5, & x < 0; \\ x^2 + 3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Discuss one-sided limits at  $x = 0$  and  $x = 1$ .

**Solution:**  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x - 5 = -5$  and  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 + 3x = 0$ .  
Similarly,  $\lim_{x \rightarrow 1^-} f(x) = 4$  and  $\lim_{x \rightarrow 1^+} f(x) = 4$ .

## 2.2 Two sided limits and Continuity

**Definition 2** We write

$$\lim_{x \rightarrow a} f(x) = L$$

and say "as  $x$  approaches  $a$ , the limit of  $f(x)$  is  $L$ ." If  $L$  is a finite number, we say that the limit exists, otherwise, the limit does not exist.

**Theorem 1**

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L.$$

**Example 20** Consider the Heaviside function

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t \rightarrow 2} H(t) = 1,$$

$$\lim_{t \rightarrow 0^+} H(t) = 1, \lim_{t \rightarrow 0^-} H(t) = 0, \Rightarrow \lim_{t \rightarrow 0} H(t) \nexists.$$

**Example 21** Show that  $\lim_{x \rightarrow 0} \frac{|x|}{x} \nexists$ .

**Solution:**  $\because \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1, \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1.$

**Example 22** Let

$$f(x) = \begin{cases} x - 5, & x < 0; \\ x^2 + 3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then  $\lim_{x \rightarrow 0} f(x) \nexists$  and  $\lim_{x \rightarrow 1} f(x) = 4$ .

Properties: Suppose that  $\lim_{x \rightarrow a} f(x) \exists$  and  $\lim_{x \rightarrow a} g(x) \exists$ .

- $\lim_{x \rightarrow a} P(x) = P(a)$ ,  $P(x)$  is a polynomial.
- $\lim_{x \rightarrow a} (cf(x) \pm dg(x)) = c \lim_{x \rightarrow a} f(x) \pm d \lim_{x \rightarrow a} g(x)$ ,  $c, d$  are constants.
- $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$ .
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ , if  $\lim_{x \rightarrow a} g(x) \neq 0$ .
- $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$ .
- $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ . When  $n$  is even, we assume that  $\lim_{x \rightarrow a} f(x) \neq 0$ .

**Example 23**

$$\lim_{x \rightarrow 1} (x^2 - 3) = 1^2 - 3 = -2, \quad \lim_{x \rightarrow 1} \frac{3x^4 + 8x - 2}{x - 2} = \frac{3(1)^4 + 8(1) - 2}{1 - 2} = -9.$$

**Special case:**

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \quad \text{where } g(a) = 0.$$

- If  $f(a) \neq 0$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  does not exist.
- If  $f(a) = 0$ , then simplify  $\frac{f(x)}{g(x)}$  first, then study the limit.

**Example 24**

$$\lim_{x \rightarrow 2} \frac{3x^4 + 8x - 2}{x - 2} \nexists, \quad \lim_{x \rightarrow 2} \frac{x - 2}{x - 2} = 1.$$

**Example 25** Find the following limits:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$$

$$\lim_{h \rightarrow 0} \frac{(h + 1)^2 - 1}{h}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x + 4} - 2}{x}$$

**Solution:**

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \rightarrow 2} (x + 2) = 4, \\ \lim_{h \rightarrow 0} \frac{(h + 1)^2 - 1}{h} &= \lim_{h \rightarrow 0} \frac{h(h + 2)}{h} = \lim_{h \rightarrow 0} (h + 2) = 2, \\ \lim_{x \rightarrow 0} \frac{\sqrt{x + 4} - 2}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{x + 4} - 2)(\sqrt{x + 4} + 2)}{x(\sqrt{x + 4} + 2)} = \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x + 4} + 2)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x + 4} + 2} = \frac{1}{4}. \end{aligned}$$

**Theorem 2** If  $f(x) \leq g(x)$  near  $x = a$ , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

**Theorem 3** The Sandwich Theorem (The Squeeze Theorem): If  $f(x) \leq g(x) \leq h(x)$  near  $x = a$ , and  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$ , then  $\lim_{x \rightarrow a} g(x) = L$ .

**Example 26** Show that

$$\lim_{x \rightarrow 0} x^4 \cos \frac{3}{x} = 0$$

by the Squeeze Theorem.

**Solution:**  $-x^4 \leq x^4 \cos \frac{3}{x} \leq x^4$ .

**Limits of trig functions:**

**Example 27**

$$\lim_{x \rightarrow 0} \sin x = 0, \quad \lim_{x \rightarrow 0} \cos x = 1.$$

**Example 28** Estimate the limit of

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}.$$

$x$	$\frac{\sin x}{x}$
1	0.84147098
0.1	0.99833417
0.01	0.99998333
0.001	0.99999983

**Example 29** Show that

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1.$$

Proof. By the inequality

$$\cos x < \frac{\sin x}{x} < 1$$

and the Squeeze Thm.

**Example 30** Find the limits:

$$\lim_{h \rightarrow 0} \frac{\cos h - 1}{h}$$

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x}.$$

**Solution:**

$$\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = \lim_{h \rightarrow 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h \rightarrow 0} \frac{\sin h}{h} \frac{\sin h}{\cos h + 1} = 0.$$

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \frac{3x}{\sin 3x} \cdot \frac{2x}{3x} = \frac{2}{3}.$$

## Continuity

**Definition 3** If  $\lim_{x \rightarrow a} f(x) = f(a)$ , then  $f(x)$  is continuous at  $x = a$ , otherwise,  $f(x)$  is discontinuous at  $x = a$ . If  $f(x)$  is continuous at any point on an interval, then  $f(x)$  is continuous on the interval. For the end points, we only need sided limits.

**Example 31** Consider  $f(x) = \frac{x^2 - 2x + 1}{x - 1}$  at  $x = 1$ .

**Solution:**  $f(x)$  is undefined at  $x = 1$ . But  $\lim_{x \rightarrow 1} f(x) = 0$ . So the discontinuous point  $x = 1$  is **removable** if we define  $f(1) = 0$ .

**Example 32** Determine the continuity of  $f(x) = \frac{|x|}{x}$ .

**Solution:**  $x = 0$  is not removable.

**Definition 4** If  $\lim_{x \rightarrow a^-} f(x) = f(a)$ , then  $f(x)$  is continuous from the left at  $x = a$ ; if  $\lim_{x \rightarrow a^+} f(x) = f(a)$ , then  $f(x)$  is continuous from the right at  $x = a$ .

**Example 33** Determine the left and right continuity at  $x = 0$ :

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases}$$

**Solution:** continuous from right at  $x = 0$ , discontinuous from left at  $x = 0$ .

**Theorem 4** If  $f(x)$  and  $g(x)$  are continuous at  $a$ , then

$$f \pm g, fg, cf \text{ (} c \text{ is a constant)}, \frac{f}{g} \text{ (if } g(a) \neq 0)$$

are continuous.

**Theorem 5** Polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions and logarithmic functions are continuous in their domain.

**Theorem 6** If  $\lim_{x \rightarrow a} g(x) = b$  and  $f(x)$  is continuous at  $b$ , then

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(b).$$

Furthermore, if  $g(x)$  is continuous at  $a$ , and  $f(x)$  is continuous at  $g(a)$ , then  $f(g(x))$  is continuous at  $a$ .

**Example 34** Find  $k$  such that  $f(x) = \begin{cases} x^3 + kx^2 - 5x, & x > 2; \\ \frac{x}{x-3}, & x \leq 2 \end{cases}$  is continuous at  $x = 2$ .

**Solution:**

$$8 + 4k - 10 = -2, \quad k = 0.$$

## 2.3 Important Theorem

**Theorem 7** (*The Intermediate Theorem*) If  $f(x)$  is continuous on  $[a, b]$ , and  $y_0$  between  $f(a)$  and  $f(b)$ , then  $\exists c \in [a, b]$  such that  $f(c) = y_0$ .

**Example 35**

$$\frac{4}{x-5} + \frac{-9x}{(x+1)(x-2)(x+3)} = 0$$

has solutions in  $(0, 1)$ .

Proof. Let  $f(x) = \frac{4}{x-5} + \frac{-9x}{(x+1)(x-2)(x+3)}$ . Then  $f(x)$  is continuous on  $[0, 1]$ .  $f(0) = -4/5$ ,  $f(1) = 1/8$ . The conclusion follows from The Intermediate Theorem.

## 2.4&2.5 Evaluating Limits at Infinity

**Definition 5** The line  $y = L$  is called a horizontal asymptote of the curve  $y = f(x)$  if either

$$\lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L.$$

**Example 36**  $f(x) = \frac{3x^2 - x - 1}{2x^2 + 3x}$  has horizontal asymptote  $y = \frac{3}{2}$ .

**Example 37**  $\lim_{x \rightarrow \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_0} = \begin{cases} 0, & \text{if } n < m; \\ \frac{a_n}{b_n}, & \text{if } n = m; \\ \pm\infty, & \text{if } n > m. \end{cases}$

**Example 38**  $\lim_{x \rightarrow \infty} \sin x$ ,  $\lim_{x \rightarrow \infty} \cos x$  do not exist.

**Example 39** Find the horizontal asymptotes of the function  $f(x) = e^x$ .

Sol:  $\lim_{x \rightarrow -\infty} e^x = 0$ . Thus, HA:  $y = 0$ .

**Example 40** Find the horizontal asymptotes of the function

$$f(x) = \sqrt{x^2 + 1} - x.$$

**Solution:**

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 1} - x)(\sqrt{x^2 + 1} + x)}{\sqrt{x^2 + 1} + x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1} + x} = 0.$$

Thus, HA:  $y = 0$ .

**Example 41** Find the horizontal asymptotes of the function

$$f(x) = \sqrt{x^2 + 5x + 1} - x.$$

**Solution:**

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 1} - x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 5x + 1} - x)(\sqrt{x^2 + 5x + 1} + x)}{\sqrt{x^2 + 5x + 1} + x} \\ &= \lim_{x \rightarrow \infty} \frac{5x + 1}{\sqrt{x^2 + 5x + 1} + x} = \frac{5}{2}. \end{aligned}$$

Thus, HA:  $y = \frac{5}{2}$ .

## 2.6 Limits equal to $\pm\infty$

**Definition 6**

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that  $f(x)$  can be arbitrarily large as  $x$  tends to  $a$ ;

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that  $f(x)$  can be arbitrarily large negative as  $x$  tends to  $a$ .

**Example 42**  $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ ,  $\lim_{x \rightarrow 1} \frac{-1}{(x-1)^2} = -\infty$ .

**Definition 7** The line  $x = a$  is called a vertical asymptote of the curve  $y = f(x)$  if at least one of the following statements is true:

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty, \lim_{x \rightarrow a^+} f(x) = \pm\infty, \lim_{x \rightarrow a} f(x) = \pm\infty.$$

**Example 43**  $f(x) = \frac{1}{x}$  has HA:  $y = 0$  and VA:  $x = 0$ .

**Theorem 8** If  $r > 0$  is a rational number, then

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0.$$

**Example 44** Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{4x - 8}.$$

**Solution:**

$$\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 + 1}}{4x - 8} = \frac{\sqrt{2}}{4}, \quad \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 + 1}}{4x - 8} = -\frac{\sqrt{2}}{4}.$$

Thus, HA:  $y = \frac{\sqrt{2}}{4}$  and  $y = -\frac{\sqrt{2}}{4}$ .

VA:  $4x - 8 = 0 \Rightarrow x = 2$ .

## Infinite limits at $\infty$

The notation  $\lim_{x \rightarrow \infty} f(x) = \infty$  is used to indicate that the values of  $f(x)$  become large as  $x$  becomes large. Similar meanings are for

$$\lim_{x \rightarrow \infty} f(x) = -\infty, \quad \lim_{x \rightarrow -\infty} f(x) = \infty, \quad \lim_{x \rightarrow -\infty} f(x) = -\infty.$$

**Example 45**  $\lim_{x \rightarrow \infty} x^5 = \infty$ ,  $\lim_{x \rightarrow -\infty} x^5 = -\infty$ ,  $\lim_{x \rightarrow \pm\infty} (x^3 - x^5) = \mp\infty$ .

## Chapter 3. The Derivative of a Function

### 3.1-3.2 Derivatives

#### Tangents and the derivative at a point

**Definition 8** The derivative of the function  $y = f(x)$  at a point  $a$  is

$$y'(a) = \frac{dy}{dx} = f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

Meaning:  $f'(a) =$

- instantaneous rate of change of  $f(x)$  at  $a$ , or
- rate of change of  $f(x)$  at  $a$ , or
- the slope of the tangent line to the curve at  $a$ .

**Example 46** Let  $f(x) = x^2$ . Calculate  $f'(5)$ .

Sol:

$$f'(5) = \lim_{h \rightarrow 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \rightarrow 0} \frac{(5+h)^2 - 5^2}{h} = 10.$$

**Definition 9** Let  $P = (a, f(a))$  be a point on the curve  $y = f(x)$ . The tangent of  $f(x)$  at  $P$  is the line through  $P$  with slope

$$m = f'(a) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

**Example 47** Find the slope and the equation of the tangent line to the curve

$$y = f(x) = 3x^2 - 6x + 1$$

at the point  $(2, 1)$ . Sketch the curve.

Sol.  $a = 2, f(a) = 1$ .

$$m = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{3x^2 - 6x + 1 - 1}{x - 2} = \lim_{x \rightarrow 2} (3x) = 6.$$

The tangent line is

$$y - 1 = 6(x - 2), \implies y = 6x - 11.$$

To sketch the curve, we complete square:  $y = 3(x - 1)^2 - 2$ .

**Example 48** Find the tangent line to the hyperbola  $xy = 4$  at the point  $(1, 4)$ .

Sol.  $a = 1, f(x) = y = 1/x$ .

$$m = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h \rightarrow 0} \frac{-1}{1+h} = -1.$$

The tangent line is

$$y = -x + b, \implies 4 = -1 + b, b = 5, \quad y = -x + 5.$$

**Example 49** Find the slope of the tangent line to the curve  $y = \frac{1}{\sqrt{x+1}}$  at the point  $(0, 1)$ .

Sol.  $a = 0$ ,

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{0+h+1}} - 1}{h} = \lim_{h \rightarrow 0} \frac{1 - \sqrt{h+1}}{h\sqrt{h+1}} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h\sqrt{h+1}(1 + \sqrt{h+1})} = -\frac{1}{2}. \end{aligned}$$

## The derivative as a function

**Definition 10** The derivative of the function  $y = f(x)$  is the function  $f'(x)$ :

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

**Definition 11** (One sided derivatives):

- Right-hand derivative at  $a$ :  $f'(a+) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$ .
- Left-hand derivative at  $a$ :  $f'(a-) = \lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h}$ .

**Example 50** Let  $f(x) = \sqrt{x-3}$ . Find  $f'(x)$  and state the domains of  $f$  and  $f'$ .

Sol:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h-3} - \sqrt{x-3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h-3} - \sqrt{x-3})(\sqrt{x+h-3} + \sqrt{x-3})}{h(\sqrt{x+h-3} + \sqrt{x-3})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h-3} + \sqrt{x-3})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h-3} + \sqrt{x-3}} = \frac{1}{2\sqrt{x-3}}. \end{aligned}$$

The domain of  $f$ :  $x - 3 \geq 0$ ,  $x \geq 3$ .

The domain of  $f'$ :  $x - 3 \geq 0$  and  $2\sqrt{x-3} \neq 0$ ,  $x > 3$ .

**Definition 12** The function  $f$  is differentiable at  $a$  if  $f'(a)$  exists. It is differentiable on an interval if  $f'(a)$  exists for any  $a$  on the interval.

**Theorem 9** If a function is differentiable at  $x = c$ , then the function is continuous at  $x = c$ .

**Example 51**  $f(x) = |x|$  is not differentiable at  $x = 0$ .

**Solution:**

$$f'(x) = \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{if } x < 0. \end{cases} \quad f'(0) \nexists.$$

## Differentiation rules

- Constant rule: If  $f(x) = c$ , then  $f'(x) = 0$  or  $\frac{d}{dx}(c) = 0$ .
- Power Rule: If  $f(x) = x^n$ ,  $n$  is any real number. Then  $f'(x) = nx^{n-1}$ .
- Constant multiple rule:  $[cf(x)]' = cf'(x)$ .
- Sum rule and difference rule:  $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
- Derivative of polynomial:  $[a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0]' = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + \dots + a_1$ .
- Product rule:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

- Quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

**Example 52** Let  $f(x) = (\sqrt{x} + x^2)(x^3 + x)$ . Calculate  $f'(1)$ .

**Solution:**

$$f'(x) = \left(\frac{1}{2\sqrt{x}} + 2x\right)(x^3 + x) + (\sqrt{x} + x^2)(3x^2 + 1), \quad f'(1) = 13.$$

**Example 53** Let  $f(x) = \frac{x^3+4x^2}{x^5+x+1}$ . Calculate  $f'(1)$ .

**Solution:**  $f'(1) = \frac{1}{3}$ .

**Example 54** At what point(s) on the curve  $y = \frac{x^2-4}{x+1}$  is the tangent line

a) parallel to  $y = 3x$ ?

b) perpendicular to  $y = -0.5x$ ?

**Solution:** By quotient rule,

$$y' = \frac{(x^2 - 4)'(x + 1) - (x^2 - 4)(x + 1)'}{(x + 1)^2} = \frac{2x(x + 1) - (x^2 - 4)1}{(x + 1)^2} = \frac{x^2 + 2x + 4}{(x + 1)^2}.$$

a) Let  $y' = 3 \Rightarrow \frac{x^2+2x+4}{(x+1)^2} = 3 \Rightarrow 2x^2 + 4x - 1 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{6}}{2}$ .

b)  $(-0.5)y' = -1 \Rightarrow -0.5 \frac{x^2+2x+4}{(x+1)^2} = -1 \Rightarrow x^2 + 2x - 2 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{3}}{2}$ .

**Higher derivatives:** Let  $y = f(x)$ . Then

$$y''(x) = f''(x) = \frac{d}{dx} \left(\frac{df}{dx}\right) = \frac{d}{dx} \left(\frac{dy}{dx}\right), \quad y^{(n)}(x) = f^{(n)}(x) = \frac{d}{dx} \left(\frac{dy^{(n-1)}}{dx}\right).$$

**Example 55** Let  $f(x) = 4x^3 + 6x^2 - 23x + 7$ . Then  $f''(x) = 24x + 12$ ,  $f'''(x) = 24$  and  $f^{(4)}(x) = 0$ .

**Example 56**

$$(x^n)^{(n)} = n!, \quad \left(\frac{1}{x}\right)^{(n)} = (-1)^n n! x^{-n-1}.$$

### 3.3 The chain rule

- Chain Rule:

$$[f(g(x))]' = f'(g(x))g'(x), \quad \frac{df(g(x))}{dx} = \frac{df(v)}{dv} \cdot \frac{dg(x)}{dx}, v = g(x), \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- General Power Rule:

$$[u(x)^n]' = nu^{n-1}u'(x).$$

**Example 57** Let  $f(x) = (x^2 - x - 1)^{100}$ . Calculate  $f'(x)$ .

**Solution:**  $f'(x) = 100(x^2 - x - 1)^{99}(x^2 - x - 1)' = 100(x^2 - x - 1)^{99}(2x - 1)$ .

**Example 58** Let  $h(x) = g(f(x))$ , where  $f'(2) = 3$ ,  $f(2) = 4$ ,  $g'(3) = -5$ ,  $g(4) = 8$ ,  $g'(4) = 7$ . Find  $h'(2)$ .

**Solution:**  $h'(x) = g'(f(x))f'(x) \Rightarrow h'(2) = g'(f(2))f'(2) = g'(4)(3) = 7(3) = 21$ .

**Example 59** Let  $y = \sqrt{x + \sqrt{x^2 + x}}$ . Calculate  $y'$ .

**Solution:**

$$\begin{aligned} y' &= \frac{1}{2} \frac{1}{\sqrt{x + \sqrt{x^2 + x}}} (x + \sqrt{x^2 + x})' \\ &= \frac{1}{2\sqrt{x + \sqrt{x^2 + x}}} \left( 1 + \frac{1}{2} \frac{1}{\sqrt{x^2 + x}} (x^2 + x)' \right) = \frac{1}{2\sqrt{x + \sqrt{x^2 + x}}} \left( 1 + \frac{2x + 1}{2\sqrt{x^2 + x}} \right) \end{aligned}$$

### 3.4 Implicit Functions and Their Derivatives

Implicit Differentiation: Assume  $f(x, y) = C$ . To find  $y'$ ,

- consider  $x$  as an independent variable,  $y$  as a dependent variable;
- differentiate both sides with respect to  $x$ ;
- isolate  $y'$ .

**Example 60** Find  $y'$  from  $y^2 + x^2 = 1$ .

**Solution:**

$$\frac{d}{dx}(y^2 + x^2) = \frac{d1}{dx}, \Rightarrow 2yy' + 2x = 0, \Rightarrow y' = -\frac{x}{y}.$$

**Example 61** *Let*

$$y^2 + x^2 = xy + 3.$$

- 1) *Find the equation of the tangent line to the curve at (1, 2).*
- 2) *Find all the points on the curve where the tangent line is either horizontal or vertical.*
- 3) *Find all the points on the curve where the tangent line is parallel to  $y = -x$ .*

**Solution:** 1)

$$\frac{d}{dx}(y^2 + x^2) = \frac{d}{dx}(xy + 3), \Rightarrow 2yy' + 2x = y + xy', \Rightarrow y' = \frac{y - 2x}{2y - x}.$$

$$m = y'|_{(1,2)} = \frac{1}{3}, \Rightarrow y = \frac{1}{3}x + \frac{5}{3}.$$

2) Horizontal tangent line:  $y' = 0 \Rightarrow y - 2x = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1, y = 2$  or  $x = -1, y = -2$ .

Vertical tangent line:  $y' = \infty \Rightarrow 2y - x = 0 \Rightarrow y^2 = 1 \Rightarrow y = 1, x = 2$  or  $y = -1, x = -2$ .

3)  $y' = -1$ , we imply that  $y = -x, x^2 = 1$ , i.e.,  $x = \pm 1$ . Thus we get the points  $(1, -1), (-1, 1)$ .

**Normal line:** The normal line of  $f(x)$  at  $(a, f(a))$  is the line which is perpendicular to the tangent line at the point.

**Example 62** *Let  $y^2 + x^2 = xy + 3$ . Find the equation of the normal line to the curve at (1, 2).*

**Solution:**

$$m_{\perp} = -\frac{1}{y'} \Big|_{(1,2)} = -3, \Rightarrow y = -3x + 5.$$

## 3.5 Derivatives of trigonometric functions

Recall the result:

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1.$$

**Derivative of Trig Functions:**

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x, \quad (\sec x)' = \sec x \tan x, \dots$$

**Example 63** Differentiate  $\csc x$ ,  $\cot x$ ,  $e^x \cos(x)$ ,  $\frac{1+\cos x}{1+\sin x}$ ,  $e^x \sin x$ .

**Example 64** Let  $y = \sin(x)$ , calculate  $y^{(10)}(x)$ .

**Example 65** Given the position function  $s = f(t) = 2 \sin(t)$ , calculate the velocity and acceleration at  $t = \frac{\pi}{3}$ .

**Example 66** Find the equation of the tangent line to the curve  $\sin(1-x)$  at  $(1, 0)$ .

**Example 67** Find  $f'(x)$ . If

$$f(x) = \sin x^2, \quad \sin^2 x, \quad e^{\sin x}, \quad \sin(\cos(\tan x)).$$

**Example 68** Find  $y'$  from  $\tan(xy+x) = x+y$ .

## 3.6 Important Results about Derivatives

**Rolle's Thm** Let  $f(x)$  be continuous on  $[a, b]$  and differentiable in  $(a, b)$ . If  $f(a) = f(b)$ , then there exists  $c \in (a, b)$  such that  $f'(c) = 0$ .

**Mean Value Thm** Let  $f(x)$  be continuous on  $[a, b]$  and differentiable in  $(a, b)$ , then there exists  $c \in (a, b)$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

**Example 69** Show that  $\sin x \leq x$  for  $x \geq 0$ .

**Solution:** Let  $f(x) = x - \sin x$ , apply Mean Value Thm to  $\frac{f(x)-f(0)}{x-0}$ .

## 3.7 Inverse functions

**One-to-one function:**  $y = f(x)$  is 1-1  $\Leftrightarrow$  for each  $y \in R$ , there is only one  $x \in D$ . Horizontal line test can be used to check this  $\Leftrightarrow f(x_1) \neq f(x_2)$  for any  $x_1 \neq x_2$  in  $D$ .

Method to test one-to-one: Horizontal line test.

**Example 70**  $f(x) = x^2$  is not 1-1;  $g(x) = \sqrt{x}$  is 1-1.

**Example 71** Let  $f(x) = \frac{3x+2}{5x-4}$ . Find the domain, and show that the function is 1:1 in the domain.

**Inverse function:** If  $f$  is one-to-one, we define the inverse  $y = f^{-1}(x)$  of  $y = f(x)$  as:  $f^{-1}(b) = a$  if  $f(a) = b$ .

- The graph of  $f^{-1}$  and the graph of  $f$  are symmetric about the line  $y = x$ .
- Cancellation:  $f(f^{-1}(y)) = y$ .
- $f^{-1}(f(x)) = x$
- $D(f) = R(f^{-1})$ ,  $R(f) = D(f^{-1})$ .

**Example 72** let  $f(x) = \frac{3x+2}{5x-4}$ , find the inverse  $f^{-1}(x)$ .

Strategy:

- 1) Write  $y = \frac{3x+2}{5x-4}$ ;
- 2) Switch  $x$  and  $y$ :  $x = \frac{3y+2}{5y-4}$ ;
- 3) Isolate  $y$ :  $y = \frac{4x+2}{5x-3}$ ;
- 4) Answer:  $y = f^{-1}(x) = \frac{4x+2}{5x-3}$ .

## Derivative of inverse functions

A general formula for the derivative of inverse functions:

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}.$$

Proof. Let  $y = f^{-1}(x)$ , then  $f(y) = x \Rightarrow f'(y)y' = 1$ .

## 3.8 Inverse Trig Functions

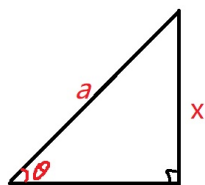
Inverse Trig Function	Domain	Restriction (Range)	Meaning
$y = \arcsin x = \sin^{-1}(x)$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$	$\sin y = x$
$y = \arccos x = \cos^{-1}(x)$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$	$\cos y = x$
$y = \arctan x = \tan^{-1}(x)$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$\tan y = x$
$y = \operatorname{arcsec} x = \sec^{-1}(x)$	$ x  \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$	$\sec y = x$
$y = \operatorname{arccsc} x = \csc^{-1}(x)$	$ x  \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$	$\csc y = x$
$y = \operatorname{arccot} x = \cot^{-1}(x)$	$-\infty < x < \infty$	$0 < y < \pi$	$\tan y = x$

**Graphs of the inverse functions:** Using the symmetry line  $y = x$ .

**Example 73** Find the exact values of the following expressions: (a)  $\arcsin(1)$  (b)  $\arctan(-1)$   
 (c)  $\tan^{-1}(\sqrt{3})$  (d)  $\sin[\cos^{-1}(\frac{\sqrt{3}}{2})]$

**Example 74** Simplify the following expression:

$$\tan \arcsin \frac{x}{a}.$$



**Solution:**

$$\tan \arcsin \frac{x}{a} = \tan \theta = \frac{x}{\sqrt{a^2 - x^2}}.$$

**Example 75**

$$\lim_{x \rightarrow 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) = \arcsin\left(\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

**Example 76**  $y = \tan^{-1} x$  has horizontal asymptotes  $y = \frac{\pi}{2}$  and  $y = -\frac{\pi}{2}$ .

$$\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2} \text{ or } \lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

### 3.9 Derivative of Inverse Trig Functions

$$\begin{aligned} \frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx} \arccos x &= \frac{1}{-\sqrt{1-x^2}}, & \frac{d}{dx} \arctan x &= \frac{1}{1+x^2}, \\ \frac{d}{dx} \operatorname{arcsec} x &= \frac{1}{|x|\sqrt{x^2-1}}, & \frac{d}{dx} \operatorname{arccsc} x &= -\frac{1}{|x|\sqrt{x^2-1}}, & \frac{d}{dx} \operatorname{arccot} x &= -\frac{1}{1+x^2}. \end{aligned}$$

**Example 77** Find  $y'$ :  $y = \sin(\arctan 2x)$ ,  $y = \arcsin\left(\frac{b+a \cos x}{a+b \cos x}\right)$ .

### 3.10 L'Hospital's Rule

In this section, we are going to deal with the limit with the form:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^\infty, \quad 0 \cdot \infty, \quad 0^0, \dots$$

L'Hospital's rule: If  $\frac{f(x)}{g(x)}$  becomes  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  as  $x \rightarrow x_0$ , where  $x_0$  is finite or  $\infty$ , then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}.$$

Remark.  $x \rightarrow x_0$  can be replaced by any of the symbols  $x \rightarrow x_0^+$ ,  $x \rightarrow x_0^-$ ,  $x \rightarrow \infty$ , or  $x \rightarrow -\infty$ .

**Example 78**

$$\lim_{x \rightarrow 1} \frac{x^{2017} - 5x^2 + 4}{x^{2018} + 5x^3 - 6}, \quad \lim_{x \rightarrow \infty} \frac{3x^2 + 2x - 5}{2x^2 + x + 1}.$$

**Example 79** Calculate

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}, \quad \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{x^2}.$$

## Chapter 4. Exponentials and Logarithms

### 4.1 Exponential functions and Their logarithms

Definition: We say that  $f(x) = a^x$  is an exponential function with base  $a$ ,  $a > 0$ ,  $a \neq 1$ .

Domain:  $x \in \mathbb{R}$ .

$F(x) = \log_a x$  is called logarithmic function with the base  $a$ . Domain =  $\{x > 0\}$ .

$$y = 10^x \xrightarrow{\text{inverse function}} y = \log_{10} x = \log x.$$

Laws of exponents:

$$a^{x+y} = a^x a^y, \quad a^{x-y} = a^x / a^y, \quad (a^x)^y = a^{xy}, \quad a^x b^x = (ab)^x.$$

**Example 80** Sketch the graph of  $y = 2^x + 5$ .

**Example 81** Sketch the graph of  $y = 2^{-x} + 5$ .

**Example 82** Solve  $4^x - 7(2^x) + 6 = 0$ .

Properties of logarithms: Let  $B, C > 0$ . Then

1.  $\log_a(BC) = \log_a B + \log_a C$ ,
2.  $\log_a\left(\frac{B}{C}\right) = \log_a B - \log_a C$ ,
3.  $\log_a(B^n) = n \log_a B$ ,
4.  $\log_a(a^x) = x$ ,  $\log_a a = 1$ ,
5.  $a^{\log_a B} = B$ ,
6.  $\log_a 1 = 0$ .
7. Change of base:  $\log_a b = \frac{\log_c b}{\log_c a}$ .

Proof. Let  $x = \log_a b$ . Then  $a^x = b \Rightarrow \log_c a^x = \log_c b \Rightarrow x \log_c a = \log_c b$ .

**Example 83** Simplify  $\log_3 18 - \log_3 2$ .

## 4.2 Euler's Number

**Euler's Number  $e$**

$$e = \lim_{x \rightarrow 0} (1+x)^{1/x} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = 2.71828\dots$$

**Example 84** Calculate

$$\lim_{x \rightarrow 0} (1-x)^{1/x}, \quad \lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x.$$

## 4.3 Euler's Exponential Function and the Natural Logarithm

$y = e^x$  is called the Euler's exponential function (natural exponential function).

The inverse  $y = \ln x$  is called natural logarithm. Domain:  $x > 0$ .

**Example 85** Convert  $a^x$  to base  $e$ .

$$a^x = e^{x \ln a}.$$

**Example 86** Solve for  $x$ :

$$3^{2x-1} = 4, \quad \ln[\ln(2x+1)] = 1, \quad \log_3 x + \log_3(x-8) = 2.$$

**Example 87** Sketch  $y = \ln(x+1) - 2$ .

**Example 88** Find the inverse of  $f(x) = \sin(2e^{3x})$ .

**Solution:** Let  $y = \sin(2e^{3x})$ . Switch  $x$  and  $y$ ,

$$x = \sin(2e^{3y}), \Rightarrow \sin^{-1} x = 2e^{3y}, \Rightarrow \ln\left(\frac{1}{2} \sin^{-1} x\right) = 3y, \Rightarrow$$

$$y = \frac{1}{3} \ln\left(\frac{1}{2} \sin^{-1} x\right), \Rightarrow f^{-1}(x) = \frac{1}{3} \ln\left(\frac{1}{2} \sin^{-1} x\right).$$

**Example 89** Evaluate

$$\lim_{x \rightarrow 0^-} e^{1/x}.$$

**Solution:** Let  $t = 1/x$ , then  $x \rightarrow 0^- \Leftrightarrow t \rightarrow -\infty$ .

$$\lim_{x \rightarrow 0^-} e^{1/x} = \lim_{t \rightarrow -\infty} e^t = 0.$$

## 4.4-4.6 Derivatives of exp. and log. functions

- Derivative of exponential function:

$$(e^x)' = e^x, \quad (a^x)' = a^x \ln a.$$

- Derivatives of log functions:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad (\ln f(x))' = \frac{f'(x)}{f(x)},$$

$$(\log_a |x|)' = \frac{1}{x \ln a}, \quad (\log_a f(x))' = \frac{f'(x)}{f(x) \ln a}, \dots$$

Change base:

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

**Example 90** *At what point(s) on the curve  $y = e^x$  is the tangent line*

- parallel to  $y = 3x - 2$ ?*
- perpendicular to  $y = -2x$ ?*

**Solution:** (a):  $(\ln 3, 3)$ .

(b):  $(-\ln 2, 1/2)$ .

**Example 91** *Differentiate  $\ln(x^2 + 1)$ .*

**Example 92** *Calculate the following limits by L'Hospital's Rule:*

(a)  $\lim_{t \rightarrow 0} \frac{e^t - t - 1}{t^2}$

(b)  $\lim_{x \rightarrow 0} x^x$

(c)  $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x^2}$

(d)  $\lim_{x \rightarrow \infty} x^2 e^{-x}$

(e)  $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x$ .

**Solution:** (d):

$$\lim_{x \rightarrow \infty} x^2 e^{-x} = \lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = \lim_{x \rightarrow \infty} \frac{0}{e^x} = 0.$$

(e): Let  $y = \left(1 - \frac{1}{x}\right)^x$ , then

$$\ln y = x \ln \left(1 - \frac{1}{x}\right) = \frac{\ln \left(1 - \frac{1}{x}\right)}{\frac{1}{x}}.$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \frac{\ln \left(1 - \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{-\frac{1}{x^2} \left(1 - \frac{1}{x}\right)} = -1. \Rightarrow \\ \lim_{x \rightarrow \infty} y &= \frac{1}{e}. \end{aligned}$$

**Example 93** Find the derivative of the function  $y = 5^x \arctan(e^{2x})$ .

**Solution:**  $y = 5^x \arctan(e^{2x}) \ln 5 + 5^x \frac{2e^{2x}}{1 + e^{4x}}.$

### Logarithmic differentiation

**Example 94** Differentiate (a):  $y = \frac{x^2+x+5}{(x+1)^2}$ , (b)  $y = x^x$ , (c)  $y = (\sin x)^x$ .

**Solution:** (b):

$$\ln y = x \ln x, \Rightarrow \frac{y'}{y} = \ln x + x \cdot \frac{1}{x}, \Rightarrow y' = x^x (\ln x + 1).$$

(c):

$$\ln y = x \ln \sin x, \Rightarrow \frac{y'}{y} = \ln \sin x + x \cdot \frac{\cos x}{\sin x}, \Rightarrow y' = x^x \left( \ln \sin x + x \cdot \frac{\cos x}{\sin x} \right).$$

## Chapter 5. Curve Sketching

### 5.1 Solving Polynomial Inequalities

**Example 95** Solve  $(x - 2)(x + 2)^4(x^2 + 3) > 0$ .

### 5.2 Solving Polynomial Inequalities

**Example 96** Solve  $\frac{(x-2)(x+2)^4(x^2+3)}{x+1} > 0$ .

### 5.3 Graphing Techniques

- Absolute (Global) Maximum and Minimum:  $f(x)$  has a Global (Absolute) Maximum at  $p$  if  $f(p) \geq f(x)$  for all  $x$  in the domain;  $f(x)$  has a Global (Absolute) Minimum at  $p$  if  $f(p) \leq f(x)$  for all  $x$  in the domain;
- Local (or relative) extreme:  $f(x)$  has a local minimum at  $p$  if  $f(p) \leq f(x)$  for points  $x$  near  $p$ ;  $f(x)$  has a local maximum at  $p$  if  $f(p) \geq f(x)$  for points  $x$  near  $p$ ;
- Critical point (critical number): A point  $p$  in the domain such that  $f'(p) = 0$  or  $f'(p)$  undefined is called a critical number,  $(p, f(p))$  is a critical point,  $f(p)$  is a critical value.

**EXTREME VALUE THEOREM:** If  $f(x)$  is continuous on a closed interval  $[a, b]$ , then  $f$  attains an absolute maximum value  $f(c)$  and an absolute minimum value  $f(d)$  at some numbers  $c$  and  $d$  in  $[a, b]$ .

**FERMAT'S THEOREM:** If  $f$  has a local maximum or minimum at  $c$ , and if  $f'(c)$  exists, then  $f'(c) = 0$ .

**Example 97** Find the critical numbers of  $f(x) = x^{3/5}(4 - x)$ .

**Solution:** the critical numbers are  $3/2$  and  $0$ .

## Global extreme values

CLOSED INTERVAL METHOD: To find a global maximum or minimum for  $f(x)$  on a closed interval  $[a, b]$ :

1. Find all the critical numbers, e.g.,  $x_1, \dots, x_n$ .
2. global minimum =  $\min\{f(x_1), \dots, f(x_n), f(a), f(b)\}$ ;  
global maximum =  $\max\{f(x_1), \dots, f(x_n), f(a), f(b)\}$ .

**Example 98** Find the global maximum and minimum of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 7, \quad [-2, 0].$$

**Solution:** Step 1)  $f'(x) = 6x^2 - 6x - 12$ ,  $f'(x) = 0 \Rightarrow x = -1, 2$ ,  $f'(x)$  is defined anywhere. Hence  $x = -1$  is the only one critical number in  $(-2, 0)$ .

Step 2) global minimum =  $\min\{f(-2), f(-1), f(0)\} = \min\{3, 14, 7\} = 3$ ;  
global maximum =  $\max\{f(-2), f(-1), f(0)\} = \max\{3, 14, 7\} = 14$ .

**Example 99** Find the global maximum and minimum of the function

$$f(x) = x^2 e^{2x}, \quad [-2, 2].$$

## Local extreme values

INCREASING/DECREASING TEST (I/D TEST): If  $f'(x) > 0$  on an interval, then  $f$  is increasing on that interval. If  $f'(x) < 0$  on an interval, then  $f$  is decreasing on that interval.

**First Derivative Test:** Let  $p$  be a critical number. If  $f'$  changes from  $-$  to  $+$  at  $p$ , then  $f$  has a local minimum at  $p$ ; If  $f'$  changes from  $+$  to  $-$  at  $p$ , then  $f$  has a local maximum at  $p$ .

**Example 100** Let  $g(x) = x + 2 \sin x$ ,  $0 \leq x \leq 2\pi$ .

(a) Find all the critical numbers.

(b) State all the intervals of increase and decrease.

(c) Find all the local minimum points and all the local maximum points.

Solution:

(a)  $g'(x) = 1 + 2 \cos x$ ,  $g'(x) = 0 \Rightarrow x = 2\pi/3, 4\pi/3$ .

(b) Look at the following table

$x$	$0 < x < 2\pi/3$	$2\pi/3 < x < 4\pi/3$	$4\pi/3 < x < 2\pi$
$f'(x)$	+	-	+
$f(x)$	increase	decrease	increase

Therefore,

The intervals of increase:  $0 < x < 2\pi/3$ ,  $4\pi/3 < x < 2\pi$ .

The intervals of decrease:  $2\pi/3 < x < 4\pi/3$

(c) Note that at  $x = 2\pi/3$ ,  $f'(x)$  changes from + to -; at  $x = 4\pi/3$ ,  $f'(x)$  changes from - to +. By the First Derivative Test,  $f(x)$  has a local maximum at  $x = 2\pi/3$  and a local minimum at  $x = 4\pi/3$ .

**Second Derivative Test:** Let  $p$  be a critical number. If  $f''(p) > 0$ , then  $f$  has a local minimum at  $p$ ; If  $f''(p) < 0$ , then  $f$  has a local maximum at  $p$ ; If  $f''(p) = 0$ , then nothing.

## Concavity

**Definition 13 (CONCAVITY)** If the graph of  $f$  lies above all of its tangents on an interval  $I$ , it is called concave upward on  $I$ . If the graph of  $f$  lies below all of its tangents on  $I$ , it is called concave downward on  $I$ .

CONCAVITY TEST: If  $f''(x) > 0$  for all  $x$  in  $I$ , then the graph of  $f$  is concave upward on  $I$ . If  $f''(x) < 0$  for all  $x$  in  $I$ , then the graph of  $f$  is concave downward on  $I$ .

**Definition 14 Point of inflection:** If  $f(x)$  changes concavity at  $p$ , then  $p$  is an inflection point, and  $f''(p) = 0$  or undefined.

**Example 101** Let  $f(x) = x^3 - 3x^2$ .

(a) Find all the critical numbers.

(b) State all the intervals of increase and decrease.

(c) Find all the local minimum points and all the local maximum points.

(d) Find all the local minimum points and all the local maximum points by Second Derivative Test.

(e) Find all the points of inflection.

(f) State intervals of concavity.

(g) Sketch the graph.

**Solution:**

(a)  $f'(x) = 3x^2 - 6x = 3x(x - 2)$ . Let  $f'(x) = 0$ . We have  $3x(x - 2) = 0$ , which gives  $x = 0, 2$ .

(b) Look at the following table

$x$	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
$f'(x)$	+	-	+
$f(x)$	increase	decrease	increase

Therefore,

The intervals of increase:  $-\infty < x < 0, 2 < x < \infty$ .

The intervals of decrease:  $0 < x < 2$

(c) Note that at  $x = 0$ ,  $f'(x)$  changes from + to -; at  $x = 2$ ,  $f'(x)$  changes from - to +. By the First Derivative Test,  $f(x)$  has a local maximum at  $x = 0$  and a local minimum at  $x = 2$ .

(d) From  $f'(x) = 3x^2 - 6x$  we get  $f''(x) = 6x - 6$ .  $f'(x) = 0 \Rightarrow x = 0, 2$ . Note that  $f''(0) = -6 < 0$   $f''(2) = 6 > 0$  By the Second Derivative Test,  $f(x)$  has a local maximum at  $x = 0$  and a local minimum at  $x = 2$ .

(e)  $f''(x) = 0 \Rightarrow x = 1$ .

$x$	$-\infty < x < 1$	$1 < x < \infty$
$f''(x)$	-	+
$f(x)$	concave down	concave up

Since  $f(x)$  changes concavity at  $x = 1$ ,  $x = 1$  is a point of inflection.

(f) Concave up:  $1 < x < \infty$ ; Concave down:  $-\infty < x < 1$ .

**Example 102** Consider the function

$$f(x) = \frac{x}{x^2 - 1}.$$

Study the concavity and find all the points of inflection.

Solution: The domain of the function:  $x \neq \pm 1$ .

$$f'(x) = \frac{-1 - x^2}{(x^2 - 1)^2}, \quad f'' = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

$$f''(x) = 0, \Rightarrow x = 0.$$

$x$	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f''(x)$	-	+	-	+
$f(x)$	concave down	concave up	concave down	concave up

**Example 103** Sketch  $f(x)$ , which satisfies:

- $D = \{x \neq 0\}$ .
- $\lim_{x \rightarrow \pm\infty} f(x) = 1$ ,  $\lim_{x \rightarrow 0^+} f(x) = -\infty$ ,  $\lim_{x \rightarrow 0^-} f(x) = \infty$ .
- $f'(2) = f'(-2) = 0$ ;  $f'(x) > 0$  when  $0 < x < 2$  and  $-2 < x < 0$ ;  $f'(x) < 0$  when  $x > 2$  and  $x < -2$ .
- $f''(-3) = f''(3) = 0$ ;  $f''(x) > 0$  when  $x > 3$  and  $-3 < x < 0$ ;  $f''(x) < 0$  when  $x < -3$  and  $0 < x < 3$ .
- $f(2) = 2, f(-2) = -1$ .

## Curve sketching

The following checklist is intended as a guide to sketching a curve  $y = f(x)$  by hand.

Not every item is relevant to every function. For instance, a given curve might not have an asymptote or possess symmetry. However, the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

- A. DOMAIN
- B. INTERCEPTS
- C. SYMMETRY
  1. EVEN FUNCTION:  $f(-x) = f(x)$  for all  $x$  in  $D$ . the curve is symmetric about the  $y$ -axis. This means that our work is cut in half.
  2. ODD FUNCTION:  $f(-x) = -f(x)$  for all  $x$  in  $D$ . the curve is symmetric about the origin. This means that our work is cut in half.

3. PERIODIC FUNCTION:  $f(x + p) = f(x)$  for all  $x$  in  $D$ , where  $p$  is a positive constant. The smallest such number  $p$  is called the period.

- D. ASYMPTOTES

- HORIZONTAL:  $\lim_{x \rightarrow \pm\infty} f(x) = L$ , then  $y = L$  is a HA.

- VERTICAL:  $\lim_{x \rightarrow a^\pm} f(x) = \pm\infty$ , then  $x = a$  is a VA.

- E. INTERVALS OF INCREASE OR DECREASE: use I/D Test.

- F. LOCAL MAXIMUM AND MINIMUM VALUES: First Derivative Test or Second Derivative Test.

- G. CONCAVITY AND POINTS OF INFLECTION

**Example 104** Consider the function

$$f(x) = [x^2(6 - x)]^{1/3}, \quad -\infty < x < \infty.$$

We have (you don't need to check the following results!)

$$f'(x) = \frac{4 - x}{\sqrt[3]{x(6 - x)^2}}, \quad f''(x) = \frac{-8}{\sqrt[3]{x^4(6 - x)^5}}.$$

a) Find all the critical numbers of  $f(x)$ .

b) Find all the intervals of increasing and decreasing, and classify all the critical numbers as local maxima, minima, or neither.

c) Study the concavity and find the point(s) of inflection.

d) Sketch the graph. (Remark.  $f(x)$  has no vertical and horizontal asymptotes).

**Solution:** a) From  $f'(x) = 0$  we get  $x = 4$ ;

From  $f'(x)$  undefined we imply that  $x = 0, 6$ .

So we have three critical numbers  $x = 4, 0, 6$ .

b) Look at the following table

$x$	$-\infty < x < 0$	$0 < x < 4$	$4 < x < 6$	$6 < x < \infty$
$f'(x)$	-	+	-	-
$f(x)$	decreasing	increasing	decreasing	decreasing

By the First Derivative Test,  $f(0) = 0$  is a local minimum,  $f(4) = \sqrt[3]{32}$  is a local maximum,  $f(6)$  is neither.

c) Note that  $f''(x)$  undefined at  $x = 0, 6$ . Look at the following table

$x$	$-\infty < x < 0$	$0 < x < 6$	$6 < x < \infty$
$f''(x)$	-	-	+
$f(x)$	concave down	concave down	concave up

$x = 6$  or  $(6, f(6))$  is a point of inflection.

**Example 105** Use the first and second derivatives of  $f(x) = e^{1/x}$ , together with asymptotes, to sketch its graph.

**Solution:** Notice that the domain of  $f$  is  $\{x|x \neq 0\}$ . So, we check for vertical asymptotes by computing the left and right limits as  $x \rightarrow 0$ .

$$\lim_{x \rightarrow 0^+} e^{1/x} = \infty, \quad \lim_{x \rightarrow 0^-} e^{1/x} = 0.$$

This shows that  $x = 0$  is a vertical asymptote.

$$\lim_{x \rightarrow \pm\infty} e^{1/x} = 1,$$

this shows that  $y = 1$  is a horizontal asymptote.

The Chain Rule gives:

$$f'(x) = -\frac{e^{1/x}}{x^2},$$

we have  $f'(x) < 0$  for all  $x \neq 0$ . Thus,  $f$  is decreasing on  $(-\infty, 0)$  and on  $(0, \infty)$ . There is no critical number. So, the function has no maximum or minimum.

$$f''(x) = \frac{e^{1/x}(2x + 1)}{x^4},$$

$f''(x) > 0$  when  $x > -1/2$  ( $x \neq 0$ ), and  $f''(x) < 0$  when  $x < -1/2$ . So, the curve is concave downward on  $(-\infty, -1/2)$  and concave upward on  $(-1/2, 0)$  and on  $(0, \infty)$ .

The inflection point is  $(-1/2, e^{-2})$ .

**Example 106** Sketch the graph of:

$$f(x) = \frac{2x^2}{x^2 - 4}, \quad f(x) = xe^x, \quad f(x) = \frac{\sin x}{2 + \cos x}, \quad f(x) = \ln(1 - x^2), \quad f(x) = x \ln x.$$

**General Extremum Test:** Let  $f'(c) = \dots = f^{(n-1)}(c) = 0$ ,  $f^{(n)}(c) \neq 0$ .

- (a) If  $n$  is even, and  $f^{(n)}(c) > 0$ , then  $f(c)$  is a local minimum;
- (b) If  $n$  is even, and  $f^{(n)}(c) < 0$ , then  $f(c)$  is a local maximum;
- (c) If  $n$  is odd, then  $c$  is a POI.

## Chapter 6. Integration

### 6.1 Antiderivatives and Indefinite Integral

**Definition 15** A function  $F$  is called an antiderivative of  $f$  on an interval  $I$  if  $F'(x) = f(x)$  for all  $x$  in  $I$ .

Some basic results:

function	antiderivative	formula
$k$	$kx + C$	$\int k dx = kx + C$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C; (n \neq -1)$
$e^{kx}$	$\frac{1}{k}e^{kx} + C$	$\int e^{kx} dx = \frac{1}{k}e^{kx} + C$
$a^{kx}$	$\frac{a^{kx}}{k \ln a} + C$	$\int a^{kx} dx = \frac{a^{kx}}{k \ln a} + C$
$\frac{1}{x}$	$\ln  x  + C$	$\int \frac{1}{x} dx = \ln  x  + C$
$\cos kx$	$\frac{1}{k} \sin kx + C$	$\int \cos kx dx = \frac{1}{k} \sin kx + C$
$\sin kx$	$-\frac{1}{k} \cos kx + C$	$\int \sin kx dx = -\frac{1}{k} \cos kx + C$
$\sec^2 kx$	$\frac{1}{k} \tan kx + C$	$\int \sec^2 kx dx = \frac{1}{k} \tan kx + C$
$\sec kx \tan kx$	$\frac{1}{k} \sec kx + C$	$\int \sec kx \tan kx dx = \frac{1}{k} \sec kx + C$
$\frac{1}{\sqrt{1-(kx)^2}}$	$\frac{1}{k} \arcsin kx + C$	$\int \frac{1}{\sqrt{1-(kx)^2}} dx = \frac{1}{k} \arcsin kx + C$
$\frac{1}{1+(kx)^2}$	$\frac{1}{k} \arctan kx + C$	$\int \frac{1}{1+(kx)^2} dx = \frac{1}{k} \arctan kx + C$
		$\int kf(x) dx = k \int f(x) dx$
		$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx.$

**Example 107**  $\int \frac{x^2-1}{x^3} dx = \int (x^{-1} - x^{-3}) dx = \ln |x| + \frac{1}{2x^2} + C.$

$$\int \sin 4x + e^{5x} dx = -\frac{1}{4} \cos 4x + \frac{1}{5} e^{5x} + C.$$

**Example 108** Given  $f'(x) = \sin x + \frac{4x^2 - 22}{x^3}.$

(a) Find  $f(x).$

(b) Find  $f(x)$  such that  $f(1) = 3.$

**Solution:** (a)

$$f'(x) = \sin x + 4x^{-1} - 22x^{-3}, \Rightarrow f(x) = -\cos x + 4 \ln |x| + 11x^{-2} + C.$$

(b)

$$f(1) = 3 \Rightarrow C = -8 \Rightarrow f(x) = 4 \ln |x| + 11x^{-2} - 8.$$

**Example 109** Let  $y = y(x)$  be the solution of the differential equation

$$\frac{dy}{dx} = \frac{\pi \sin \pi x + 4x}{9y^2}$$

subject to the initial condition  $y(0) = 1$ . Find  $y(1)$ .

**Solution:**  $3y^3 = -\cos \pi x + 2x^2 + C$ ,  $C = 4$ ,  $3y^3 = -\cos \pi x + 2x^2 + 4$ ,  $y(1) = \sqrt[3]{\frac{7}{3}}$ .

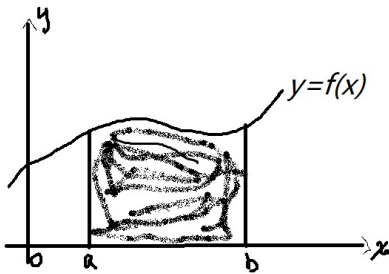
## 6.2 Definite Integrals

**Definition 16** The integral  $\int_a^b f(x)dx$  is called *Definite Integral of  $f(x)$  over  $[a, b]$* .

Geometrically, if  $f(x) \geq 0$ , then the integral  $\int_a^b f(x)dx$  is the area of the region below the continuous function  $y = f(x)$ , over  $[a, b]$ .

Generally,

$$\int_a^b f(x)dx = \text{area above } x\text{-axis} - \text{area below } x\text{-axis}.$$



**The Fundamental Theorem of Calculus :**

- $\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$ , where  $F'(x) = f(x)$ .
- $\frac{d}{dx} \int_a^x f(t)dt = f(x)$ .

Generally we have

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

Some basic properties about definite integral:

- $\int_a^b cdx = c(b - a)$ ;
- $\int_a^b f(x)dx = - \int_b^a f(x)dx$ ;
- $\int_a^a f(x)dx = 0$ ;
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ ;

- $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$ ;
- Constant multiple:  $\int_a^b cf(x) dx = c \int_a^b f(x) dx$ ;
- Comparison of Definite Integrals: If  $f(x) \leq g(x)$  for  $a \leq x \leq b$ , then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

In particular, if  $m \leq f(x) \leq M$ , then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

**Example 110** Let  $\int_1^5 f(x) dx = 3$ ,  $\int_1^5 g(x) dx = 5$ . Calculate  $\int_1^5 [2f(x) - g(x) - 1] dx$ .

**Solution:**

$$\begin{aligned} \int_1^5 [2f(x) - g(x) - 1] dx &= 2 \int_1^5 f(x) dx - \int_1^5 g(x) dx - \int_1^5 1 dx \\ &= 2(3) - 5 - 1(5 - 1) = -3. \end{aligned}$$

**Example 111** Calculate  $\int_0^2 3^t dt$ .

**Solution:** Let  $f(t) = 3^t$ , then  $F(t) = \frac{1}{\ln 3} 3^t + C$ .

$$\int_0^2 3^t dt = F(2) - F(0) = \frac{8}{\ln 3}.$$

**Example 112** Let  $f'(x) = x^3$ ,  $f(0) = 1$ . Find the value of  $f(2)$ .

**Solution:**

$$\begin{aligned} f(b) - f(a) &= \int_a^b f'(x) dx \Rightarrow \\ f(2) &= f(0) + \int_0^2 f'(x) dx = 1 + \frac{1}{4} x^4 \Big|_0^2 = 5. \end{aligned}$$

**Example 113**

$$\begin{aligned} \frac{d}{dx} \int_0^x \sin(t^2) dt &= \sin(x^2), \\ \frac{d}{dx} \int_0^{x^2} \sin(t^2) dt &= \frac{d}{du} \int_0^u \sin(t^2) dt \cdot \frac{du}{dx} = 2x \sin(u^2) = 2x \sin(x^4), \quad u = x^2, \\ \frac{d}{dx} \int_{x^2}^{x^3} f(t) dt &= \frac{d}{dx} \left( \int_{x^2}^0 f(t) dt + \int_0^{x^3} f(t) dt \right) = -2x f(x^2) + 3x^2 f(x^3). \end{aligned}$$

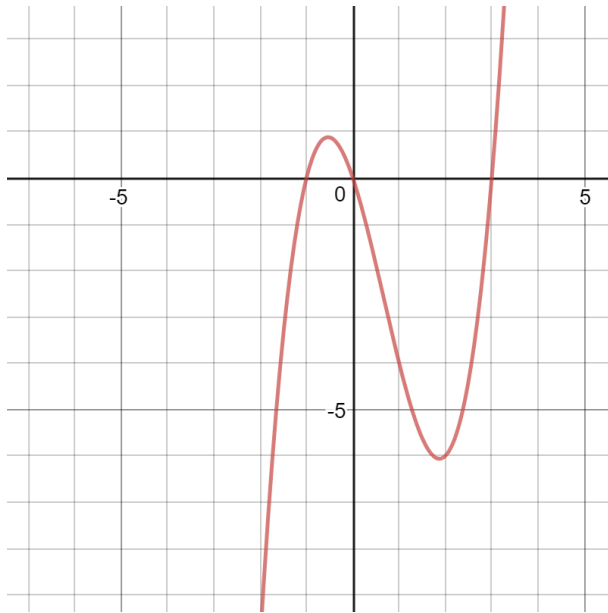
**Example 114** Let  $g(x) = \int_a^x e^{t^2} dt$ . Calculate  $g'(2)$  and  $g''(2)$ .

**Solution:**

$$g'(x) = e^{x^2}, \Rightarrow g'(2) = e^4.$$

$$g''(x) = 2xe^{x^2}, \Rightarrow g''(2) = 4e^4.$$

**Example 115** Find the area of the region between the  $x$ -axis and the graph of  $f(x) = x^3 - 2x^2 - 3x$ ,  $-1 \leq x \leq 3$ .



**Solution:**

Step 1: Find zeros of  $f(x)$ : Let  $f(x) = 0$ , then  $x = -1, 0, 3$ ;

Step 2:

$$\begin{aligned} \text{total area} &= \int_{-1}^0 f(x) dx + \int_0^3 -f(x) dx \\ &= \left( \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2 \right) \Big|_{-1}^0 - \left( \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2 \right) \Big|_0^3 = \frac{7}{12} + \frac{45}{4} = \frac{61}{6}. \end{aligned}$$

## Chapter 7. Technics of Integration

### 7.2 The Substitution Rule

- For indefinite integral:  $\int f(g(x))g'(x)dx = \int f(u)du$ ,  $u = g(x)$ . In the final result, we have to replace  $u$  by  $g(x)$ ;
- For definite integral:  $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$ .

**Example 116** Evaluate

$$\int (2x - 1)(x^2 - x)^{100} dx.$$

**Solution:** Let  $u = x^2 - x$ . Then  $du = (2x - 1)dx$ . Thus

$$\int (2x - 1)(x^2 - x)^{100} dx = \int u^{100} du = \frac{u^{101}}{101} + C = \frac{(x^2 - x)^{101}}{101} + C.$$

**Example 117**

$$\int x\sqrt{x^2 + 1} dx.$$

**Solution:** Let  $u = x^2 + 1$ . Then  $du = (2x)dx$ . Thus

$$\int x\sqrt{x^2 + 1} dx = \int \frac{1}{2}\sqrt{u} du = \frac{1}{2} \int u^{1/2} du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{3}(x^2 + 1)^{3/2} + C.$$

**Example 118**

$$\int_0^1 x\sqrt{x^2 + 1} dx.$$

**Solution:** Let  $u = x^2 + 1$ . Then  $du = (2x)dx$ ,  $x = 0 \leftrightarrow u = 1$ ,  $x = 1 \leftrightarrow u = 2$ . Thus

$$\int_0^1 x\sqrt{x^2 + 1} dx = \int_1^2 \frac{1}{2}\sqrt{u} du = \frac{1}{2} \int u^{1/2} du = \frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right]_1^2 = \frac{2\sqrt{2} - 1}{3}.$$

**Example 119** Find

$$\int \frac{1}{e^{-x} + 1} dx.$$

**Solution:**

$$\int \frac{1}{e^{-x} + 1} dx = \int \frac{e^x}{1 + e^x} dx.$$

Let  $u = 1 + e^x$ , then  $du = e^x dx$ .

$$\int \frac{e^x}{1 + e^x} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |1 + e^x| + C = \ln(1 + e^x) + C.$$

**Example 120** Evaluate

$$\int x^2 e^{x^3+1} dx.$$

**Solution:** Let  $u = x^3 + 1$ ,  $du = 3x^2 dx$ .

$$\int x^2 e^{x^3+1} dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3+1} + C.$$

**Example 121** Evaluate

$$\int \frac{x}{\sqrt{1-x}} dx.$$

**Solution:** Let  $u = 1 - x$ ,  $du = -dx$ .

$$\begin{aligned} \int \frac{x}{\sqrt{1-x}} dx &= \int \frac{1-u}{\sqrt{u}} (-du) = \int \left( -\frac{1}{\sqrt{u}} + \sqrt{u} \right) du = -2\sqrt{u} + \frac{2}{3}u^{3/2} + C \\ &= -2\sqrt{1-x} + \frac{2}{3}(1-x)^{3/2} + C. \end{aligned}$$

**Example 122** Calculate

$$\int \tan x dx.$$

**Solution:** Note that  $\tan x = \frac{\sin x}{\cos x}$ , we let  $u = \cos x$ . Then  $du = -\sin x dx$ . Thus

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = - \int \frac{1}{u} du = -\ln |u| + C = -\ln |\cos x| + C.$$

**Example 123** Evaluate

$$\int \frac{\arcsin x}{\sqrt{1-x^2}} dx.$$

**Solution:** Let  $u = \arcsin x$ ,  $du = \frac{1}{\sqrt{1-x^2}} dx$ ,

$$\int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} (\arcsin x)^2 + C.$$

**Example 124** Evaluate

$$\int \frac{(\ln x)^k}{x} dx, \quad k \neq -1.$$

**Solution:** Let  $u = \ln x$ ,  $du = \frac{1}{x} dx$ ,

$$\int \frac{(\ln x)^k}{x} dx = \int u^k du = \frac{1}{k+1} u^{k+1} + C = \frac{1}{k+1} (\ln x)^{k+1} + C.$$

## 7.3 Integration by Parts

Integration by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

**Example 125** Evaluate

$$\int (x^2 + x + 1)e^x dx.$$

**Solution:** Integration by parts

$$\begin{aligned} \int (x^2 + x + 1)e^x dx &= \int (x^2 + x + 1) de^x = (x^2 + x + 1)e^x - \int (2x + 1)e^x dx \\ &= (x^2 + x + 1)e^x - \int (2x + 1) de^x = (x^2 + x + 1)e^x - (2x + 1)e^x + 2 \int e^x dx \\ &= (x^2 + x + 1)e^x - 2xe^x + 2e^x + C \end{aligned}$$

**Example 126** Evaluate

$$\int x^2 e^{bx} dx, \quad b \neq 0.$$

**Solution:** Let  $t = bx$ . Then  $dt = bdx$ .

$$\int x^2 e^{bx} dx = \frac{1}{b^3} \int t^2 e^t dt = \frac{1}{b^3} (t^2 e^t - 2te^t + 2e^t + c).$$

**Example 127** Evaluate

$$\int 4x^3 \ln x dx.$$

**Solution:** Integration by parts

$$\begin{aligned}\int 4x^3 \ln x \, dx &= \int \ln x \, dx^4 = x^4 \ln x - \int x^4 d(\ln x) \\ x^4 \ln x - \int x^4 \frac{dx}{x} &= x^4 \ln x - \int x^3 \, dx \\ &= x^4 \ln x - \frac{1}{4}x^4 + C\end{aligned}$$

**Example 128** Evaluate

$$\int_0^1 \arctan x \, dx.$$

**Solution:** Step 1: Calculate

$$\begin{aligned}\int \arctan x \, dx &= x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \int \frac{d(1+x^2)}{1+x^2} \\ &= x \arctan x - \frac{1}{2} \ln(1+x^2).\end{aligned}$$

Step 2:

$$\int_0^1 \arctan x \, dx = [x \arctan x - \frac{1}{2} \ln(1+x^2)]_0^1 = \arctan 1 - \frac{1}{2} \ln 2 = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

**Example 129** Evaluate  $I = \int \sin(\ln(x^5)) \, dx$ .

**Solution:**

$$\begin{aligned}I &= \int \sin(\ln(x^5)) \, dx = \int \sin(5 \ln x) \, dx \\ &= x \sin(5 \ln x) - 5 \int \cos(5 \ln x) \, dx \\ &= x \sin(5 \ln x) - 5 \{x \cos(5 \ln x) + 5 \int \sin(5 \ln x) \, dx\} \\ &= x \sin(5 \ln x) - 5x \cos(5 \ln x) - 25 \int \sin(5 \ln x) \, dx \\ &= x \sin(5 \ln x) - 5x \cos(5 \ln x) - 25I\end{aligned}$$

Thus

$$\begin{aligned}26I &= x \sin(5 \ln x) - 5x \cos(5 \ln x) + C, \\ I &= \frac{x}{26} \{ \sin(5 \ln x) - 5 \cos(5 \ln x) \} + C.\end{aligned}$$

**Example 130** Evaluate  $\int x \sin(bx) dx$ .

**Solution:** Method 1:

$$\begin{aligned}\int x \sin(bx) dx &= \int x d\left(-\frac{\cos bx}{b}\right) = x\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right) dx \\ &= -\frac{1}{b}x \cos bx + \frac{1}{b} \int \cos bx dx = -\frac{1}{b}x \cos bx + \frac{1}{b^2} \sin bx dx + c\end{aligned}$$

Method 2:

$$\begin{aligned}\int x \sin(bx) dx &= \int x \left(-\frac{\cos bx}{b}\right)' dx = x\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right) (x)' dx \\ &= -\frac{1}{b}x^2 \cos bx + \frac{1}{b} \int \cos bx dx\end{aligned}$$

Method 3: Let  $t = bx$ , then  $dt = bdx$ ,  $dx = \frac{1}{b}dt$ .

$$\int x \sin(bx) dx = \frac{1}{b^2} \int t \sin t dt.$$

Remark. Similarly, we can solve the integral  $\int e^{ax} \cos(bx) dx$ .

**Example 131** Find the function  $f(x)$ , such that

$$f''(x) = 2 \ln(x) + 2, \quad f(1) = 1, \quad f'(1) = 0.$$

**Solution:**

$$\begin{aligned}f'(x) &= \int f''(x) dx = \int (2 \ln(x) + 2)(x)' dx = (2 \ln(x) + 2)x - \int (2 \ln(x) + 2)'(x) dx \\ &= (2 \ln(x) + 2)x - \int 2 dx = 2x \ln x + c.\end{aligned}$$

$$f'(1) = c = 0, \Rightarrow f'(x) = 2x \ln x.$$

$$\begin{aligned}f(x) &= \int f'(x) dx = \int 2x \ln(x) dx = \int (x^2)' \ln(x) dx = x^2 \ln(x) - \int x^2 [\ln(x)]' dx \\ &= x^2 \ln(x) - \int x dx = x^2 \ln x - \frac{1}{2}x^2 + c.\end{aligned}$$

$$f(1) = -\frac{1}{2} + c = 1, \Rightarrow c = 1.5. \text{ Thus}$$

$$f(x) = x^2 \ln x - \frac{1}{2}x^2 + 1.5.$$

## 7.4 Partial Fractions

Consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}.$$

By long division,

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where  $P(x)$ ,  $Q(x)$ ,  $S(x)$  and  $R(x)$  are polynomials,  $\deg R < \deg Q$ .

Partial fractions:

- If  $Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$ , a product of distinct linear factors, then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

- If some linear factors are same, for example,  $Q(x) = (a_1x + b_1)^n(a_2x + b_2) \cdots (a_kx + b_k)$ , then

$$\frac{R(x)}{Q(x)} = \frac{A_{11}}{a_1x + b_1} + \frac{A_{12}}{(a_1x + b_1)^2} + \cdots + \frac{A_{1n}}{(a_1x + b_1)^n} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

- If  $Q(x)$  has an irreducible factor  $ax^2 + bx + c$  without repeating, then

$$\frac{R(x)}{Q(x)} = \frac{Ax + B}{ax^2 + bx + c} + \cdots.$$

- If  $Q(x)$  has an irreducible factor  $ax^2 + bx + c$  with repeating, e.g.,  $(ax^2 + bx + c)^n$ , then

$$\frac{R(x)}{Q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n} + \cdots.$$

**Example 132** Evaluate

$$\int \frac{4x + 5}{(2x + 3)(3x + 4)} dx$$

**Solution:** Let

$$\frac{4x + 5}{(2x + 3)(3x + 4)} = \frac{A}{2x + 3} + \frac{B}{3x + 4} = \frac{(3A + 2B)x + 4A + 3B}{(2x + 3)(3x + 4)}.$$

Hence

$$4x + 5 = (3A + 2B)x + 4A + 3B, \Rightarrow 4 = 3A + 2B, 5 = 4A + 3B, \Rightarrow A = 2, B = -1.$$

Thus

$$\int \frac{4x + 5}{(2x + 3)(3x + 4)} dx = \int \left( \frac{2}{2x + 3} + \frac{-1}{3x + 4} \right) dx = \ln |2x + 3| - \frac{1}{3} \ln |3x + 4| + C.$$

**Example 133** Evaluate

$$\int \frac{x^3 + 2x^2 - 5x}{x^2 + 2x - 8} dx$$

**Solution:** Since

$$\frac{x^3 + 2x^2 - 5x}{x^2 + 2x - 8} = x + \frac{3x}{x^2 + 2x - 8} = x + \frac{3x}{(x+4)(x-2)} = x + \frac{2}{x+4} + \frac{1}{x-2}.$$

Hence

$$\begin{aligned} \int \frac{x^3 + 2x^2 - 5x}{x^2 + 2x - 8} dx &= \int \left( x + \frac{2}{x+4} + \frac{1}{x-2} \right) dx \\ &= \frac{1}{2}x^2 + \int \frac{2}{x+4} dx + \int \frac{1}{x-2} dx \\ &= \frac{1}{2}x^2 + 2 \ln|x+4| + \ln|x-2| + C. \end{aligned}$$

**Example 134** Evaluate

$$\int \frac{4x^2 + 8x}{(x+1)^2(x-1)} dx$$

**Solution:** Let

$$\frac{4x^2 + 8x}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}, \Rightarrow A = 1, B = 2, C = 3.$$

Hence

$$\begin{aligned} \int \frac{4x^2 + 8x}{(x+1)^2(x-1)} dx &= \int \left( \frac{1}{x+1} + \frac{2}{(x+1)^2} + \frac{3}{x-1} \right) dx \\ &= \ln|x+1| - \frac{2}{x+1} + 3 \ln|x-1| + C. \end{aligned}$$

**Example 135** Evaluate

$$\int \frac{3x^2 + 5x + 2}{x(x^2 + 2x + 2)} dx$$

**Solution:** Let

$$\frac{3x^2 + 5x + 2}{x(x^2 + 2x + 2)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2x + 2}, \Rightarrow A = 2, B = 1, C = 1.$$

Hence

$$\begin{aligned} \int \frac{3x^2 + 5x + 2}{x(x^2 + 2x + 2)} dx &= \int \left( \frac{2}{x} + \frac{x+1}{x^2 + 2x + 2} \right) dx \\ &= 2 \ln|x| + \frac{1}{2} \ln(x^2 + 2x + 2) + C. \end{aligned}$$

## 7.5 Products of Trigonometric Functions

Method to solve  $\int \sin^m x \cos^n x dx$ .

- If  $m$  is odd, then let  $u = \cos x$ .
- If  $n$  is odd, then let  $u = \sin x$ .
- If  $m$  and  $n$  are even, then use half-angle formula.

**Example 136** Evaluate

$$\int \sin^2(x) \cos^3 x dx.$$

**Solution:** Use  $\cos^2(x) = 1 - \sin^2(x)$  and then the substitution  $u = \sin(x)$ ,  $du = \cos(x) dx$ :

$$\begin{aligned} \int \sin^2(x) \cos^3 x dx &= \int \sin^2(x)(1 - \sin^2(x)) \cos(x) dx \\ &= \int u^2(1 - u^2) du \\ &= \int u^2 - u^4 du \\ &= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c \\ &= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + c \end{aligned}$$

**Example 137** Evaluate

$$\int \sin^2(x) dx.$$

**Solution:** Use  $\sin^2 x = \frac{1 - \cos 2x}{2}$ .

$$\int \sin^2(x) dx = \int \frac{1 - \cos 2x}{2} dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

**Example 138** Evaluate

$$\int \sin^2 x \cos^4 x dx.$$

**Solution:**

$$\int \sin^2 x \cos^4 x dx = \int (\sin x \cos x)^2 \cos^2 x dx = \int \left(\frac{1}{2}\sin 2x\right) \frac{1}{2}(1 + \cos 2x) dx$$

$$\begin{aligned}
&= \frac{1}{4} \int (\sin 2x + \sin 2x \cos 2x) dx = \frac{1}{4} \left( \int \sin 2x dx + \int \frac{1}{2} \sin 4x dx \right) \\
&= -\frac{1}{8} \cos 2x - \frac{1}{16} \cos 4x + C.
\end{aligned}$$

## 7.6 Trigonometric Substitution

Trigonometric substitutions:

1.  $\sqrt{a^2 - x^2}$ : Let  $x = a \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ ;
2.  $\sqrt{a^2 + x^2}$ : Let  $x = a \tan \theta$ ,  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ;
3.  $\sqrt{x^2 - a^2}$ : Let  $x = a \sec \theta$ ,  $0 \leq \theta < \frac{\pi}{2}$  or  $\pi \leq \theta < \frac{3\pi}{2}$ .

**Example 139** Evaluate

$$\int \sqrt{4 - x^2} dx.$$

**Solution:** Let  $x = 2 \sin \theta$ . Then

$$\sqrt{4 - x^2} = 2 \cos \theta, \quad dx = 2 \cos \theta d\theta.$$

Hence

$$\begin{aligned}
\int \sqrt{4 - x^2} dx &= \int (2 \cos \theta)(2 \cos \theta d\theta) \\
&= 4 \int \cos^2 \theta d\theta.
\end{aligned}$$

Note that

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2},$$

hence

$$\begin{aligned}
\int \sqrt{4 - x^2} dx &= 4 \int \cos^2 \theta d\theta = 4 \int \frac{1 + \cos 2\theta}{2} d\theta = 2 \int (1 + \cos 2\theta) d\theta \\
&= 2\theta + \sin 2\theta + C \\
&= 2 \arcsin \frac{x}{2} + \sin \left( 2 \arcsin \frac{x}{2} \right) + C.
\end{aligned}$$

**Example 140** Evaluate

$$\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1-x^2}} dx.$$

**Solution:** Let  $x = \sin t$ . Then  $x = 0 \leftrightarrow t = 0$ ,  $x = 1/\sqrt{2} \leftrightarrow t = \pi/4$  and

$$\frac{x^2}{\sqrt{1-x^2}} = \frac{\sin^2 t}{\cos t}, \quad dx = \cos t dt.$$

Therefore

$$\begin{aligned} \int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1-x^2}} dx &= \int_0^{\pi/4} \frac{\sin^2 t}{\cos t} \cos t dt \\ &= \int_0^{\pi/4} \sin^2 t dt = \int_0^{\pi/4} \frac{1 - \cos 2t}{2} dt \\ &= \frac{1}{2} \left[ t - \frac{\sin 2t}{2} \right]_0^{\pi/4} = \boxed{\frac{\pi}{8} - \frac{1}{4}}. \end{aligned}$$

**Example 141** Evaluate

$$\int \frac{1}{\sqrt{x^2-4}} dx$$

**Solution:** Let  $x = 2 \sec t$ , then  $dx = 2 \sec t \tan t dt$ ,  $\sqrt{x^2-4} = 2 \sec t$ .

**Example 142** Evaluate

$$\int \frac{1}{\sqrt{4+x^2}} dx$$

**Solution:** Let  $x = 2 \tan t$ , then  $dx = 2 \sec t dt$ .

## 7.7 Improper Integrals

The following three are called Type I improper integrals:

$$\begin{aligned} \int_a^\infty f(x) dx &= \lim_{t \rightarrow \infty} \int_a^t f(x) dx, \\ \int_{-\infty}^b f(x) dx &= \lim_{t \rightarrow -\infty} \int_t^b f(x) dx, \\ \int_{-\infty}^\infty f(x) dx &= \int_c^\infty f(x) dx + \int_{-\infty}^c f(x) dx. \end{aligned}$$

**Definition:**

- If the limit is a finite number, we say that the integral is convergent;
- If the limit is  $\pm\infty$ , we say that the integral converges to  $\pm\infty$ ;
- If the limit does not exist at all, we say that the integral is divergent.

**Example 143** Determine the values of  $p$  for which the following improper integral

$$\int_1^{\infty} \frac{1}{x^p} dx$$

converges or diverges.

**Solution:**

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p} dx = \begin{cases} \lim_{t \rightarrow \infty} \frac{1}{1-p} x^{1-p} \Big|_1^t, & \text{if } p \neq 1; \\ \lim_{t \rightarrow \infty} \ln |x| \Big|_1^t, & \text{if } p = 1. \end{cases} \\ &= \begin{cases} \lim_{t \rightarrow \infty} \frac{1}{1-p} (t^{1-p} - 1), & \text{if } p \neq 1; \\ \lim_{t \rightarrow \infty} (\ln |t|), & \text{if } p = 1. \end{cases} = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1; \\ \infty, & \text{if } p \leq 1. \end{cases} \end{aligned}$$

**Example 144** Evaluate the following improper integral:  $\int_{-\infty}^1 \frac{1}{1+x^2} dx$

**Solution:**

$$\int_{-\infty}^1 \frac{1}{1+x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{1+x^2} dx = \lim_{t \rightarrow \infty} (\arctan t - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{3\pi}{4}.$$

**Example 145** Evaluate the following improper integral:  $\int_{-\infty}^{\infty} e^{-|x|} dx = 2$ .

**Solution:**

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-|x|} dx &= \int_{-\infty}^0 e^{-|x|} dx + \int_0^{\infty} e^{-|x|} dx = \int_{-\infty}^0 e^x dx + \int_0^{\infty} e^{-x} dx \\ &= \lim_{t \rightarrow -\infty} \int_t^0 e^x dx + \lim_{t \rightarrow -\infty} \int_0^t e^{-x} dx = \lim_{t \rightarrow -\infty} (1 - e^t) + \lim_{t \rightarrow -\infty} (e^{-t} - 1) = 2. \end{aligned}$$

**Example 146** Determine if the integral

$$I = \int_2^{\infty} x e^{-x} dx$$

is convergent or divergent and evaluate if it is convergent.

**Solution:**  $I = \lim_{t \rightarrow \infty} \int_2^t x e^{-x} dx$  (integration by parts: let  $u = x$  and  $dv = e^{-x} dx$ )

$$\begin{aligned}
&= \lim_{t \rightarrow \infty} \left( x(-e^{-x}) \Big|_2^t - \int_2^t -e^{-x} dx \right) \\
&= \lim_{t \rightarrow \infty} (-x e^{-x} - e^{-x}) \Big|_2^t \\
&= \lim_{t \rightarrow \infty} -(x+1)e^{-x} \Big|_2^t \\
&= \lim_{t \rightarrow \infty} (-(t+1)e^{-t} + 3e^{-2}) \\
&= 3e^{-2} \text{ (where } \lim_{t \rightarrow \infty} (t+1)e^{-t} = 0 \text{ by L'Hospital's Rule).}
\end{aligned}$$

## Chapter 8. Applications of the Integral

### 8.2 Finding The Area Between Two Curves

- Vertical slice area = (difference in the  $y$ -coordinates)·(width)  
= (difference in the  $y$ -coordinates)·( $dx$ ).
- Horizontal slice area = (difference in the  $x$ -coordinates)·(width)  
= (difference in the  $x$ -coordinates)·( $dy$ ).

**Example 147** *A region in the  $xy$ -plane has a vertical slice with coordinates  $(x, x^2e^x)$  and  $(x, e^{3x})$  with  $0 \leq x \leq 1$ . What is its area?*

**Solution:** With  $0 \leq x \leq 1$ ,  $x^2e^x \leq e^{3x}$ . Thus

$$\text{Vertical slice area} = (e^{3x} - x^2e^x) \cdot (dx).$$

**Theorem 10** *If  $f(x) \geq g(x)$  for  $a \leq x \leq b$ , then the area of the region bounded by*

$$y = f(x), y = g(x), x = a, x = b$$

*is*

$$\int_a^b (f(x) - g(x)) dx.$$

**Example 148** *Calculate the area of the region bounded by*

$$y = x^2 - 4x + 7, y = -x^2 + 4x + 1.$$

**Solution:**

Step 1. Find intersections: Let

$$(-x^2 + 4x + 1) = (x^2 - 4x + 7), \Rightarrow$$

$$x^2 - 4x + 3 = 0, \Rightarrow x = 1, x = 3.$$

Step 2. In (1, 3):  $-x^2 + 4x + 1 > x^2 - 4x + 7$ . Therefore

$$\begin{aligned} \text{area} &= \int_1^3 [(-x^2 + 4x + 1) - (x^2 - 4x + 7)]dx \\ &= \int_1^3 (-2x^2 + 8x - 6)dx = \left(-\frac{2}{3}x^3 + 4x^2 - 6x\right) \Big|_1^3 \\ &= \left(-\frac{2}{3}(3)^3 + 4(3)^2 - 6(3)\right) - \left(-\frac{2}{3}(1)^3 + 4(1)^2 - 6(1)\right) \\ &= 0 - \left(-\frac{8}{3}\right) = \frac{8}{3}. \end{aligned}$$

**Example 149** Calculate the area of the region bounded by

$$y = x^2 - 4x + 7, y = -x^2 + 4x + 1, x = 0, x = 2.$$

**Solution:** Step 1. Find intersections: Let  $(-x^2 + 4x + 1) = (x^2 - 4x + 7)$ ,  $\Rightarrow x^2 - 4x + 3 = 0$ ,  $\Rightarrow x = 1, x = 3$ .

Step 2. By using the intersections, the interval (0, 2) is divided into (0, 1) and (1, 2). In (0, 1):  $x^2 - 4x + 7 > -x^2 + 4x + 1$ ; In (1, 2):  $-x^2 + 4x + 1 > x^2 - 4x + 7$ . Therefore

$$\begin{aligned} \text{area} &= \int_0^1 [(x^2 - 4x + 7) - (-x^2 + 4x + 1)]dx + \int_1^2 [(-x^2 + 4x + 1) - (x^2 - 4x + 7)]dx \\ &= \int_0^1 (2x^2 - 8x + 6)dx + \int_1^2 (-2x^2 + 8x - 6)dx. \end{aligned}$$

**Example 150** Find the area of the region between  $y = x^{1/2}$  and  $y = x^{1/3}$  for  $0 \leq x \leq 1$ .

**Example 151** Find the area of the region bounded by the parabolas  $y = 2x - x^2$  and  $y = x^2$ .

**Solution:** Step 1. Find intersections:  $2x - x^2 = x^2 \Rightarrow x = 0, x = 1$ .

Step 2. For  $0 < x < 1$ ,  $2x - x^2 > x^2$ .

$$\text{area} = \int_0^1 [2x - x^2 - x^2]dx = 1 - \frac{2}{3} = \frac{1}{3}.$$

**Example 152** Calculate the area of the region bounded by

$$y = \sin x, y = \cos x, x = 0, x = \frac{\pi}{2}.$$

**Solution:**  $A = 2\sqrt{2} - 2$ .

**Example 153** Find the area of the region bounded by

$$y = e^x, y = 4e^{-x}, y = 1.$$

**Example 154** Calculate the area of the region bounded by

$$y = \sin x, y = \cos x, \quad \text{between } x = 0 \text{ and } x = \pi/2.$$

**Theorem 11** If the region is bounded by

$$x = f(y), x = g(y), y = c, y = d$$

where  $f(y) \geq g(y)$  for  $c \leq y \leq d$ , then the area of the region is

$$A = \int_c^d (f(y) - g(y)) dy.$$

**Example 155** Calculate the area of the region enclosed by

$$y = x - 1, y^2 = 2x + 6.$$

**Solution:**

$$A = \int_{-2}^4 [(y - 1) - (\frac{1}{2}y^2 - 3)] dy = 18.$$

## 8.3 The Volume of a Solid of revolution

**Case 1:** We rotate a region bounded by  $y = f(x)$ ,  $y = g(x)$ ,  $x = a$ ,  $x = b$  **around the line**  $y = k$  ( $k = 0 \iff$  **x-axis**), where  $f(x) \geq g(x) \geq 0$ . Then the cross section which is perpendicular to  $x$ -axis has the area

$$A(x) = \pi r_{out}^2 - \pi r_{in}^2,$$

where

$$r_{in} = g(x) - k, \quad r_{out} = f(x) - k, \quad dV = A(x)dx,$$

$$V = \int_a^b [\pi[f(x) - k]^2 - \pi[g(x) - k]^2] dx.$$

**Case 2:** We rotate a region bounded by  $x = f(y)$ ,  $x = g(y)$ ,  $y = c$ ,  $y = d$  **around the line**  $x = k$  ( $k = 0 \iff$  **y-axis**), where  $f(y) \geq g(y) \geq 0$ . Then the cross section which is perpendicular to  $y$ -axis has the area

$$A(y) = \pi r_{out}^2 - \pi r_{in}^2,$$

where

$$r_{in} = g(y) - k, \quad r_{out} = f(y) - k, \quad dV = A(y)dy,$$

$$V = \int_a^b [\pi[f(y) - k]^2 - \pi[g(y) - k]^2]dy.$$

**Example 156** Find the volume of the solid obtained by rotating about the  $x$ -axis the region under the curve  $y = \sqrt{x}$  from 0 to 1.

**Solution:**  $A(x) = \pi(\sqrt{x})^2$ .

**Example 157** Find the volume of the solid obtained when the region bounded by  $y = x^2$ ,  $y = 0$ ,  $x = 1$  and  $x = 2$  is rotated around the line  $y = -3$ .

**Solution:** The cross-sectional area at  $x$  is:

$$A(x) = \pi R^2 - \pi r^2, \quad R = x^2 - (-3) = x^2 + 3, r = 0 - (-3) = 3.$$

$$\Rightarrow A(x) = \pi(3 + x^2)^2 - \pi 3^2 = \pi(x^4 + 6x^2).$$

$$V = \int_1^2 A(x)dx = \int_1^2 \pi(x^4 + 6x^2)dx = 20.2\pi.$$

**Example 158** Find the volume of the solid obtained by rotating the region bounded by  $y = x^3$ ,  $y = 8$ , and  $x = 0$  about the  $y$ -axis.

**Solution:**  $A(y) = \pi x^2 = \pi y^{2/3}$ .

### Case 3: Volumes by Cylindrical Shells

(1) If we rotate a region bounded by  $y = f(x)$ ,  $y = 0$ ,  $x = a$ ,  $x = b$  about the  $y$ -axis, then small shell has the base area  $\pi(x + dx)^2 - \pi x^2$  and the height  $f(x)$ . Thus

$$dV = \{\pi(x + dx)^2 - \pi x^2\}f(x) \approx 2\pi x f(x).$$

$$V = 2\pi \int_a^b x f(x)dx.$$

(2) If we rotate a region bounded by  $y = f(x)$ ,  $y = g(x)$ ,  $x = a$ ,  $x = b$  (where  $f(x) \geq g(x)$  for  $x \in (a, b)$ ) about the  $y$ -axis, then

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

(3) If we rotate a region bounded by  $x = f(y)$ ,  $x = g(y)$ ,  $y = c$ ,  $y = d$  (where  $f(y) \geq g(y)$  for  $y \in (c, d)$ ) about the  $x$ -axis, then

$$V = 2\pi \int_c^d y[f(y) - g(y)]dy.$$

**Example 159** Find the volume of the solid by rotating the region enclosed by  $y = 4x^2 - x^3$  and  $x$ -axis about  $y$ -axis.

**Solution:** If we use Case 2 method, then we have difficulty to find  $r_{in}$  and  $r_{out}$  through solving  $x$  from the function. By cylindrical shell method,

$$V = \int_0^4 2\pi x(4x^2 - x^3)dx.$$

**Example 160** Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by  $y = x^3$ ,  $y = 8$ , and  $x = 0$  about the  $y$ -axis.

**Solution:** When  $y = 8$ ,  $x = 2$ . Thus

$$V = 2\pi \int_0^2 x[8 - x^3]dx = 19.2\pi.$$

**Example 161** Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by  $y = \sqrt[3]{x}$ ,  $y = 0$ , and  $x = 8$  about the  $x$ -axis.

**Solution:** When  $x = 8$ ,  $y = 2$ . From  $y = \sqrt[3]{x}$  we have  $x = y^3$ . Thus

$$V = 2\pi \int_0^2 y[8 - y^3]dy = 19.2\pi.$$

**Example 162** Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by  $y = x - x^2$  and  $y = 0$  about the line  $x = 4$ .

**Solution:** The intersection is  $x = 0, 1$ . The radius of the cylindrical shell at  $x$  is  $4 - x$ , the circumference is  $2\pi(4 - x)$ , and the height is  $y = x - x^2$ . Thus

$$V = \int_0^1 2\pi(4 - x)(x - x^2)dx.$$

**Example 163** Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by  $y = 3x - x^2$  and  $y = 2x$  about the line  $x = 4$ .

**Solution:** The intersection is  $x = 0, 1$ . The radius of the cylindrical shell at  $x$  is  $4 - x$ , the circumference is  $2\pi(4 - x)$ , and the height is  $y = x - x^2$ . Thus

$$V = \int_0^1 2\pi(4 - x)(x - x^2)dx.$$