MA 235

Extra Problems for Chapter 1

1. Let $W = \{1, 2\}$. Construct a relation B on W that is neither reflexive nor irreflexive. (Express the relation B using a table with two rows and two columns.)

2. Prove that, if a relation B on a set W is asymmetric, then it is also irreflexive. [Hint: For a proof by contradiction, suppose that xBx and set $y = x$ in the definition of *asymmetric*.]

3. Let $W = \{1, 2\}$. Construct (in tabular form) relations B_1, B_2, and B_3 on W so that B_1 is reflexive and symmetric, B_2 is irreflexive and symmetric, and B_3 is symmetric, but neither reflexive nor irreflexive. Your answers should be relations that are neither empty (i.e., contain no pair $(x, y) \in W \times W$) nor exhaustive (contain every pair $(x, y) \in W \times W$) .

4. Prove that, if a relation B on a set W is complete, then it is also reflexive.

5. Let $W = \{1, 2, ..., 10\}$ and define the relation M ("is a multiple of" on W by $x \mid y$ (pronounced "x divides y") if and only if there exists an integer k so that $y = kx$. For example, $2\ M\ 6$ because $6 = 3 \times 2$ but $\neg\ (3\ M\ 7)$ because there is no integer k so that $7 = k3$. Show that
   (a) M is a reflexive relation.
   (b) M is an antisymmetric relation.
   (c) M is not a complete relation. (Hint: Find a counterexample!)
   (d) Is M a transitive relation? Either prove that it is, or find a counterexample.
   (e) Of the four special orderings in *Definition* 1.3, which, if any, applies to M?

6. Let $W = \{a, b, c, d\}$ and suppose that a DM's preference on W is given by $b \sim d \succ a \succ c$. Suppose that the DM has a von Neumann-Morgenstern utility function $u: W \to \mathbb{R}$, and that $u(a) = 10$ and $u(b) = 20$. What can you deduce about $u(c)$ and $u(d)$?

7. In a static non-stochastic decision problem, a DM must choose an action from the set $A = \{1, 2, 3, 4, 5\}$. The possible outcomes are as in #6, and the link between actions and outcomes is given by the function $f: A \to W$ defined by $f(1) = c$, $f(2) = a$, $f(3) = d$, $f(4) = d$, and $f(5) = b$. Find *all* optimal actions for the DM.

8. With reference to the outcomes of #6, consider the lotteries

$$L_1 = \langle a, 0.2; b, 0.3; c, 0.4, d, 0.1\rangle,$$
$$L_2 = \langle a, 0.6; b, 0; c, 0.4, d, 0\rangle,$$
$$L_3 = \langle a, 0; b, 0.6; c, 0.4, d, 0\rangle.$$

   For $i = 1$, 2, and 3, calculate the expected utility, $E[u(L_i)]$. Use any information you deduced about $u(c)$ and $u(d)$ in #6. Determine, if possible, the DM's preference relation among the lotteries. Which one does the DM prefer?

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9. You are going to win a prize: \$20, \$200, or a set of cookware. You are risk-neutral with respect to money, so your utility for a gain of \$$x$ is $u(\$x) = x$. The specific prize you win will be determined by lottery, and you may choose the lottery. The possibilities are

$$L_1 = \langle \$20, 0.4; \textit{cookware}, 0.2; \$200, 0.4 \rangle,$$
$$L_2 = \langle \$20, 0.3; \textit{cookware}, 0.4; \$200, 0.3 \rangle.$$

You feel indifferent between L_1 and L_2. You are therefore indifferent between the cookware and what amount of money?

10. You have won a prize, which will be either a car, a bicycle, or a pair of roller blades. Your preference is Car $\succ$ Bicycle $\succ$ Roller Blades. Moreover, you feel indifferent between the Bicycle (for certain) and the lottery $L = \langle \textit{car}, 0.2; \textit{roller blades}, 0.8 \rangle$. Assuming that all appropriate axioms are satisfied, find a von Neumann-Morgenstern utility for your preferences. Express the utility in 0-1 normalization.

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