

Solution to Midterm Test 2 (version A)

MAT1341E, Fall 2018

Total = 15 marks

Part I. Multiple Choice Questions ($4 \times 4 = 16$ marks)

Answers: ECEED

1. Let $U = \{(x, y, z) \mid x, y, z \in \mathbb{R}, xy = 0\}$. Then

- (A) The zero vector $\mathbf{0} = (0, 0, 0)$ is in U , and it is closed under addition.
- (B) The zero vector $\mathbf{0} = (0, 0, 0)$ is in U , but it is not closed under scalar multiplication.
- (C) U is closed under addition and scalar multiplication.
- (D) U is closed under addition, but it is not closed under scalar multiplication.
- (E) U is not closed under addition, but it is closed under scalar multiplication.
- (F) None of the above is true.

Solution. (E) The vector $(x, y, z) = (0, 0, 0)$ satisfies the condition $xy = 0$. The zero vector is in U .

If $\mathbf{u} = (x, y, z) \in U$, then $xy = 0$. For any scalar c , $c\mathbf{u} = (cx, cy, cz)$. Since $(cx)(cy) = c^2xy = 0$, $c\mathbf{u}$ is in U . U is closed under scalar multiplication.

U is not closed under addition. For example, let $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 1, 0)$. Obviously, $\mathbf{u}, \mathbf{v} \in U$. However, $\mathbf{u} + \mathbf{v} = (1, 1, 0)$ is not in U .

2. Which of the following are subspaces of \mathbb{R}^3 ?

$$U = \{(x, y, z) \mid 2x - 3y + z = 0, x, y \in \mathbb{R}\};$$

$$V = \{(x, y, z) \mid (x - z)(x - y) = 0, x, y, z \in \mathbb{R}\};$$

$$W = \{(x + y, 2y, x - y) \mid x, y \in \mathbb{R}\};$$

$$X = \{(x, xy, y) \mid x, y \in \mathbb{R}\};$$

- (A) U and V only;
- (B) U and X only;
- (C) U and W only;
- (D) V and W only;
- (E) V and X only;
- (F) W and X only.

Solution. (C) V is not closed under addition: $\mathbf{u} = (1, 2, 1)$ and $\mathbf{v} = (1, 1, 2) \in W$, but $\mathbf{u} + \mathbf{v} = (2, 3, 3)$ is not in V . X is not closed under scalar multiplication: $\mathbf{u} = (1, 2, 2)$ is in X , but $2\mathbf{u} = (2, 4, 4)$ is not in X .

3. Let $M_{2 \times 2}$ be the vector space of all 2×2 matrices with real entries. Which one of the following is a subspace of $M_{2 \times 2}$?

(A) $U = \left\{ \begin{bmatrix} a & 2 \\ b & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$.

(B) $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ac = 2, a, b, c, d \in \mathbb{R} \right\}$.

(C) $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, \text{ and } c \text{ are integers} \right\}$.

(D) $X = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ab = 0, a, b, c, d \in \mathbb{R} \right\}$.

(E) $Y = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid c + d = 0, a, b, c, d \in \mathbb{R} \right\}$.

(F) None of the above.

Solution. (E). U and V do not contain the zero matrix.

W is not closed under scalar multiplication: $\mathbf{u} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is in W , but $0.5\mathbf{u} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0 \end{bmatrix}$ is not in W .

X is not closed under addition: $\mathbf{u} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ are in X , but $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not in X .

4. Let U be a subspace of \mathbb{R}^{10} . Suppose U has a spanning set with 9 vectors and it also has a linearly independent set with 7 vectors. Which one of the following statements is always true?

- (A) $\dim U < 7$; (B) $\dim U > 7$; (C) $\dim U < 9$;
(D) $\dim U > 9$; (E) $\dim U \leq 9$; (F) $\dim U \leq 7$.

Solution. (E) A spanning set of 9 vectors contains a basis of U . A basis of U has at most 9 vectors.

5. Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be non-zero vectors in a vector space V such that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, and $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Which one of the following statements is always true?

- (A) $\{\mathbf{u}, \mathbf{w}\}$ is linearly dependent.
(B) $\{\mathbf{v}, \mathbf{w}\}$ is linearly dependent.
(C) $\mathbf{v} \in \text{span}\{\mathbf{u}, \mathbf{w}\}$.
(D) $\mathbf{w} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$.
(E) $\mathbf{u} \in \text{span}\{\mathbf{v}, \mathbf{w}\}$.
(F) None of the above.

Solution. (D). If \mathbf{w} is not in $\text{span}\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent. Hence $\mathbf{w} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$. (A) and (E) are false if $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$, and $\mathbf{w} = (0, 2)$. (B) and (C) are false if $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$, and $\mathbf{w} = (2, 0)$.

Part II. True / False Question (6 marks)

6. Determine whether each of the following statements is always true or possibly false. **Justify your answer.**

- (a) If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a linearly dependent set of a vector space V , then $\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is also linearly dependent.

Solution. The statement is false. Let $\mathbf{u}_1 = (0, 0, 0)$, $\mathbf{u}_2 = (1, 0, 0)$, $\mathbf{u}_3 = (0, 1, 0)$, and $\mathbf{u}_4 = (0, 0, 1)$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent, but $\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent.

(b) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3,$ and $\mathbf{v}_4,$ be non-zero vectors in a vector space $V,$ and $U = \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \}.$ Then $\dim U = 4.$

Solution. The statement is false. For example, $\mathbf{v}_1 = (1, 1), \mathbf{v}_2 = (2, 2), \mathbf{v}_3 = (3, 3),$ and $\mathbf{v}_4 = (4, 4),$ then $\dim (\text{span}\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \}) = 1.$

(c) If U and W are subspaces of $\mathbb{R}^3,$ and $V = U \cap W = \{ \mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} \in U \text{ and } \mathbf{v} \in W \}$ is closed under scalar multiplication. I.e., if $\mathbf{u} \in V,$ and c is a scalar, then $c\mathbf{u} \in V.$

Solution. This statement is true.

Because U and W are closed under scalar multiplication, $c\mathbf{u}$ in both U and $W,$ i.e., $c\mathbf{u} \in V.$

(d) Let $U = \text{span} \{ x^2 \sin^2(2x), x^2 \cos^2(2x), x^2 \}$ be a subspace of $\mathbf{F}(\mathbb{R}),$ the vector space of all one-variable functions defined for all real numbers. Then $\dim U = 3.$

Solution. This is False. Because $x^2 \sin^2(2x) + x^2 \cos^2(2x) = x^2,$ the set $\{ x^2 \sin^2(2x), x^2 \cos^2(2x), x^2 \}$ is linearly dependent, and $\dim U < 3.$

Part III. Detailed-Answer Questions (4 marks)

7. Consider the vector space $\mathbf{P}_2 = \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \}$ of polynomials with degree at most 2. Let $U = \{ f \in \mathbf{P}_2 \mid f(-1) = 0 \}.$

(a) Use the subspace test to show that U is a subspace of $\mathbf{P}_2.$ (**Pay attention to your notation!**)

Solution. The zero vector in \mathbf{P}_2 is the zero function: $f_0(x) = 0$ for all $x.$ Since $f_0(-1) = 0, f_0$ is in $U.$

If $f \in U,$ then $f(-1) = 0.$ Let c be any scalar. Then $(cf)(-1) = c(f(-1)) = c \times 0 = 0.$ Hence, $cf \in U.$

If $f_1, f_2 \in U,$ then $f_1(-1) = f_2(-1) = 0.$ $(f_1 + f_2)(-1) = f_1(-1) + f_2(-1) = 0 + 0 = 0.$ Hence $f_1 + f_2 \in U.$

U contains the zero function, and it is closed under addition and scalar multiplication, it is a subspace of $\mathbf{P}_2.$

(b) Show that $U = \text{span} \{ 1 + x, x + x^2 \}.$

Solution. Let $f(x) = ax^2 + bx + c \in U.$ Then $f(-1) = a - b + c = 0, b = a + c.$ Hence,

$$f(x) = a + bx + cx^2 = a + (a + c)x + cx^2 = a(1 + x) - c(x + x^2) \in \text{span} \{1 + x, x + x^2\}.$$

On the other hand, if $f(x) \in \text{span} \{1 + x, x + x^2\}$, then $f(x) = s(1 + x) + t(x + x^2)$, and $f(-1) = s(1 - 1) + t(1 - 1^2) = 0$. Hence, $f \in U$.

Therefore, $U = \text{span} \{1 + x, x + x^2\}$.

(c) Find a basis of U and explain why this is a basis.

Solution. $B = \{1 + x, x + x^2\}$ is a basis of U . We have shown that $\text{span } B = U$. We can also show that B is linearly independent. Indeed, if $a(1 + x) + b(x + x^2) = a + (b + a)x + bx^2 = 0$, then we must have $a = b = 0$. Then B is linearly independent, and it is a basis.