2. Function Review

Lec 1 mini review.

functions:	independent variable, dependent variable, domain, range		
graphs:	vertical line test, symmetry (even/odd), periodicity, transformations intervals of increase/decrease		
linear functions:	slope, intercepts		
polynomials:	any degree $n \geq 0$ (constant, linear, quadratic, cubic,...), coefficients		
power functions:	$f(x) = x^n$	**root functions**:	$f(x) = x^{1/n} = \sqrt[n]{x}$
rational functions	**algebraic functions**	**absolute value**:	$f(x) = \|x\|$

Composition

Let f and g be functions. If all numbers in the range of g are in the domain of f, then the **composition** $f \circ g$ is a function defined by $(f \circ g)(x) = f(g(x))$

Example 2.1. Find the composition $f \circ g$, where $f(x) = \dfrac{x-1}{x+1}$ and $g(x) = \dfrac{1}{\sqrt{x}}$

$$f \circ g(x) = f(g(x)) = f\left(\frac{1}{\sqrt{x}}\right) = \frac{\left(\frac{1}{\sqrt{x}}\right) - 1}{\left(\frac{1}{\sqrt{x}}\right) + 1} = \frac{\frac{1}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}}}{\frac{1}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}}} = \frac{\frac{1-\sqrt{x}}{\sqrt{x}}}{\frac{1+\sqrt{x}}{\sqrt{x}}} = \frac{(1-\sqrt{x})\cancel{(\sqrt{x})}}{(1+\sqrt{x})\cancel{(\sqrt{x})}} = \frac{1-\sqrt{x}}{1+\sqrt{x}}$$

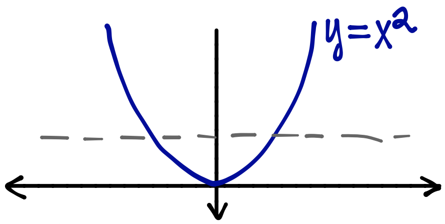
cancels because $x \neq 0$

$$\boxed{\therefore f \circ g(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Inverse

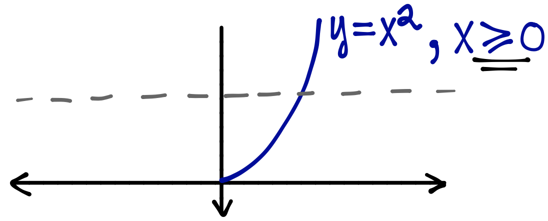
Horizontal Line Test: Let $y = f(x)$ be a function. If every horizontal line crosses the graph of f at most once, then $f(x)$ is a **one-to-one (injective)** function and f has an inverse.

Ex.



fails H.L.T. $\Rightarrow$ not one-to-one

Ex.



passes H.L.T. $\Rightarrow$ it's one-to-one!

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Inverse: Let $y = f(x)$ be a function. If f passes the Horizontal Line Test, then the map f^{-1} defined by the rule

$$f^{-1}(y) = x \iff f(x) = y$$

is a function called the **inverse** of f. Informally, f^{-1} undoes f

Composition of a function with its inverse: $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$

Example 2.2. Find the inverse of $g(x) = \dfrac{2x-1}{3x+2}$ and verify that $(g \circ g^{-1})(x) = x = (g^{-1} \circ g)(x)$.

1. Write $y = g(x)$: $y = \dfrac{2x-1}{3x+2}$

2. Interchange x's and y's: $x = \dfrac{2y-1}{3y+2}$

3. Isolate 'new' y: $x(3y+2) = (2y-1)$

$\Rightarrow 3xy + 2x = 2y - 1$

$\Rightarrow 3xy - 2y = -2x - 1$

$\Rightarrow y(3x-2) = -2x-1$

$\Rightarrow y = \dfrac{-2x-1}{3x-2}$

$$\boxed{\therefore\ g^{-1}(x) = \frac{-2x-1}{3x-2}}$$

$$\begin{aligned}
(g \circ g^{-1})(x) &= g(g^{-1}(x)) \\
&= g\left(\frac{-2x-1}{3x-2}\right) \\
&= \frac{2\left(\frac{-2x-1}{3x-2}\right) - 1}{3\left(\frac{-2x-1}{3x-2}\right) + 2} \\
&= \frac{\frac{-4x-2}{3x-2} - \frac{(3x-2)}{(3x-2)}}{\frac{-6x-3}{3x-2} + \frac{2(3x-2)}{3x-2}} \\
&= \frac{\frac{-7x}{\cancel{3x-2}}}{\frac{-7}{\cancel{3x-2}}} \\
&= \frac{\cancel{-7}x}{\cancel{-7}} \\
&= x \ \checkmark
\end{aligned}$$

similarly, we can also verify that $(g^{-1} \circ g)(x) = x$.

Exercise 2.3. Use the table to evaluate each of the following expressions.

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	6	5
$g(x)$	5	3	2	6	2	3

a. $f^{-1}(1)$ $= 2$

b. $(g \circ f)(3)$ $= g(f(3)) = g(4) = 6$

c. $(f \circ g)(6)$ $= f(g(6)) = f(3) = 4$

d. $(f \circ f^{-1})(4)$ $= f(3) = 4$

e. $g(g(1))$ $= g(5) = 2$

f. $(g \circ f)(1)$ $= g(f(1)) = g(3) = 2$

g. $(f^{-1} \circ f^{-1})(4)$ $= f^{-1}(f^{-1}(4)) = f^{-1}(3) = 1$

h. $(f^{-1} \circ f)(6)$ $= 6$

i. $(g \circ f^{-1})(1)$ $= g(f^{-1}(1)) = g(2) = 3$

Example 2.4. Find the inverse of $f(x) = \sqrt{x-2}$ and sketch the graphs of f and f^{-1}.

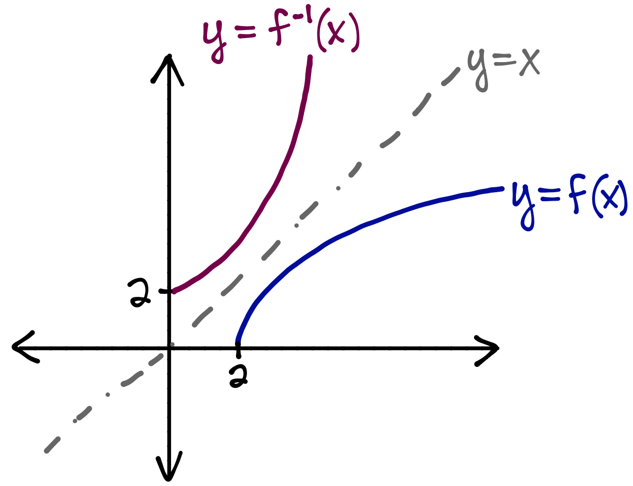
1. $y = \sqrt{x-2}$
2. $x = \sqrt{y-2}$ $(\therefore x \geq 0)$
3. $x^2 = y - 2$

$\Rightarrow y = x^2 + 2$

$$\boxed{\therefore f^{-1}(x) = x^2 + 2 \quad \text{with } x \geq 0.}$$

since $f(x) = y \Leftrightarrow f^{-1}(y) = x$,

[(x,y) is a point on the graph of f] $\Leftrightarrow$ [(y,x) is a point on the graph of f^{-1}]



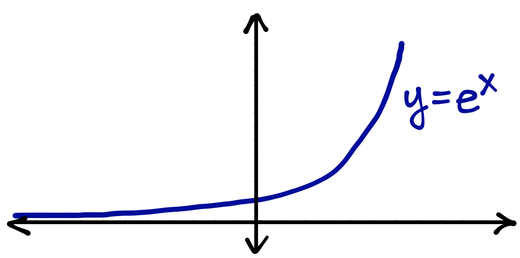
$\therefore$ f^{-1} is a reflection of f about the line $y=x$

Catalogue of Important Functions: Exponential & Logarithmic

Exponential Functions: $f(x) = a^x$ where $a > 0$ is a positive constant, $a \neq 1$

a is called the <u>base</u>

Natural Base: $f(x) = e^x$ $\quad e \approx 2.718281\ldots$



domain $(-\infty, \infty)$

range $(0, \infty)$

Laws of Exponents:

$a^x a^y = a^{x+y}$ $\qquad a^1 = a$ $\qquad (ab)^x = a^x b^x$

$(a^x)^y = a^{xy}$ $\qquad a^0 = 1$ $\qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

$a^{-x} = \frac{1}{a^x}$ $\qquad \frac{a^x}{a^y} = a^{x-y}$

Example 2.5. Solve for x in the equation $2^{x+3} = 16^{2x-1}$.

$2^{x+3} = 16^{2x-1}$

$\Rightarrow 2^{x+3} = (2^4)^{2x-1}$

$\Rightarrow 2^{x+3} = 2^{8x-4}$

$\Rightarrow x + 3 = 8x - 4$

$\Rightarrow 7 = 7x$

$\Rightarrow x = 1$

Check:

$LS = 2^{1+3} = 16$

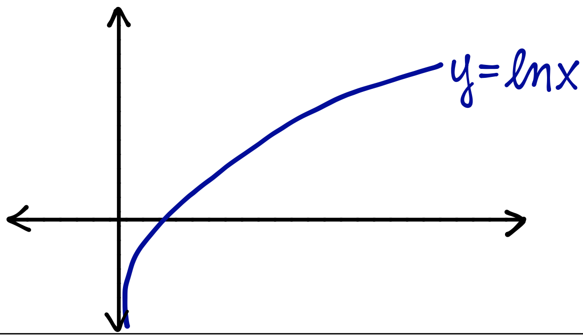
$RS = 16^{2(1)-1} = 16$

$\therefore LS = RS$ ✓

Logarithmic Functions: $f(x) = \log_a(x)$ base $a > 0$, $a \neq 1$, domain: $(0, \infty)$

$$\boxed{\log_a(x) = y \iff a^y = x \quad (x > 0)}$$

Natural Logarithm: $f(x) = \ln(x)$ ← base e gets its own notation: $\log_e(x) = \ln x$



domain: $\{x \in \mathbb{R} : x > 0\}$

range: $\mathbb{R}$

Laws of Logs:

$\ln(xy) = \ln(x) + \ln(y)$

$\ln(x^p) = p\ln(x)$

$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

$\ln(e) = 1$

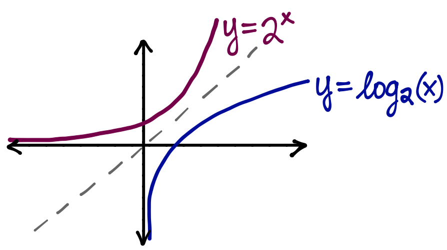
$\ln(1) = 0$

Change of base:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Inverse Relationship between a^x and $\log_a(x)$: $\boxed{\log_a(x) = y \iff a^y = x \quad (x > 0)}$

$\Longrightarrow$ $y = \log_a(x)$ is the inverse of $y = a^x$ and vice versa



composition of inverses

$\boxed{\log_a(a^y) = y}$ $\boxed{\ln(e^y) = y}$

$\boxed{a^{\log_a(x)} = x}$ $\boxed{e^{\ln x} = x}$

Example 2.6. Solve for x in the equation $\log(x+1) + \log(x+4) = 1$.

First, observe that $x > -1$ and $x > -4$ $\therefore$ $x > -1$

in order for the equation to make sense, we must respect the domains of both $\log(x+1)$ and $\log(x+4)$

$$\log(x+1) + \log(x+4) = 1$$

$$\Longrightarrow \log((x+1)(x+4)) = 1$$

$$\Longrightarrow \log(x^2+5x+4) = 1$$

$$\Longrightarrow 10^{\log(x^2+5x+4)} = 10^1$$

$$\Longrightarrow x^2+5x+4 = 10$$

$$\Longrightarrow x^2+5x-6 = 0$$

$$\Longrightarrow (x+6)(x-1) = 0$$

$$\Longrightarrow \cancel{\boxed{x=-6}} \text{ or } \boxed{x=1} \checkmark$$

reject because x has to be > -1

$\therefore$ the equation has one solution: $x = 1$

Example 2.7. A bacterial population grows according to the model $b(t) = 1.8^t b_0$ where t represents time in hours, $b(t)$ represents the number of bacteria in the population at time t, and b_0 represents the initial population at time $t = 0$ (assume $b_0 > 0$).

How long will it take for the initial population to triple in size?

At what time t is $b(t) = 3b_0$?

$$3b_0 = 1.8^t b_0$$

$$\Longrightarrow 3 = 1.8^t \quad (\text{since } b_0 \neq 0)$$

$$\Longrightarrow \ln(3) = \ln(1.8^t)$$

$$\Longrightarrow \ln(3) = t \cdot \ln(1.8)$$

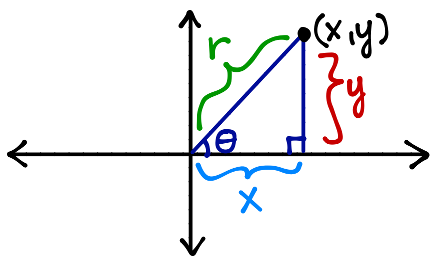
$$\Longrightarrow t = \frac{\ln(3)}{\ln(1.8)} \approx 1.869\ldots$$

$\therefore$ it will take

$$t = \frac{\ln 3}{\ln 1.8} \approx 1.869 \text{ hours}$$

for the initial population to triple.

Trigonometric Ratios:



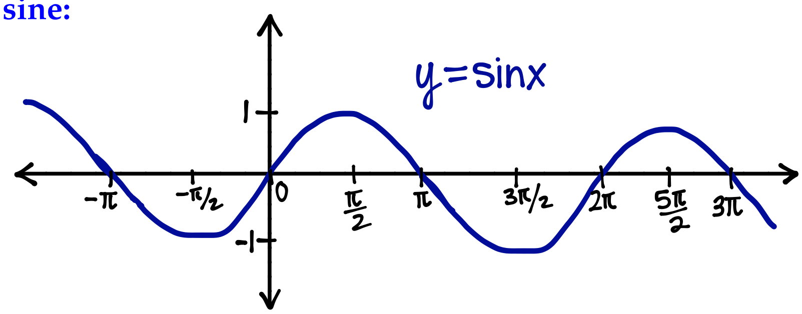
$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{y}{r} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{x}{r} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$$

Basic Trigonometric Functions

sine:



domain: $(-\infty, \infty)$

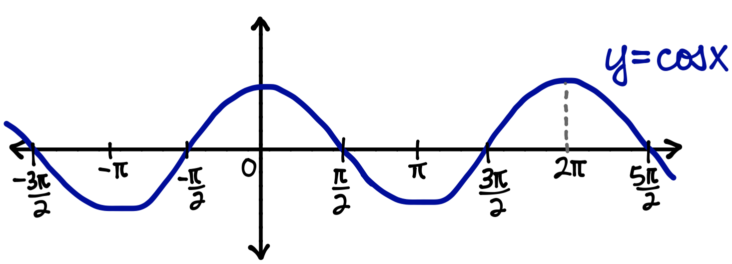
range: $[-1, 1]$

period: 2π

roots: at $x = k\pi$, $k \in \mathbb{Z}$

(K is an integer)

cosine:



domain: $(-\infty, \infty)$

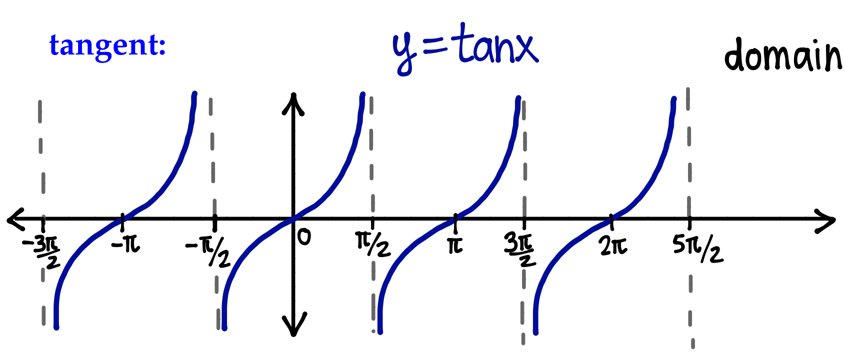
range: $[-1, 1]$

period: 2π

roots: at $x = (2k+1)\cdot\frac{\pi}{2}$, $k \in \mathbb{Z}$

(odd multiples of $\frac{\pi}{2}$)

tangent:



domain: $\{x \in \mathbb{R} : x \neq (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$

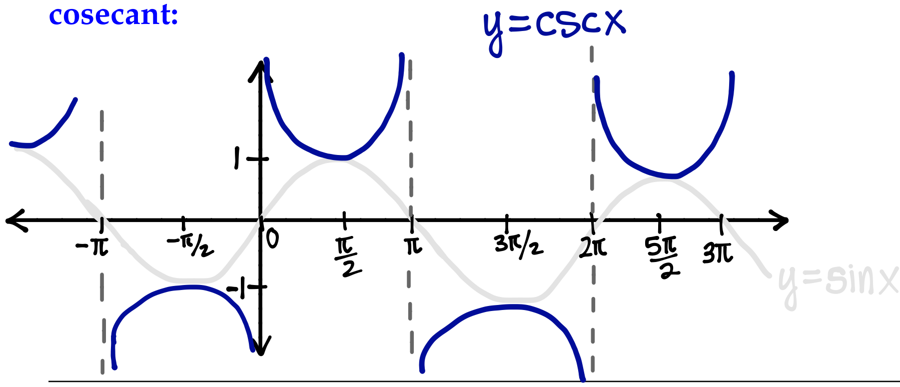
(Vertical asymptotes at all odd integer multiples of $\frac{\pi}{2}$)

range: $(-\infty, \infty)$ period: π

roots: at $x = k\pi$, $k \in \mathbb{Z}$

Reciprocal Trig Functions

cosecant:



domain: $\{x \in \mathbb{R} : x \neq k\pi,\ k \in \mathbb{Z}\}$

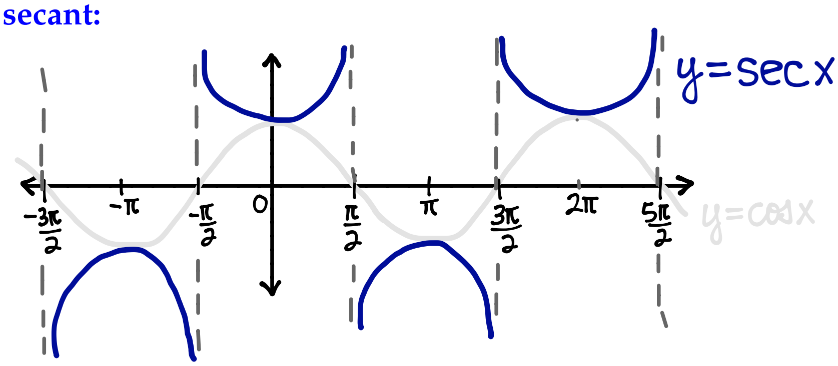
↳ all real #s except the roots of sinx

range: $(-\infty, -1] \cup [1, \infty)$

period: 2π

cscx has no roots

secant:



domain: $\{x \in \mathbb{R} : x \neq (2k+1)\frac{\pi}{2},\ k \in \mathbb{Z}\}$

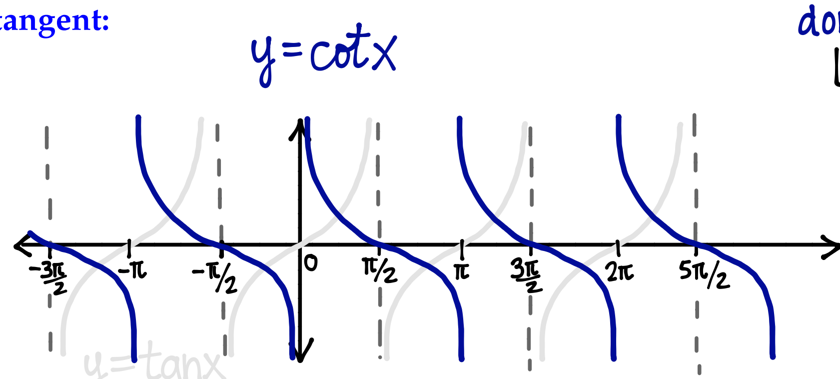
↳ all real #s except the roots of cosx

range: $(-\infty, -1] \cup [1, \infty)$

period: 2π

secx has no roots

cotangent:



domain: $\{x \in \mathbb{R} : x \neq k\pi,\ k \in \mathbb{Z}\}$

↳ all real #s except the roots of sinx

range: $(-\infty, \infty)$

period: π

roots at $x = (2k+1)\frac{\pi}{2},\ k \in \mathbb{Z}$

(at roots of cosx)

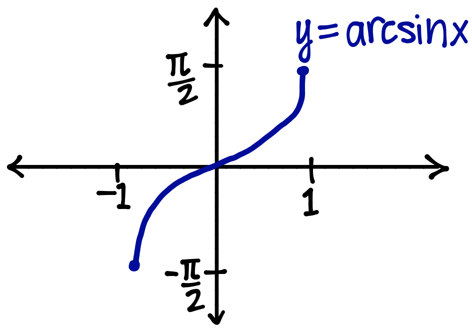
Useful Trig Identities

1. $\boxed{\sin^2\theta + \cos^2\theta = 1}$
2. $\boxed{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$
3. $\boxed{\tan^2\theta + 1 = \sec^2\theta}$
4. $\boxed{1 + \cot^2\theta = \csc^2\theta}$
5. $\sin(x+y) = \sin x \cos y + \cos x \sin y$
6. $\cos(x+y) = \cos x \cos y - \sin x \sin y$
7. $\sin(2x) = 2\sin x \cos x$
8. $\cos(2x) = \cos^2 x - \sin^2 x$

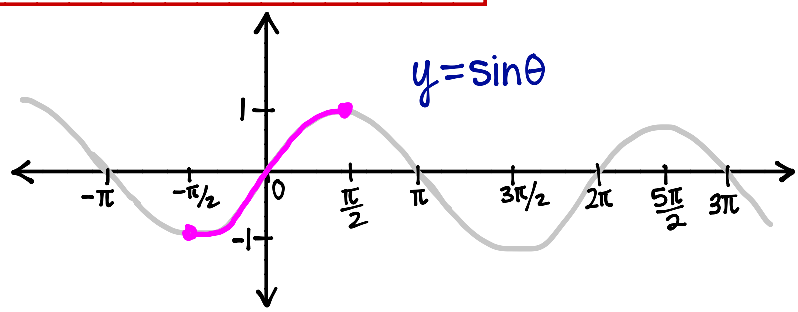
⋮

Inverse Trig Functions

arcsine: $\boxed{\arcsin x = \theta \iff \sin\theta = x \text{ and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$



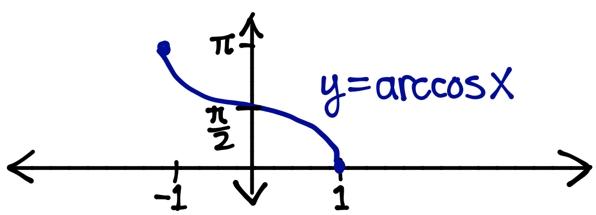
domain: $[-1,1]$ range: $[-\pi/2, \pi/2]$



$y=\sin\theta$ is not one-to-one (fails H.L.T.)

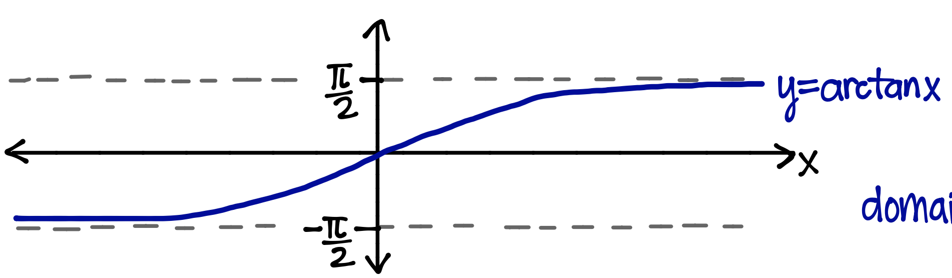
To fix this issue, we restrict to a representative one-to-one chunk of $y=\sin\theta$

arccosine: $\boxed{\arccos x = \theta \iff \cos\theta = x \text{ and } 0 \leq \theta \leq \pi}$ ← same idea as with sinθ and arcsinx



domain: $[-1,1]$ range: $[0,\pi]$

arctangent: $\boxed{\arctan x = \theta \iff \tan\theta = x \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}}$ ← same idea as with sinθ and arcsinx



domain: $(-\infty,\infty)$ range: $(-\frac{\pi}{2},\frac{\pi}{2})$

Study Guide

Stewart, 8th ed.	§1.2 pg. 33 # 5, 6 §1.3 pg. 43 # 13, 19, 23, 29b, 33, 35, 37, 38, 43–47, 52, 53, 55 §1.4 pg. 53 # 1–3, 11–15, 17, 19–21, 30–32, 37 §1.5 pg. 66 # 3–13, 15, 17, 21–25, 29, 35–41, 47, 49–53, 55, 63–71 App. D pg. A32 # 1, 3, 7, 11, 23, 29, 65, 67, 69, 81