[**3**] **1.** Let $f(x) = \sqrt{x^2+3}$. Estimate $f(1.02)$, using the linearization $L(x)$ of the function $f(x)$ at the point $a = 1$.

solution: We first find $f'(x) = \dfrac{x}{\sqrt{x^2+3}}$.

Then we want $f(1) = \sqrt{1^2+3} = 2$ and $f'(1) = \frac{1}{\sqrt{1^2+3}} = \frac{1}{2}$.

Finally $L(x) = f(a) + f'(a)(x-a) = 2 + \frac{1}{2}(x-1)$, so $L(1.02) = 2 + \frac{1}{2}(1.02-1) = 2.01$.

[**2**] **2.** Give the Riemann sum using $n = 4$ rectangles for $\displaystyle\int_1^3 e^{-x}\,\mathrm{d}x$. Use right-hand heights for the rectangles. You should give an explicit formula, but you do not need to evaluate it numerically. You may use sigma-notation if you wish.

solution: We have $\Delta x = (b-a)/4 = 1/2$. The i-th point is $a + i\Delta x = 1 + i/2$.

$$\frac{1}{2}\sum_{i=1}^{n} e^{-(1+i/2)} = \frac{1}{2}\sum_{i=1}^{n} e^{-1-i/2}$$

Alternatively, just write the whole thing out.

$$\frac{1}{2}\left(e^{-(1.5)} + e^{-(2)} + e^{-(2.5)} + e^{-(3)}\right)$$
$$=\frac{1}{2}\left(e^{-1.5} + e^{-2} + e^{-2.5} + e^{-3}\right)$$

[**2**] **3.** A particle moves along a straight line, with velocity $v(t) = 2t - 3$.

Find the net displacement between $t = 1$ and $t = 3$.

solution: The net displacement is

$$\begin{aligned}\int_1^3 v(t)\,\mathrm{d}t &= \int_1^3 2t - 3\,\mathrm{d}t \\ &= t^2 - 3t\,\Big|_1^3 \\ &= (0) - (-2) \\ &= 2\end{aligned}$$

[**2**] **4.** Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{x^2}^{x} \cos(t)\,\mathrm{d}t$.

solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{x} \cos(t)\,\mathrm{d}t &= \cos(x) - \cos(x^2)(x^2)' \\ &= \cos(x) - \cos(x^2)2x\end{aligned}$$

[**9**] **5.** Evaluate each of the following definite integrals.

a) $\displaystyle\int_1^2 \cos(2t+1)\,\mathrm{d}t$

solution:

$$\int_1^2 \cos(2t+1)\,\mathrm{d}t = \left(\frac{\sin(2t+1)}{2}\right)\Bigg|_1^2$$
$$\frac{\sin 5}{2} - \frac{\sin 3}{2}$$

b) $\displaystyle\int_0^1 \frac{x^3}{(x^2+1)^{12}}\,\mathrm{d}x$

solution: Substitution $u = x^2+1$ giving $\mathrm{d}u = 2x\,\mathrm{d}x$. Then $x^2 = u-1$. Changing limits.

$$x = 0 \implies u = 1$$
$$x = 1 \implies u = 2$$

We get the new integral.

$$\begin{aligned}\int_0^1 \frac{x^3}{(x^2+1)^{12}}\,\mathrm{d}x &= \int_0^1 \frac{x^2}{(x^2+1)^{12}}x\,\mathrm{d}x \\ &= \int_1^2 \frac{u-1}{u^{12}}\frac{1}{2}\,\mathrm{d}u \\ &= \frac{1}{2}\int_1^2 u^{-11} - u^{-12}\,\mathrm{d}u \\ &= \frac{1}{2}\left(\frac{u^{-10}}{-10} - \frac{u^{-11}}{-11}\right)\Bigg|_1^2\end{aligned}$$

c) $\displaystyle\int_1^2 \frac{e^{\sqrt{3x+1}}}{\sqrt{3x+1}}\,\mathrm{d}x$

solution: Substitution with $u = \sqrt{3x+1}$ and $\mathrm{d}u = \dfrac{3\,\mathrm{d}x}{2\sqrt{3x+1}}$. We change the limits.

$$t = 1 \implies u = 2$$
$$t = 2 \implies u = \sqrt{7}$$

This gives the integral.

$$\frac{2}{3}\int_2^{\sqrt{7}} e^u\,\mathrm{d}u = \frac{2}{3}e^u\Big|_2^{\sqrt{7}} = \frac{2}{3}e^{\sqrt{7}} - \frac{2}{3}e^2$$

[**6**] **6.** Determine the indefinite integrals. You must show your work!

a) $\int x^3 \ln(x^2)\,\mathrm{d}x$

solution: Integration by parts.

$$u = \ln\left(x^2\right) \qquad \mathrm{d}v = x^3$$

$$\mathrm{d}u = \frac{2x^1}{x^2}\,\mathrm{d}x = \frac{2}{x}\,\mathrm{d}x \qquad v = \frac{x^4}{4}$$

We get the following

$$\begin{aligned}\int x^3 \ln(x^2) &= \ln\left(x^2\right)\frac{x^4}{4} - \int \frac{x^4}{4}\frac{2}{x}\,\mathrm{d}x \\ &= \ln\left(x^2\right)\frac{x^4}{4} - 2\int x^3\,\mathrm{d}x \\ &= \ln\left(x^2\right)\frac{x^4}{4} - 2\frac{x^4}{4} + C\end{aligned}$$

b) $\int e^{-x}\sin(2x)\,\mathrm{d}x$

solution:

$$\int e^{-x}\sin(2x)\,\mathrm{d}x$$

$$u = e^{-x}\,, \qquad \mathrm{d}v = \sin(2x)\,\mathrm{d}x$$

$$\mathrm{d}u = -e^{-x}\,\mathrm{d}x\,, \qquad v = -\frac{1}{2}\cos(2x)$$

$$= e^{-x}\left(-\frac{1}{2}\cos(2x)\right) - \int -\frac{1}{2}\cos(2x)\left(-e^{-x}\right)\,\mathrm{d}x$$

$$= \frac{-e^{-x}}{2}\cos(2x) - \frac{1}{2}\int e^{-x}\cos(2x)\,\mathrm{d}x$$

$$u = e^{-x}\,, \qquad \mathrm{d}v = \cos(2x)\,\mathrm{d}x$$

$$\mathrm{d}u = -e^{-x}\,\mathrm{d}x\,, \qquad v = \frac{1}{2}\sin(2x)$$

$$= \frac{-e^{-x}}{2}\cos(2x) - \frac{1}{2}\left(e^{-x}\left(\frac{1}{2}\sin(2x)\right) - \int \frac{1}{2}\sin(2x)\left(-e^{-x}\right)\,\mathrm{d}x\right)$$

$$= \frac{-e^{-x}}{2}\cos(2x) + \frac{-e^{-x}}{4}\sin(2x) - \frac{1}{4}\int e^{-x}\sin(2x)\,\mathrm{d}x$$

Then we solve for the integral.

$$\int e^{-x}\sin(2x)\,\mathrm{d}x = \frac{-e^{-x}}{2}\cos(2x) + \frac{-e^{-x}}{4}\sin(2x) - \frac{1}{4}\int e^{-x}\sin(2x)\,\mathrm{d}x$$

$$\frac{5}{4}\int e^{-x}\sin(2x)\,\mathrm{d}x = \frac{-e^{-x}}{2}\cos(2x) + \frac{-e^{-x}}{4}\sin(2x)$$

$$\int e^{-x}\sin(2x)\,\mathrm{d}x = e^{2x}\left(\frac{2}{5}\cos(2x) + \frac{1}{5}\sin(2x)\right)$$

+C

[6] **7.** Determine the indefinite integrals. You must show your work!

a) $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}}$

solution: Trig substitution, using $x = 3\sec\theta$ and $\mathrm{d}x = 3\sec\theta\tan\theta\,\mathrm{d}\theta$.

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}} &= \frac{1}{9}\int \frac{\sec\theta\tan\theta\,\mathrm{d}\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}}\\ &= \frac{1}{9}\int \frac{\sec\theta\tan\theta\,\mathrm{d}\theta}{\sec^2\theta\tan\theta}\\ &= \frac{1}{9}\int \cos\theta\,\mathrm{d}\theta\\ &= \frac{1}{9}\sin\theta\end{aligned}$$

Now drawing a triangle, we see that if $x = 3\sec\theta$ then $\sin\theta = \sqrt{x^2-9}/x$

$$\frac{\sqrt{x^2-9}}{9x} + C$$

b) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{2x^2+4x+10}}$

solution: First complete the square.

$$\begin{aligned}2x^2+4x+10 &= 2(x^2+2x)+10\\ &= 2(x^2+2x+1)+8\\ &= 2(x+1)^2+8\\ &= 8\left[\left(\frac{x+1}{2}\right)^2+1\right]\end{aligned}$$

Now we substitute as follows.

$$\begin{aligned}\frac{x+1}{2} &= \tan\theta\\ \frac{\mathrm{d}x}{2} &= \sec^2\theta\,\mathrm{d}\theta\end{aligned}$$

We get the integral.

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{2x^2+4x+10}} &= \int \frac{\mathrm{d}x}{\sqrt{8\left[\left(\frac{x+1}{2}\right)^2+1\right]}}\\ &= \int \frac{2\sec^2\theta\,\mathrm{d}\theta}{8\sqrt{\tan^2\theta+1}}\\ &= 16\int \sec\theta\,\mathrm{d}\theta\\ &= 16\ln\left(\sec\theta+\tan\theta\right)+C\end{aligned}$$

Using a triangle, we find that

$$\sec\theta = 8\left[\left(\frac{x+1}{2}\right)^2 + 1\right] = 2x^2 + 4x + 10$$

So our integral is

$$16\ln\left(2x^2 + 4x + 10 + \frac{x+1}{2}\right) + C$$