CONCORDIA UNIVERSITY
Department of Computer Science
COMP 238

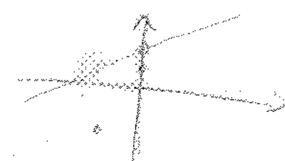
1. Consider functions $f : A \to B$ and $g : B \to C$. Prove each of the following statements, if the statement is true, or give a counterexample if the statement is false.

   a) (5 marks) if $g \circ f$ and f are both one-to-one then g must be one-to-one.

   b) (5 marks) if $g \circ f$ is onto then g must be onto.

2. (5 marks) Determine whether function $f : \mathcal{R} \to \mathcal{R}$ defined below is one-to-one and onto. If f is bijective find its inverse.

$$f(x) = \begin{cases} 2x+3 & \text{for } x \neq -1, x \neq 0 \\ 3 & \text{for } x = -1 \\ 1 & \text{for } x = 0 \end{cases}$$



3. (5 marks) Prove by contradiction that $\log_2 3$ is irrational.

4. (5 marks) Prove or disprove: Any two consecutive integers are relatively prime.

5. (10 marks) Derive a formula for

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)}$$

by examining the values of the formula for small values of $n, n \geq 1$. Use mathematical induction to prove it for any positive integer.

6. A (10 marks) Construct a circuit using inverters, OR gates and AND gates that gives output 1 if and only if not all three input variables have the same value.

   B (5 marks) Write the Boolean function, $F(x, y, x)$, representing the circuit in Part A.

7. Let $a, b, q, r, \in Z^+$.

   (15 marks) Prove. If $a = bq + r$ then $G.C.D.(a, b) = G.C.D.(b, r)$.

8. Let $f_n, n = 1, 2, 3, 4, \ldots$ be a Fibonacci sequence.

   (10 marks) Prove $f_n < 2^n \quad n \geq 1$.

TOTAL: 75

1. Determine whether the following are true or false for all sets A, B and C. If they are true prove them using **set laws**, if they are false give a counterexample.

   a) (3 pts)

$$(A-B)-C = A-(B-C).$$

   b) (3 pts)

$$(A-B)-C = (A-C)-(B-C)$$

   c) (3 pts)

$$P(A)-P(B) = P(A-B)$$

   where P(A) is the power set of A.

2. (8 pts) Let A and B be sets, and let $P(x)$ and $Q(x)$ denote the statements "$x \in A$" and "$x \in B$". Use $P(x)$ and $Q(x)$ to write

   $A \oplus B = (A \cup B) - (A \cap B)$

$$\underbrace{A \cap B = \varnothing}_{p} \iff \underbrace{A \oplus B = A \cup B}_{q}$$

   as a statement in predicate logic, then use propositional calculus to prove the statement is true.

3. (8 pts) Determine whether the following is tautology, contradiction or contingency. Justify your answer using propositional calculus.

$$[(p \longrightarrow q) \wedge (r \longrightarrow s)] \longrightarrow [(p \wedge r) \longrightarrow (q \wedge s)]. \qquad T$$

4. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ and $g : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x, y) = (x+y, x-2y+3)$ and $g(x, y) = x - y$.

   does it mean the range or domain?

   a) (2 pts) Find $h = g \circ f$.

   does it mean the pair set of [1, 3] when they are plugged into h^{-1}

   b) (4 pts) Find set $h^{-1}([1, 3])$ and graph it.

   c) (6 pts) Let $t : A \to B$, and let $S \subseteq A$. Prove $S \subseteq t^{-1}(t(S))$. Can the last inclusion be replaced by equality? Justify your answer.

5. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as follows

$$f(x) = \begin{cases} 4 & \text{for } x = 0 \\ 4 + \ln x & \text{for } 0 < x < 1 \\ x^2 + 2x + 1 & \text{for } x > 1. \end{cases}$$

   a) (3 pts) Determine whether f is one-to-one. $f(0)=4$

   b) (3 pts) Determine whether f is onto.

   c) (4 pts) Find f^{-1} if it exists.

3. Prove $Log_2 3$ is irrational.

Proof:

assume $Log_2 3$ is rational. ∴, $\Rightarrow \exists\, p, q \in \mathbb{Z}^+ \ni$
G.C.D.$(p, q) = 1$.

∴, $Log_2 3 = \frac{p}{q}$

$3 = 2^{p/q}$

$3^q = 2^p \Rightarrow$ a violation of the Fundamental Theorem of arithmetic.

Hence, $Log_2 3$ is irrational.

Proved!

4. Prove. any two consecutive integers are relatively prime.

Proof:

Let $n, n+1$ be the two integers.

Two numbers are said to be relatively prime if and only if their G.C.D. is 1.

G.C.D.$(n, n+1)$: $n+1 = 1(n) + 1$ (remainder not 0)

$n = n(1) + 0$ (remainder 0, ⇒ G.C.D. is

Proved!

5. A. $\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + \cdots + \left(\frac{1}{2n-1}\right)\left(\frac{1}{2n+1}\right) = \frac{n}{2n+1}$.

B. $n=1$: $\frac{1}{(1)(3)} = \frac{1}{2(1)+1}$

$\frac{1}{3} = \frac{1}{3}$ true.

$n=2$: $\cancel{\frac{1}{(3)(5)}}\ \frac{1}{(1)(3)} + \frac{1}{(3)(5)} = \frac{2}{2(2)+1}$

$\frac{1}{3} + \frac{1}{15} = \frac{2}{5}$

$\frac{6}{15} = \frac{2}{5}$

$\frac{2}{5} = \frac{2}{5}$ true.

$n=k$: assumed true.

$$\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + \cdots + \left(\frac{1}{2k-1}\right)\left(\frac{1}{2k+1}\right) = \frac{k}{2k+1}$$

$n=k+1$: must prove that:

$$\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + \cdots + \left(\frac{1}{2(k+1)-1}\right)\left(\frac{1}{2(k+1)+1}\right) = \frac{k+1}{2k+3}$$

now then;

$$\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + \cdots + \left(\frac{1}{2(k+1)-1}\right)\left(\frac{1}{2(k+1)+1}\right)$$

$$= \underbrace{\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + \cdots + \left(\frac{1}{2k-1}\right)\left(\frac{1}{2k+1}\right)} + \left(\frac{1}{2(k+1)-1}\right)\left(\frac{1}{2(k+1)+1}\right)$$

$$= \frac{k}{2k+1} + \left(\frac{1}{2k+1}\right)\left(\frac{1}{2k+3}\right)$$

I.A.

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

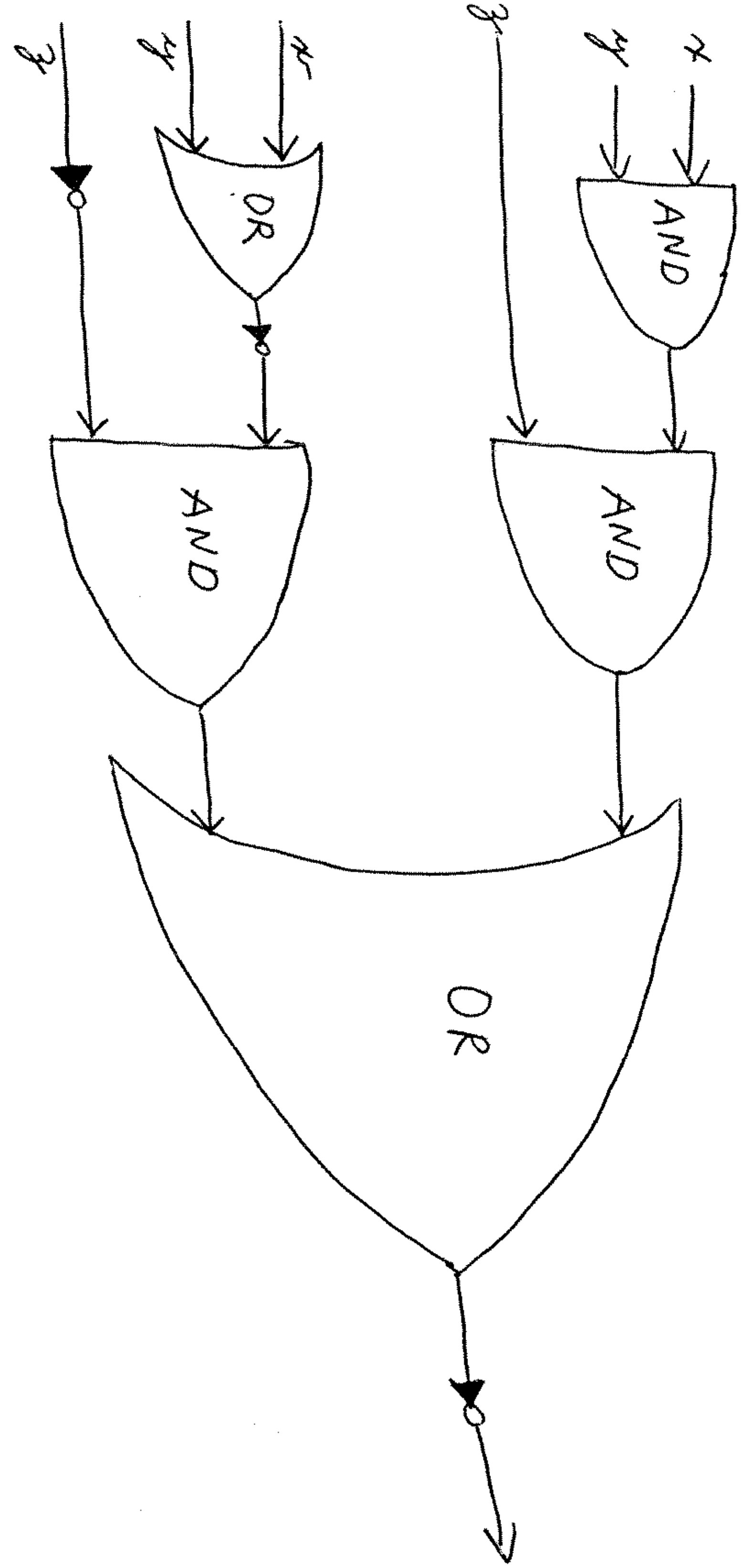
$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$ which is what had to be proved!

Since $n = k+1$ is true, $\Rightarrow P(n)$ is true $\forall_n$!

Proved!

6.A.



3. $F(x,y,z) = (\overline{xyz})(\overline{\bar{x}\,\bar{y}\,\bar{z}})$.

7. Prove. If $a = bq + r$ then $G.C.D.(a,b) = G.C.D.(b,r)$.

<u>Proof:</u>

Let $d = G.C.D.(a,b) \Rightarrow d/a$ and d/b.

Since $d/a \Rightarrow d/(bq+r)$. and since $d/b \Rightarrow d/r$

$\Rightarrow$ ~~b~~ $d / G.C.D.(b,r)$.

now then, Let $e = G.C.D.(b,r) \Rightarrow e/(bq+r) \Rightarrow e/a$

$\Rightarrow e / G.C.D.(a,b) \Rightarrow e/d$; and since $d/e \Rightarrow$

$d = e$, or $G.C.D.(a,b) = G.C.D.(b,r)$.

<u>Proved!</u>

8. Prove $b_n < 2^n$.

<u>Proof:</u> use Complete Induction.

$n=1$: $b_1 < 2^1$

$1 < 2$ true.

$n=2$: $b_2 < 2^2$

$1 < 4$ true.

$n \leq k$: assumed true.

$b_k < 2^k \quad \forall n \leq k$.

$n = k+1$: must Prove that: $b_{(k+1)} < 2^{(k+1)}$.

$$b_{(k+1)} = b_k + b_{(k-1)}$$
$$< 2^k + 2^{(k-1)} \quad \text{I.A.}$$
$$\leq 2^k\left(1 + \tfrac{1}{2}\right)$$

$$b_{(k+1)} \leq 2^{k}\left(\frac{3}{2}\right)$$

$$\downarrow \; < 2^{k}(2)$$

$$b_{(k+1)} \leq 2^{k+1}$$

$\therefore b_{(k+1)} < 2^{(k+1)}$ which is what had to be Proved.

Since $n = k+1$ is true, $P(n)$ is true $\forall n$!

Proved!

Concordia University Department of Computer Science
COMP 238 Mathematics for Computer Science I

Fall 2001
Assignment 1

1. This is a problem about an island in which the inhabitants are all either knights or knaves. Knights always tell the truth and knaves always lie. Suppose we have three people, A, B, and C, each of whom is a knight or a knave. A and B make the following statements:

   A: All of us are knaves.
   B. Exactly one of us is a knight.

   What are A, B, and C? Explain your reasoning.

2. Classify each of the following as a contradiction, a tautology, or a contingency, using truth tables or logical equivalences.

   (a) $p \rightarrow (q \rightarrow p)$
   (b) $\neg[(p \wedge q) \rightarrow p]$
   (c) $p \rightarrow (q \wedge p)$
   (d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

3. Prove the following logical equivalences:

   (a) $[p \rightarrow (q \rightarrow r)] \Leftrightarrow [(p \wedge q) \rightarrow r]$
   (b) $(p \oplus q) \Leftrightarrow \neg(p \leftrightarrow q)$

4. Let $P(x, y)$ be the predicate $x^2 < y + 2$. Determine the truth values of the following. The universe of discourse is the set of integers.

   (a) $\forall x \exists y \ P(x, y)$ T
   (b) $\exists x \forall y \ P(x, y)$ F
   (c) $\exists x \exists y \ P(x, y)$ T
   (d) $\forall x \forall y \ P(x, y)$ F
   (e) $\forall y \exists x \ P(x, y)$ T
   (f) $\exists y \forall x \ P(x, y)$ F

5. Let $I(x)$ be the statement "x has an internet connection" and $C(x, y)$ be the statement "x and y have chatted with each other over the internet," where the universe of discourse for the variables x and y is the set of all students in your class. Express each of the following using logical operations and logical quantifiers. Then form the negation of each statement so that no negation is to the left of a quantifier.

   (a) Not everyone in your class has an internet connection.

   (b) Everyone except one student in your class has an internet connection.

   (c) Everyone in your class with an internet connection has chatted over the internet with at least one other student in your class.

   (d) Someone in your class has an internet connection but has not chatted with anyone over the internet.

   (e) There are two students in your class who have not chatted with each other over the internet.

   (f) Noone in your class has chatted with everyone else in the class over the internet.

   (g) There are at least two students in your class who have not chatted with the same person over the internet.

   (h) There are two students in your class who between them have chatted with everyone else in the class over the internet.

6. The notation $\exists! x P(x)$ denotes the proposition:

   "There exists a unique x such that $P(x)$ is true." **If the universe of discourse is the set of integers, what are the truth values of the following statements:**

   (a) $\exists! x(x > 1)$

   (b) $\exists! x(x^2 = 1)$

   (c) $\exists! x(x + 3 = 2x)$

   (d) $\exists! x(x = x + 1)$

   Express the quantification $\exists! x P(x)$ using existential and universal quantifiers and logical operators.

Concordia University   Department of Computer Science
COMP 238   Mathematics for Computer Science I

Fall 2001
Assignment 2

1. A, B, and C are three suspects being questioned by the police about a certain theft. After many hours of interrogation, the Chief District Attorney summed up his findings to the judge as follows:

   (a) If either B is the crook or C is not, then A is.

   (b) The following statement is not true: If it is impossible that neither B nor C is the crook, then either both A and B are, or both A and C are.

   Replace these statements with logical symbols and determine which of the three suspects is the crook.

2. Negate each of the following statements:

   (a) $\exists x \forall y \, [P(x) \vee Q(y)]$

   (b) $\forall x \exists y \, [P(x, y) \rightarrow Q(x, y)]$

   (c) $\forall x \forall y \, [P(x, y) \rightarrow \exists z [z \neq x \wedge Q(y, z)]]$

3. Among 54 students taking examinations in mathematics, programming, and logic, 37 passed mathematics, 28 programming, and 43 logic. Further, 19 passed mathematics and programming, 29 mathematics and logic, and 20 programming and logic. How many passed all three subjects? Assume that all students passed at least one examination.

4. Using set identities, simplify the set $\overline{A} \cap (\overline{\overline{A} \cup B})$. Verify your result with a Venn diagram.

5. Let Z^+ denote the positive integers. For $i \geq 2$ denote

$$X_i = \{ik \mid k \in Z^+\}$$

   Describe the following sets in words.

   (a) X_2

   (b) $X_2 \cap X_3$

   (c) $X_2 \cup X_3$

   (d) $Z^+ - X_2$.

6. Assume the sets X, Y, Z are subsets of a universal set U and that the universal set for Cartesian products is $U \times U$. For each of the following, say whether the statement is true or false for all sets X, Y, and Z. If true, then you must give a proof, and if false, you must give a counterexample.

   (a) $X \cap (Y - Z) = (X \cap Y) - (X \cap Z)$

   (b) $(Y - X) \cup (Z - X) = (Y \cup Z) - X$

   (c) $\overline{X \times Y} = \overline{X} \times \overline{Y}$

   (d) $(X \cap Y) \cup (X - Y) = X$

   (e) $(X - Y) \cup (Y - X) = \emptyset$

   (f) $(X - Z \cap (Z - Y) = \emptyset$

Concordia University Department of Computer Science

COMP 238 Mathematics for Computer Science I

FALL 2001
ASSIGNMENT 3

1) Classify each of the following functions f as one–to–one, onto, both or neither. R denotes the set of all real numbers and R+ denotes the set of non–negative real numbers.

   i) f: R → R, g: R+ → R, h: R → R+.
   $f = g \circ h$, where $g(x) = \sqrt{x}$ and $h(x) = x^2$

   ii) f: R → R.
   $f(x) = 3(x^{1/3}) + 13$

   iii) f: D → D. D = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}.
   f(x) = the last digit of x^3.

   iv) f: B → C. B = the set of bit strings of length ten. C = the set of bit strings of length nine.
   f(x) = the complement of the last nine bits of x.

   v) f: R → R.
   $f(x) = \lfloor x \rfloor$

2) a) Let S be the set of all 5–bit binary strings. The function f: S → Z is defined by the
rule

$$f(x) = \text{the number of zeros in } x$$

for each binary string $x \in S$. Find

(i) f(01001);
(ii) the set of pre–images of 4;
(iii) the range of f.

b) Decide whether the function f defined in part (a) has either the one–to–one property or the onto property, justifying your answer.

3) Decide, for each of the functions, f, g, h defined below, whether it is invertible. If so, give the inverse function; if not, give a reason why no inverse function exists.

(i) f : R → R defined by $f(x) = 3x - 1$

(ii) g: $Z \to Z$ defined by $g(r) = 3r - 1$
(iii) h: $R \to Z$ defined by $h(x) = \lfloor x \rfloor$ (where $\lfloor x \rfloor$ denotes the floor of x).

4) Find all integers congruent to 9 modulo 7.

5) Give the prime factorizations for 72, 1815. Find GCD(72,1815), LCM(72,1815), GCD(200,$(200)^2$), LCM(20!,12!)

6) Prove that $\sqrt{3}$ is an irrational number

7) Draw the graph of
i) $2+3\lfloor (n-1)/4 \rfloor$

ii) $\lceil x/2 \rceil + \lceil x/2 \rceil$

DEPARTMENT OF COMPUTER SCIENCE
COMP 238 - Fall 2001
ASSIGNMENT 4

1. Let g: A → B, f: B → C be functions. Prove or disprove the following:

(a) If f is one-to-one and $f \circ g$ is one-to-one, then g is one-to-one.
(b) If $f \circ g$ is one-to-one and g is one-to-one, then f is one-to-one.
(c) If f is onto and g is onto, then $f \circ g$ is onto.
(d) If g is onto and $f \circ g$ is onto, then f is onto.

2. Prove the following for all n ≥ 1 by mathematical induction

(a) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(b) $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

(c) $1(1!) + 2(2!) + 3(3!) + \ldots + n(n!) = (n + 1)! + 1$

(d) $3^1 + 3^2 + {}^3 + \ldots + 3^n = \frac{3^{n+1} - 3}{2}$

3. Give a recursive definition of the sequence $\{a_n\}$, n = 1,2,3,... if

(a) $a_n = 1 + (-1)^n$ (b) $a_n = n^2$

4. If fn denotes the nth Fibonacci number, prove the following:
$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ whenever n is a positive integer.

5. Let R1 and R2 be relations on a set A represented by the matrices

$$M_{R1} = \begin{bmatrix} 0101 \\ 1011 \\ 1101 \\ 0111 \end{bmatrix} \qquad M_{R2} = \begin{bmatrix} 0100 \\ 1111 \\ 1010 \\ 0110 \end{bmatrix}$$

Find the matrices that represent
(a) $R1 \cup R2$ (b) $R1 \cap R2$ (c) $R1 \circ R2$ (d) $R2 \circ R1$

6. Give the directed graph of the relation {(a,b) | b ≡ 1 (mod a)} on the set {2,3,5,7,10,11}.

Concordia University Department of Computer Science
COMP 238 Mathematics for Computer Science I

Fall 2001
Assignment 5

1. A relation R on a set A is said to be *irreflexive* if for every $a \in A$, $(a,a) \notin R$. Which of the following relations on Z are irreflexive? Give reasons for your answers.
   (a) $xy \geq 1$
   (b) $x = y + 1$ or $x = y - 1$

2. Let R be a relation from set A to set B. The *inverse* relation from B to A, denoted R^{-1} is the set of ordered pairs $\{(b,a) \mid (a,b) \in R\}$. The *complementary* relation $\overline{R}$ is the set of ordered pairs $\{(a,b) \mid (a,b) \notin R\}$.
   (a) Let $R_1 = \{(a,b) \mid a < b\}$ on the set of integers. Find R_1^{-1} and $\overline{R_1}$.
   (b) Let $R_2 = \{(a,b) \mid a \text{ divides } b\}$ on the set of positive integers. Find R_2^{-1} and $\overline{R_2}$.
   (c) Show that a relation R on a set A is symmetric if and only if $R = R^{-1}$.
   (d) Show that a relation R on a set A is reflexive if and only if $\overline{R}$ is irreflexive.
   (e) Show that a relation R on a set A is anti-symmetric if and only if $R \cap R^{-1} \subseteq \{(a,a) \mid a \in A\}$.

3. Let $R = \{(a,b),(a,c),(c,d),(a,a),(b,a)\}$. What are R^{-1} and $R \circ R$? Specify which (if any) of R, R^{-1} or $R \circ R$ is a function.

4. Find the transitive closures of the following relation on $\{a,b,c,d,e\}$. Draw their graphs.
   (a) $R_1 = \{(a,c),(b,d),(c,a),(d,b),(e,d)\}$
   (b) $R_2 = \{(b,c),(b,e),(c,e),(d,a),(e,b),(e,c)\}$

5. Answer the following questions regarding the poset $(\{2,4,6,9,12,18,27,36,48,60,72\}, |)$.
   (a) Draw the Hasse diagram.
   (b) Find the maximal and minimal elements.
   (c) Is there a greatest element or a least element?
   (d) Find all upper bounds of $\{2,9\}$, and the least upper bound if it exists.
   (e) Find all lower bounds of $\{60,72\}$, and the greatest lower bound if it exists.

6. Let R be the relation defined on the set of all logical propositions as follows: pRq if $p \Leftrightarrow q$. Show that R is an equivalence relation.

7. Use a Karnaugh map to find a minimal expansion as a Boolean sum of Boolean products for each of the following functions in the variables x, y, z, and w.

   (a) $wxyz + wx\overline{y}z + wx\overline{y}\,\overline{z} + w\overline{x}\,\overline{y}z$

   (b) $wxyz + wxy\overline{z} + wx\overline{y}z + w\overline{x}yz + w\overline{x}y\overline{z} + \overline{w}xyz + \overline{w}\,\overline{x}yz + \overline{w}\,\overline{x}y\overline{z} + \overline{w}\,\overline{x}\,\overline{y}z$