SYSC 3600 - Fall 2018

Assignment # 2

Solutions

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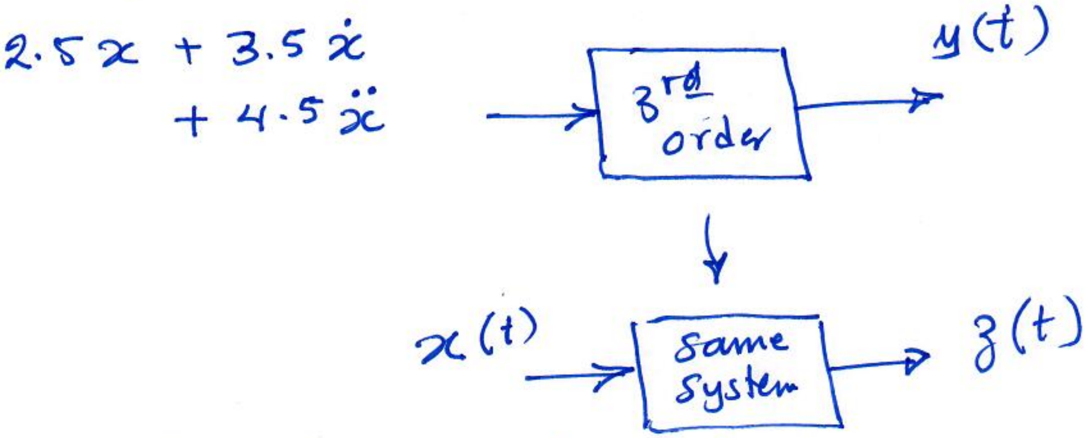
(1) $$2\dddot{y} + 6\ddot{y} + 3\dot{y} + y = 5x + 7\dot{x} + 9\ddot{x}$$

First divide by 2 throughout

$$\dddot{y} + 3\ddot{y} + 1.5\dot{y} + 0.5y = 2.5x + 3.5\dot{x} + 4.5\ddot{x}$$

There are several ways to solve this problem but the one discussed in class goes like this:



Solve this problem first

$$\dddot{z} + 3\ddot{z} + 1.5\dot{z} + 0.5z = x$$

The state variables are:

$$s_1 = z \;,\; s_2 = \dot{z} \;,\; s_3 = \ddot{z}$$

The corresponding state equations are:

$$\dot{s}_1 = s_2$$

$$\dot{s}_2 = s_3$$

$$\dot{s}_3 = x - (3\ddot{z} + 1.5\dot{z} + 0.5z)$$

$$= x - (3s_3 + 1.5s_2 + 0.5s_1)$$

Now, we apply the superposition

if the input is x $\longrightarrow$ The output is $\underline{s_1}$

~ ~ ~ ~ $2.5x$ $\longrightarrow$ ~ ~ ~ $\underline{2.5s_1}$

~ ~ ~ ~ $3.5\dot{x}$ $\longrightarrow$ ~ ~ ~ $3.5\dot{s}_1 = \underline{3.5}\, s_2$

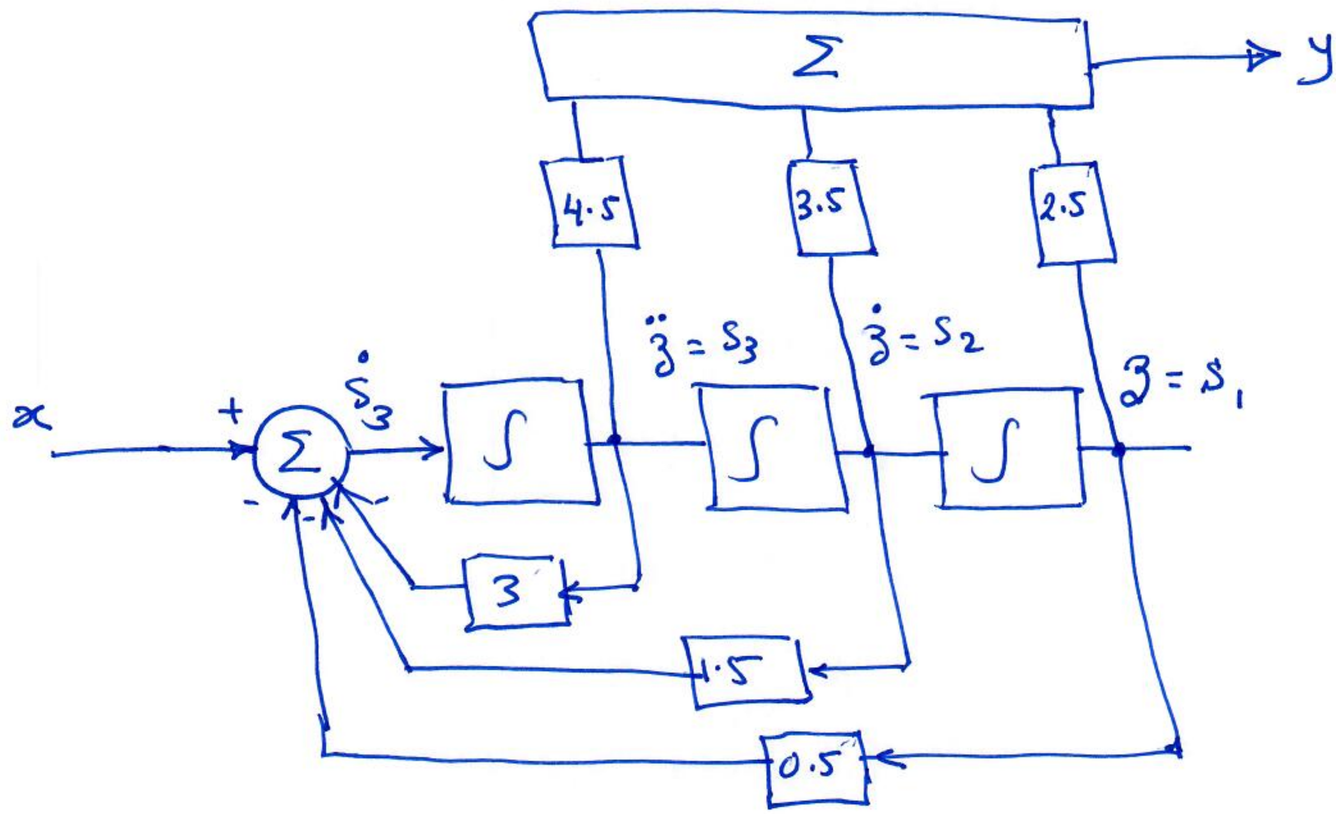
~ ~ ~ ~ $4.5\ddot{x}$ $\longrightarrow$ ~ ~ ~ $4.5\ddot{s}_1 = 4.5\dot{s}_2 = \underline{4.5}\, s_3$

If the input is:

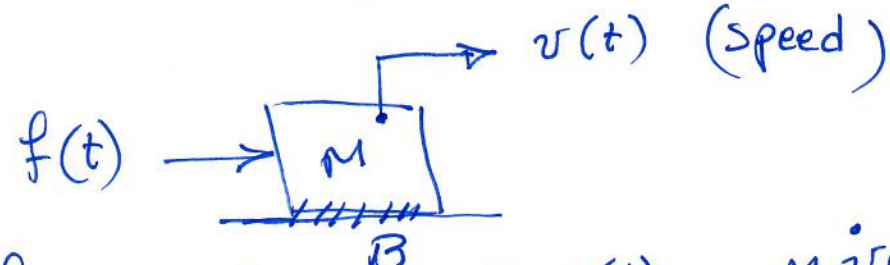
$$2.5x + 3.5\dot{x} + 4.5\ddot{x}$$

Then the output is:

$$2.5s_1 + 3.5s_2 + 4.5s_3$$



Problem 2 This is a first order system



input force $= f(t) = B\,v(t) + M\,\dot{v}(t)$

(a) $\dot{v}(t) + \frac{B}{M} v(t) = \frac{1}{M} f(t)$

a b

The impulse response is $h(t) = b\,e^{-at}\,u(t)$

$B = 2,\ M = 1 \Rightarrow a = 2,\ b = 1$

$$h(t) = e^{-2t}\,u(t)$$

The speed equation is:

(b) $$v(t) = v_{nat.}(t) + v_{forced}(t)$$

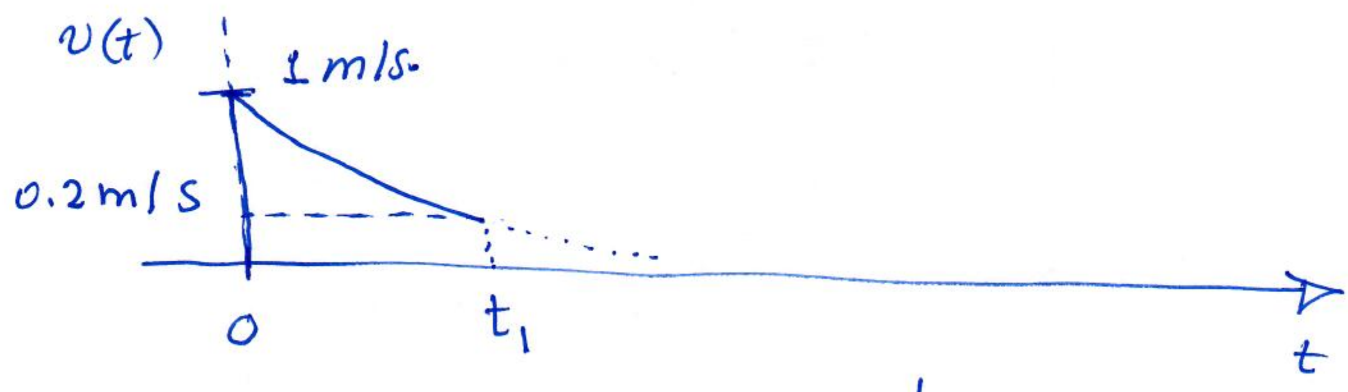
at $\underline{t=0}$ the initial condition is $v(0) = 0$

$$\therefore v_{nat.}(t) = 0$$

$t > 0$ $$v(t) = h(t) * f(t)$$

since $f(t) = \delta(t)$

$$\therefore v(t) = h(t) = e^{-2t}$$



at $t = t_1$ $v(t_1) = 0.2$ m/s

$$\therefore 0.2 = e^{-2t_1} \Rightarrow t_1 = 0.8047 \text{ sec.}$$

(c) at $t = t_1 = 0.8047$ sec. another impulse force is applied to keep the speed above 0.2 m/s

$\underline{t \geq t_1}$

$$v(t) = v_{nat.}(t) + v_{forced}(t)$$

The initial condition at $t = t_1$ is $v(t_1) = 0.2$ m/s

$$v_{nat.}(t) = 0.2\, e^{-2(t-t_1)}\, u(t-t_1)$$

$$v_{forced}(t) = \delta(t-t_1) * e^{-2(t-t_1)}\, u(t-t_1)$$

$$= e^{-2(t-t_1)}\, u(t-t_1)$$

$$v(t) = 1.2\, e^{-2(t-t_1)}\, u(t-t_1)$$

(d) at $t = t_2$ $\quad v(t_2) = 0.2$ m/s

$$v(t_2) = 1.2\, e^{-2(t_2-t_1)} = 0.2$$

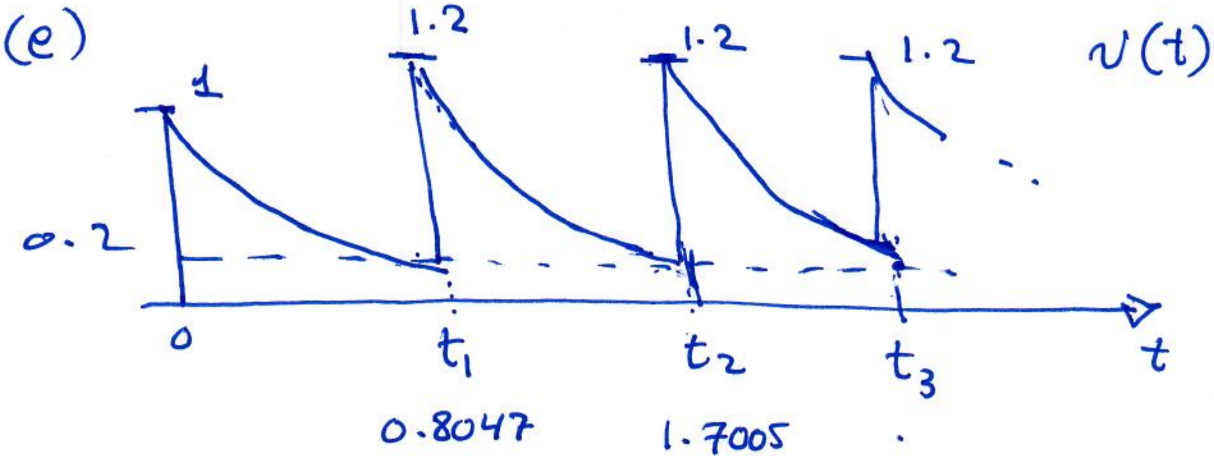
$$t_2 - t_1 = \Delta t \Rightarrow 1.2\, e^{-2\Delta t} = 0.2$$

$$\ln\left(\frac{0.2}{1.2}\right) = -2\Delta t$$

$\Delta t = 0.8958$ sec. $\quad \therefore t_2 = t_1 + \Delta t$

$t_2 = 0.8047 + 0.8958 = 1.7005$ sec.

(e)



If we need third, fourth, ... pulses the time ~~space~~ between successive pulses is $\Delta t = 0.8958$.

To compute how many pulses are needed till the mass M hits the wall at 1.5 m. We need to convert time into distance by integrating $v(t)$

$$x_1 = \int_0^{t_1} v(t)\,dt = \int_0^{0.8047} e^{-2t}\,dt$$

$$= \frac{e^{-2t}\Big)_0^{t_1}}{-2} = \frac{1}{2}\left(1 - e^{-2 \times 0.8047}\right)$$

$\cong 0.4$ meters

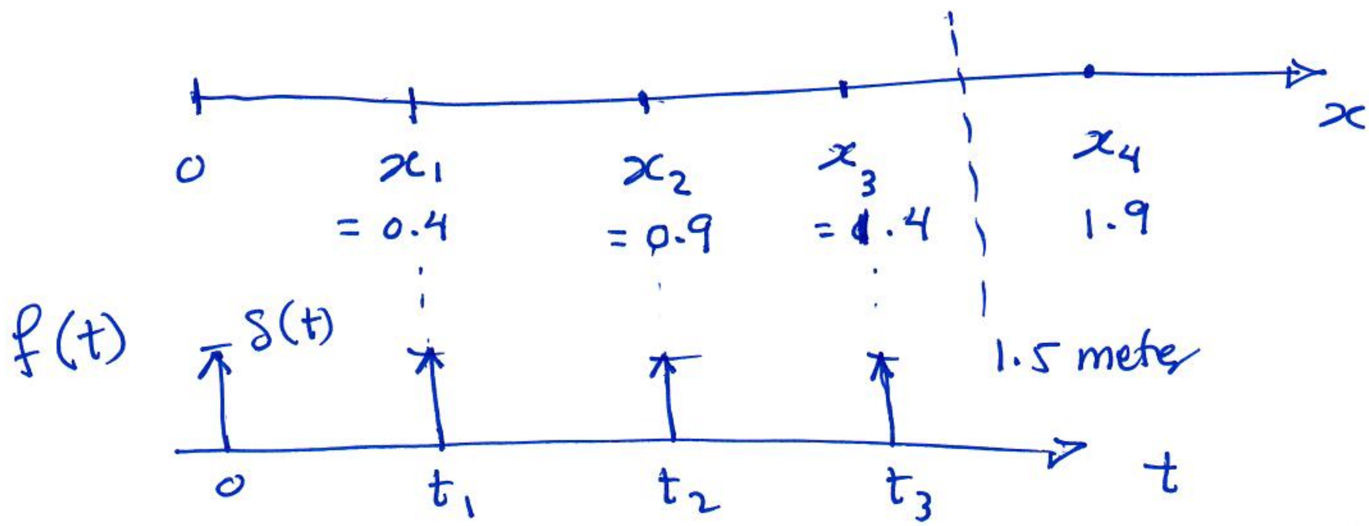
at $t = t_2 \quad x = x_2 = \int_0^{\Delta t} 1.2 e^{-2t}\, dt + x_1$

$$\Delta x = x_2 - x_1 = \int_0^{0.8958} 1.2\, e^{-2t}\, dt$$

$$\Delta x = \frac{1.2}{2} \cdot \Big[e^{-2t} \Big]_{0.8958}^{0}$$

$$= 0.6 \left[1 - e^{-2 \times 0.8958} \right]$$

$$= 0.4999 \text{ m} \simeq 0.5 \text{ m.}$$



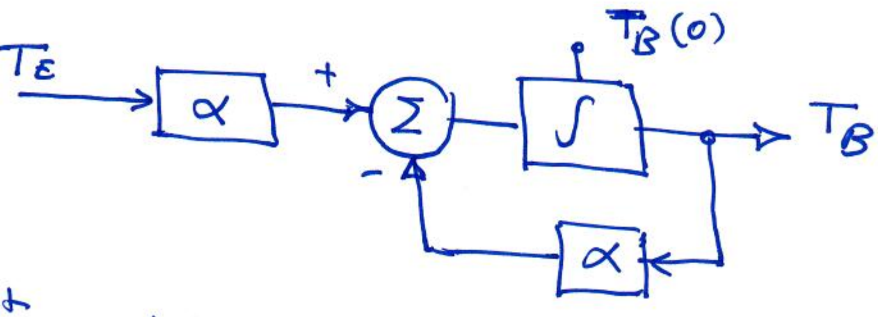
4 impulses are needed

Problem (3)

$$\dot{T}_B(t) + \alpha T_B(t) = \alpha T_E(t)$$

$$\alpha = 6 \times 10^{-4} \ \text{sec}^{-1}$$

(a)



1^{st} order system $\quad a = b = \alpha$

(b) $\quad h(t) = \alpha e^{-\alpha t} u(t)$

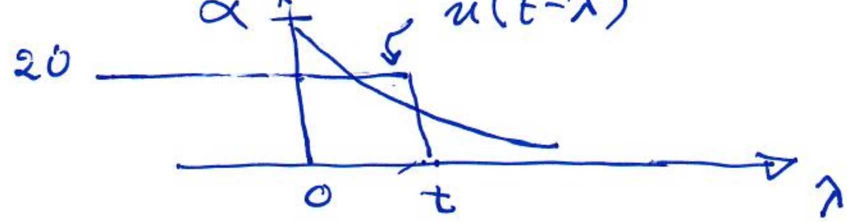
system impulse response.

$$T_B = T_{B_{nat.}}(t) + T_{B_{forced}}(t)$$

at $t = 0$, $T_B(0) = 4\,^{o}c$

$$T_B = 4 e^{-6\times10^{-4} t} + 20\, u(t) * h(t)$$

$$T_{B_{forced}}(t) = 20 \int \alpha e^{-\alpha \lambda} u(t-\lambda)\, d\lambda$$



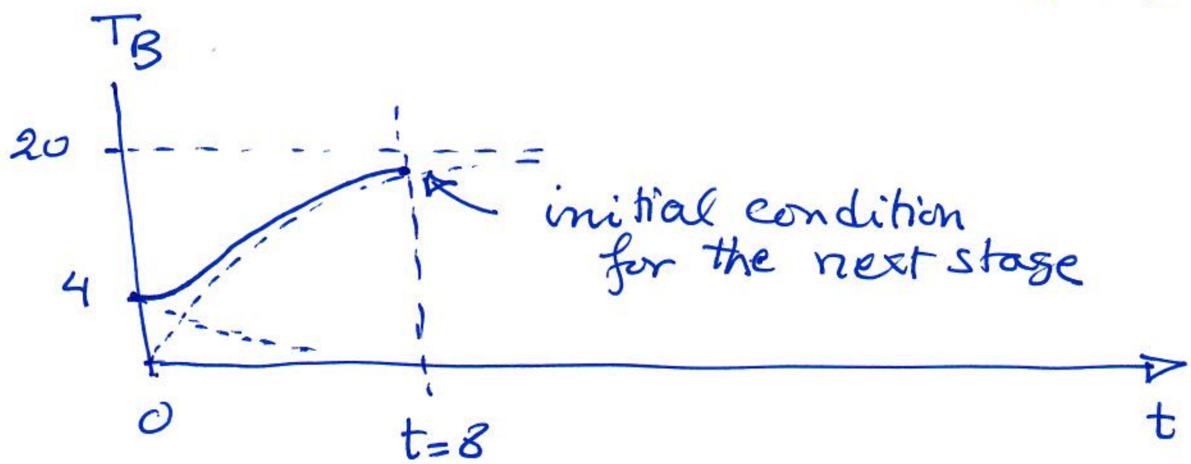
$$T_{B_{forced}}(t) = 20 \times 6 \times 10^{-4} \int_0^t e^{-6\times10^{-4}\lambda}\, d\lambda$$

$$= 20 \times \cancel{6 \times 10^{-4}} \; \frac{e^{-6\times10^{-4}\lambda}\Big)_{\lambda=0}^{\lambda=t}}{-\cancel{6\times10^{-4}}}$$

$$= 20\left[1 - e^{-6\times10^{-4}t}\right] \; ; \; t \geq 0$$

$$T_B(t) = 4\, e^{-6\times10^{-4}t} + 20\left[1 - e^{-6\times10^{-6}t}\right] \; ; \; t \geq 0$$

(c)



$$T_B(8) = 4 \times e^{-6\times10^{-4}\times8\times60} + 20\left[1 - e^{-6\times10^{-4}\times8\times60}\right] \simeq 8\ ^{o}C$$

$\underline{t \geq 8}$

$$T_B(t) = 8\, e^{-\alpha\Delta t} + 4\left[1 - e^{-\alpha\Delta t}\right] \; ; \; t \geq 8$$

$$\Delta t = t - 8$$

at $t = 15$ min $\quad \Delta t = 7$ min

$$T_B(15) = 8e^{-\alpha \times 7 \times 60} + 4\left[1 - e^{-\alpha \times 7 \times 60}\right]$$

$$= 7.11\ ^{o}C$$

(e)