The ABC's of Calculus, Volume 2, August 12, 2017 Edition

Solutions to Exercise Sets

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Exercise Set 1.

1. $\vec{PQ} = (-2,-1), \vec{PR} = (1,-3)$. So $\vec{PQ} + \vec{PR} = (-1,-4)$.
2. $\vec{PR} = \vec{R} - \vec{P}$ (with points (insert word) as vectors) and so $\vec{PR} = (2,-1) - (1,2) = (1,-3)$, while $\vec{RS} = \vec{S} - \vec{R} = (0,1) - (2,-1) = (-2,2)$. $\therefore 2\vec{PR} - \vec{RS} = 2(1,-3) - (-2,2) = (4,-8)$
3. $\vec{PQ} + \vec{QR} + \vec{RS} = \vec{PS}$ (addition rule for vectors) so $\vec{PS} = \vec{S} - \vec{P} = (0,1) - (1,2) = (-1,-1)$.
4. $3\vec{RQ} - 2\vec{PS} = 3(\vec{Q} - \vec{R}) - 2(\vec{S} - \vec{P}) = (-7,8)$.
5. $\vec{SQ} - 5\vec{PQ} + \vec{QR} = \vec{Q} - \vec{S} - 5(\vec{Q} - \vec{P}) + \vec{R} - \vec{Q} = (12,3)$.
6. $\vec{SR} + \vec{SQ} - \vec{QP} = (-1,-3)$.
7. $\vec{QS} + 2\vec{SQ} - 3\vec{PS} = (2,3)$
8. $\vec{PR} + \vec{QR} - \vec{QS} = (3,-5)$.
9. $\vec{RQ} - \vec{RP} + 3\vec{SQ} = \vec{Q} - \vec{R} - (\vec{P} - \vec{R}) + 3(\vec{Q} - \vec{S}) = (-5,-1)$
10. $\vec{QR} + \vec{RS} - \vec{SP} + \vec{PR} = (1,-4)$
11. $|\vec{PQ}| = |\vec{Q} - \vec{R}| = |(-2,-1)| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$
12. $\because |\vec{PR} + \vec{RS}| = |\vec{PS}|, |\vec{PS}| = |\vec{S} - \vec{P}| = \sqrt{2}$
13. $2|\vec{QS}| - |\vec{PS}| = 2|(\vec{S} - \vec{Q}| - |\vec{S} - \vec{P}| = 2 - \sqrt{2}$

14. $|2\vec{PQ} + 3\vec{QR}| = |2(\vec{Q} - \vec{P}) + 3(\vec{R} - \vec{Q})| = \sqrt{89}$

15. $\because \vec{PR} = -\vec{RP}, |\vec{PR}| = |\vec{RP}| \therefore |\vec{PR}| - |\vec{RP}| = 0.$

16. $|2\vec{PQ} + 2\vec{QP}| = |2 \cdot \vec{0}| = 0$, since $\vec{PQ} = -\vec{QP}$.

17. $|3\vec{PR} - 3|\vec{QR}| = |3(\vec{R} - \vec{P})| - 3|\vec{R} - \vec{Q}| = 3\sqrt{10} - 3\sqrt{13}.$

18. $2|\vec{PS}| - |2\vec{SP}| = 2|\vec{PS}| - 2|\vec{SP}| = 0$, (Since $|\vec{SP}| = | - \vec{PS}| = |\vec{PS}|$).

19. $|\vec{SQ}| + 2|\vec{PR}| - |\vec{PQ}| = 1 + 2\sqrt{10} - \sqrt{5}$

20. $|\vec{QS} + \vec{SQ} - \vec{RS}| = | - \vec{RS}| = |\vec{RS}| = 2\sqrt{2}$ (Since $\vec{QS} + \vec{SQ} = 0$).

21. $v = (2, 3) \to \tan\theta = b/a$ and $(2, 3)$ is in Q1, so $\theta = \arctan(b/a) = \arctan(1.5) = 0.9828$ radians.

22. $(-1, 2)$ is in Q2 $\to \theta = \arctan(-2) + \pi = -\arctan(2) + \pi = -1.1071 + 3.1416 = 2.0344$ rads.

23. $\vec{PQ} = \vec{Q} - \vec{P} = (-1, 0) - (0, 1) = (-1, 1)$ is in Q3. Thus, $\theta = \arctan(1) + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$ radians.

24. $(2, -5)$ is in Q4, hence $\theta = \arctan(-2.5) + 2\pi = -\arctan(2.5) + 2\pi = 5.0929$ radians.

25. $\vec{v}$ lies along y-axis, hence $\theta = \pi/2$ radians.

26. $(-2, 4)$ is in Q2 so $\theta = \arctan(-2) + \pi = -\arctan(2) + \pi = 2.0344$ radians.

27. $(3, -4)$ is in Q4 so $\theta = \arctan(-4/3) + 2\pi = -\arctan(4/3) + 2\pi = 5.3559$ radians

28. $\vec{PQ} = (0, 1) - (2, -1) = (-2, 2)$ is in Q2, so $\theta = \arctan(-1) + \pi = -\arctan(1) + \pi = -\pi/4 + \pi = \frac{3\pi}{4}$.

29. The point is in Q3 so $\theta = \arctan(1) + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

30. Here the point is in Q4 so $\theta = \arctan(-1) + 2\pi$ (insert word)

Exercise Set 2.

1. Yes. $\vec{v} \cdot \vec{w} = (0, -1, 1) \cdot (1, 1, 1) = 0 \cdot 1 + (-1) \cdot 1 + (1)(1) = 0$

2. $\vec{v} \cdot \vec{z} = (0, -1, 1) \cdot (0, 1, -1) = 0 - 1 - 1 = -2$. On the other hand $|\vec{v}| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}$ while $|\vec{w}| = \sqrt{2}$ also $\therefore |\vec{v}||\vec{w}| = (\sqrt{2})^2 = 2$ so that $\vec{v} \cdot \vec{z} = -|\vec{w}||\vec{v}|$. Hence $\vec{v}, \vec{z}$ are antiparallel (even though they are actually negatives of each other here).

3. $\vec{x} = \frac{1}{2}\vec{w}$ so $\vec{x}$ is a multiple of $\vec{w}$ of $\vec{x}, \vec{w}$ must be parallel.

4. No. $(\vec{v} + \vec{w}) \cdot \vec{z} = (1, 0, 2) \cdot (0, 1, -1) = 1 \cdot 0 + 0 \cdot 1 + 2(-1) = -2$ so they cannot be orthogonal.

5. We want $(\alpha\vec{v} + \vec{w}) \cdot \vec{z} = 0$. So this means that $(\alpha(0, -1, 1) + (1, 1, 1)) \cdot (0, 1, -1) = 0$ or $((0, -\alpha, \alpha) + (1, 1, 1)) \cdot (0, 1, -1) = 0$, i.e., $(1, 1 - \alpha, 1 + \alpha) \cdot (0, 1, -1) = 0$. Evaluating the dot product we get $1 \cdot 0 + (1 - \alpha) - (1 + \alpha) = 0$ or $-2\alpha = 0$ which forces $\alpha = 0$. Hence $\vec{w} \perp \vec{z}$, or $\alpha\vec{v} + \vec{w} \perp \vec{z}$ for $\alpha = 0$.

6. Yes. $(\vec{v}+\vec{z})\cdot\vec{z} = ((0,-1,1)+(0,1,-1))\cdot(0,1,-1) = (0,0,0)\cdot(0,1,-1) = 0$. (Note that $\vec{0}\cdot\vec{z} = 0$ for **any** vector $\vec{z}$.)

7. Look at the vectors $\vec{PR}$ and $\vec{PQ}$. Since R is to be the midpoint of the line segment from P to Q it's clear that $|\vec{PR}| = \frac{1}{2}|\vec{PQ}|$. On the other hand (draw a picture) we see that $\vec{PR} = \frac{1}{2}\vec{PQ}$ too! Hence, $(x-a, y-b, z-c) = \vec{PR} = \frac{1}{2}\vec{PQ} = \frac{1}{2}(x_1 - x_0, y_1 - y_0, z_1 - z_0) = \left(\frac{x_1-x_0}{2}, \frac{y_1-y_0}{2}, \frac{z_1-z_0}{2}\right)$ from which $x - a = \frac{1}{2}(x_1 - x_0), y - b = \frac{y_1-y_0}{2}, z - c = \frac{z_1-z_0}{2}$. Solving for a, b, c we get the result.

8. This is similar to #7 previously except that we replace "$\frac{1}{2}$" there by "α".

9. -2. $\vec{w} + \vec{z} - \vec{x} = (1/2, 3/2, -1/2)$ so $\vec{v}\cdot(\vec{w} + \vec{z} - \vec{x}) = (0,-1,1)\cdot(1/2, 3/2, -1/2) = -3/2 - 1/2 = -2$.

10. $-7/2$. $\vec{v}-\vec{w} = (-1,-2,0)$, $\vec{x}+\vec{z} = (1/2, 3/2, -1/2)$ and the result follows.

11. This question should have read: Evaluate $(\mathbf{x} - \frac{1}{2}\mathbf{u})\cdot(\mathbf{w}+\mathbf{x})$. The solution is the scalar, $9/4$. This is because $\frac{1}{2}\vec{\mathbf{u}}\cdot(\vec{w}+\vec{x}) = 0$ and $\mathbf{x}\cdot(\mathbf{w}+\mathbf{x}) = 9/4$.

12. $-9/4$. Note that $\vec{x}\cdot\vec{x} = \frac{3}{4}$ while $\vec{w}\cdot\vec{w} = 3$

13. -4. Observe that $\vec{v}\cdot\vec{z} = -2, \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{2}$ and $\sqrt{\vec{w}\cdot\vec{w}} = \sqrt{2}$ Yes, this is **always true** and called the **Schwarz...** (insert word)

14. $-3/2$. Here $\vec{x}\cdot\vec{v} = 0, \vec{x}\cdot\vec{x} = 3/4, \vec{v}\cdot\vec{v} = 2$.

15. 2. Observe that $\vec{x}+\vec{v} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}\right)$ and $(\vec{x}+\vec{v})\cdot(\vec{x}+\vec{v}) = 11/4$ while $\vec{x}\cdot\vec{x} = 3/4$

16. $\sqrt{17}$

17. $2\sqrt{13}$

18. $(\frac{3}{5}, \frac{4}{5})$

19. $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$

20. $-\mathbf{i} - 3\,\mathbf{j}$

21. -18

22. $-\frac{15}{4}$

23. $|\mathbf{u}| = \sqrt{14}$; $\mathbf{u}+\mathbf{v} = (3,8,6)$; $\mathbf{u}-\mathbf{v} = (3,-4,-8)$; $2\mathbf{u} = (6,4,-2)$; $3\mathbf{u}+4\mathbf{v} = (9,30,25)$.

Exercise Set 3.

1. We need to find a vector $\vec{u}$ in the direction of $\vec{v}$. So let $\vec{u} = \vec{v}/|\vec{v}|$ where $|\vec{v}| = \sqrt{(-1)^2 + 2^2 + 0^2} = \sqrt{5}$. Then $\vec{u} = \frac{1}{\sqrt{5}}(-1,2,0) = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) \therefore \cos\alpha = -\frac{1}{\sqrt{5}}, \cos\beta = \frac{2}{\sqrt{5}}, \cos\gamma = 0$.

2. $\vec{u}$ is already a unit vector so its coordinates are its direction cosines. $\cos\alpha = -\frac{1}{\sqrt{2}}, \cos\beta = 0, \cos\gamma = -\frac{1}{\sqrt{2}}$.

3. As before we let $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ where $|\vec{v}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$ so that $\vec{u} = \frac{1}{\sqrt{6}}(2, -1, 1) = \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$. So, $\cos\alpha = \frac{2}{\sqrt{6}}, \cos\beta = -\frac{1}{\sqrt{6}}, \cos\gamma = \frac{1}{\sqrt{6}}$.

4. Here $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ where $|\vec{v}| = \sqrt{0^2 + (-2)^2 + 1^2} = \sqrt{5}$. So, $\vec{u}$, the unit vector in the direction of $\vec{v}$, has direction cosines given by $\cos\alpha = 0, \cos\beta = -\frac{2}{\sqrt{5}}, \cos\gamma = \frac{1}{\sqrt{5}}$.

5. Here $|\vec{v} = \sqrt{(-2)^2 + 3^2 + 4^2} = \sqrt{29}$ so that $\vec{u}$, given by $\vec{u} = \frac{1}{\sqrt{29}}(-2, 3, 4)$, is a unit vector in the same direction as $\vec{v}$. It follows that $\cos\alpha = -\frac{2}{\sqrt{29}}, \cos\beta = \frac{3}{\sqrt{29}}, \cos\gamma = \frac{4}{\sqrt{29}}$.

6. Since $\vec{u}$ is already a unit vector we know that $\cos\alpha = \frac{1}{\sqrt{3}}, \cos\beta = -\frac{1}{\sqrt{3}}$ and $\cos\gamma = \frac{1}{\sqrt{3}}$. So $\alpha = \arccos(1/\sqrt{3}) = 0.9553, \beta = \arccos(-1/\sqrt{3}) = 2.1863$ and $\gamma = \arccos(1/\sqrt{3}) = 0.9553$ rads.

   Remember that α, β, γ must all belong to the interval from 0 to π

7. $\vec{v} = (-3, 0, 3)$ means that $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ is a unit vector in the same direction as $\vec{v}$ where $|\vec{v}| = \sqrt{18}$ (why?) So, $\vec{u} = \frac{1}{\sqrt{18}}(-3, 0, 3)$ means that $\cos\alpha = -\frac{3}{\sqrt{18}}, \cos\beta = 0$, and $\cos\gamma = \frac{3}{\sqrt{18}}$. Using our calculator we get $\alpha = \arccos\left(-\frac{3}{\sqrt{18}}\right) = 2.3562 rads$; $\beta = \pi/2$, and $\gamma = \arccos\left(\frac{3}{\sqrt{18}}\right) = 0.7854$ rads.

   Note: We could reduce $3/\sqrt{18}$ to $1/\sqrt{2}$ using the simplifications $\frac{3}{\sqrt{18}} = \frac{3}{\sqrt{9 \cdot 2}} = \frac{3}{\sqrt{9}\sqrt{2}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$ but since we need to use our calculator anyhow, this isn't necessary.

8. $\vec{v} = (1, -1, 2)$ so $|\vec{v}| = \sqrt{6}$ and $\vec{u} = \frac{1}{\sqrt{6}}(1, -1, 2)$ is in the same direction as $\vec{v}$. It follows that $\cos\alpha = 1/\sqrt{6}, \cos\beta = -1/\sqrt{6}, \cos\gamma = \frac{2}{\sqrt{6}}$ or $\alpha = \arccos\left(\frac{1}{\sqrt{6}}\right) = 1.1503$; $\beta \arccos\left(-\frac{1}{\sqrt{6}}\right) = 1.9913$; $\gamma = \arccos\left(\frac{2}{\sqrt{6}}\right) = 0.6155$ rads.

9. $\vec{v} = (-3, -2, -1)$ has $|\vec{v}| = \sqrt{9 + 4 + 1} = \sqrt{14}$ so $\cos\alpha = -\frac{3}{\sqrt{14}}, \cos\beta = -\frac{2}{\sqrt{14}}, \cos\gamma = -\frac{1}{\sqrt{14}}$ and $\alpha = \arccos\left(-\frac{3}{\sqrt{14}}\right) = 2.5011$; $\beta \arccos\left(-\frac{2}{\sqrt{14}}\right) = 2.1347$; $\gamma = \arccos\left(-\frac{1}{\sqrt{14}}\right) = 1.8413$ rads.

10. $\vec{v} = (-1, 2, -2)$ has $|\vec{v}| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$. Hence, $\cos\alpha = -1/3, \cos\beta = 2/3, \cos\gamma = -2/3$ which imply that $\alpha = \arccos(-1/3) = 1.9106$; $\beta = \arccos(2/3) = 0.8411$; $\gamma = \arccos(-2/3) = 2.3005$ radians.

11. $\vec{u} = \left(\frac{1}{\sqrt{2}}, 0, c\right)$ means the $\cos\alpha = \frac{1}{\sqrt{2}}, \cos\beta = 0$ and $\cos\gamma = c$. But $|\vec{u}| = 1$ also (insert word) that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ or $\frac{1}{2} + 0 + \cos^2\gamma = 1$ or $\cos^2\gamma = 1/2$ which implies that $\cos\gamma = \pm\frac{1}{\sqrt{2}}$. However $c < 0$ (given), so $\cos\gamma = -\frac{1}{\sqrt{2}}$ or $\gamma = 3\pi/4$ (or 2.3562 radians (insert word))

12. $\vec{u} = \left(\frac{1}{\sqrt{3}}, b, -\frac{1}{\sqrt{3}}\right)$ means that $\cos\beta = b$. But $|\vec{u}| = 1$ also implies that $\frac{1}{3} + \cos^2\beta + \frac{1}{3} = 1$ or $\cos^2\beta = 1/3$, i.e., $\cos\beta = \pm 1/\sqrt{3}$. Since $b > 0$ is given we get $\beta = \arccos(1/\sqrt{3}) = 0.9553$ radians.

13. $\vec{u} = (a, -1/2, 1/2)$ and $|\vec{u}| = 1$ implies that $\cos^2\alpha + \frac{1}{4} + \frac{1}{4} = 1$ or $\cos^2\alpha = 1/2$, i.e., $\cos\alpha = \pm\frac{1}{\sqrt{2}}$. Since $a < 0$ we get $\cos\alpha = -\frac{1}{\sqrt{2}}$ or $alpha = 3\pi/4$.

14. $\vec{u} = \left(0, \frac{1}{4}, c\right)$ and $|\vec{u}| = 1$ both imply that $\frac{1}{16} + \cos^2\gamma = 1$ or $\cos^2\gamma = 15/16$, i.e., $\cos\gamma = \frac{1}{4}\sqrt{15}$ (since $c = \cos\gamma > 0$ is given). Hence $\gamma = \arccos\left(\frac{\sqrt{15}}{4}\right) = 0.2527$ radians.

15. $\vec{u} = \left(\frac{1}{2}, b, -\frac{1}{2}\right), |\vec{u}| = 1$ gives us $\frac{1}{4} + \cos^2\beta + \frac{1}{4} = 1$ or $\cos^2\beta = 1/2$, i.e., $b = \cos\beta = \pm\frac{1}{\sqrt{2}}$, i.e., $\cos\beta = \frac{1}{\sqrt{2}}$ since $b > 0$. Hence, $\beta = \pi/4$ radians.

16. $\vec{w} = (a, b, c)$ and we know that $\frac{b}{\sqrt{10}} = \cos\beta$, $\frac{c}{\sqrt{10}} = \cos\gamma$. But $\cos\beta = 0 \Longrightarrow b = 0$ while $\cos\gamma = -\frac{1}{\sqrt{10}} \Longrightarrow c = -1$. But $\vec{w} \perp (1, -1, 3)$ means that $(a, b, c) \cdot (1, -1, 3) = 0$, i.e., $a - b + 3c = 0$. Combining our results we get $a = b - 3c = 0 - (-3) = 3$, $\therefore \vec{w} = (3, 0, -1)$.

Exercise Set 4.

1. $\vec{v} \times \vec{w} = (1, 2, 3)$ (Use the expansion of the determinant)
2. $\vec{w} \times \vec{v} = (-3, 1, 2)$
3. $(2\vec{v}) \times \vec{w} = (2, 8, 4)$
4. $\vec{v} \times (3\vec{w}) = (-24, -3, 6)$
5. $\vec{v} \times \vec{w} = (2, 1, 3)$
6. $\mathbf{n}_1 = \vec{v} \times \vec{w} = (0, 3, 0)$ and $\mathbf{n}_2 = \vec{w} \times \vec{v} = (0, -3, 0)$
7. $\mathbf{n}_1 = \vec{v} \times \vec{w} = (3, -1, 2)$ and $\mathbf{n}_2 = \vec{w} \times \vec{v} = (-3, 1, -2)$
8. $\mathbf{n}_1 = \vec{v} \times \vec{w} = (\frac{1}{12}, -\frac{1}{2}, -\frac{1}{8})$ and $\mathbf{n}_2 = \vec{w} \times \vec{v} = (-\frac{1}{12}, \frac{1}{2}, \frac{1}{8})$
9. $\mathbf{n}_1 = \vec{v} \times \vec{w} = (1, \frac{1}{2}, \frac{1}{4})$ and $\mathbf{n}_2 = \vec{w} \times \vec{v} = (-1, -\frac{1}{2}, -\frac{1}{4})$
10. $\mathbf{n}_1 = \vec{v} \times \vec{w} = (-0.32, 0.2032, 0.48)$ and $\mathbf{n}_2 = \vec{w} \times \vec{v} = (0.32, -0.2032, -0.48)$
11. Since $\mathbf{w} = -2\mathbf{v}$ the vectors are antiparallel.
12. Since $\mathbf{v} = 2\mathbf{w}$ the vectors are parallel.
13. Since $\mathbf{v} \times \mathbf{w} = (1, 2, 1) \neq \mathbf{0}$ the vectors are neither parallel nor antiparallel.
14. Since $\mathbf{v} = 2\mathbf{w}/3$, the vectors are parallel.
15. Since $\mathbf{w} = -2\mathbf{v}$ the vectors are antiparallel.
16. $\mathbf{0}$, the zero vector.
17. 1
18. -1
19. $\mathbf{0}$, the zero vector.
20. -1
21. $\theta = \text{Arcsin}\,(1/2) = \pi/6$.
22. $\theta = \text{Arcsin}\,(\sqrt{3}/2) = \pi/3$
23. $\theta = \text{Arcsin}\,(\sqrt{3}/\sqrt{5}) = 0.8861$ rads.
24. Find c first: $c = \pm\sqrt{8} = \pm 2\sqrt{2}$. It follows that $\mathbf{v} \cdot \mathbf{w} = \pm\sqrt{8}$. Since $|\mathbf{v}| = \sqrt{5}$ and $|\mathbf{w}| = 3$ we get $\theta = \text{Arccos}\,(\pm\sqrt{8}/(3\sqrt{5})) = 1.1355$ or 2.006 rads.
25. $\mathbf{v} \times \mathbf{w} = (b+a, -b+a, -2)$ and $\mathbf{v} \times \mathbf{w} = (2, 2, -2)$ both imply that $b+a = 2$ and $-b+a = 2$. So, $a = 2, b = 0$. Thus, $\theta = \text{Arcsin}\,(1) = \pi/2$.
26. Since the area is equal to $(1/2)|\mathbf{v} \times \mathbf{w}|$ we get Area $= (1/2)|(2, -1, 4)|$ $= \sqrt{21}/2$.
27. Area $= \sqrt{54}/2$. Same idea as the previous one.
28. We don't really need S since the area of the parallelogram is simply twice the area of ΔPQR. So, Area $= 2(1/2)|(6, 2, -8)| = \sqrt{104}$.

29. Area $= (1/2)(\sqrt{150} + \sqrt{72})$

30. $\vec{OP} \times \vec{OQ} = (2,2,2)$ and the required volume is $(1,1,1) \cdot (2,2,2) = 6$.

31. 10. Same idea as the preceding one.

32. Expand both sides and simplify.

33. Expand both sides and simplify.

34. Expand both sides and simplify.

35. Expand both sides and simplify.

36. Expand both sides and simplify.

37. Expand both sides and simplify.

38. Expand both sides and simplify.

39. $|\mathbf{v} \times \mathbf{w}|^2 = \sin^2\theta |\mathbf{v}|^2 |\mathbf{w}|^2$ and $|\mathbf{v} \cdot \mathbf{w}|^2 = \cos^2\theta |\mathbf{v}|^2 |\mathbf{w}|^2$.
$\therefore$, $|\mathbf{v} \times \mathbf{w}|^2 + |\mathbf{v} \cdot \mathbf{w}|^2 = |\mathbf{v}|^2 |\mathbf{w}|^2 (\sin^2\theta + \cos^2\theta) = |\mathbf{v}|^2 |\mathbf{w}|^2$.

40. Expand both sides and simplify.

Exercise Set 5.

1. $8x + 13y + z = 32$

2. $2x - 3y + z = 2$

3. $z = 0$

4. $z - y = 3$

5. $y = 2$

6. $x = -2$

7. $2x - y + 3z = 4$

8. $-\frac{x}{2} + \frac{y}{2} - z = 2$

9. $x + y - z = 0$

10. $x + 2y + z = 0$

11. Parallel

12. Not parallel

13. Not parallel

14. Parallel

15. 0.9553 rads (or 54.7^o)

16. 1.3806 rads (or 79.1^o)

17. 1.3508 rads (or 77.4^o)

18. 0.5148 rads (or 29.5^o)

19. $\dfrac{x}{-2} = \dfrac{y-1/2}{1} = \dfrac{z+1/2}{3}$.

20. $\dfrac{x-1}{1} = \dfrac{y-1}{2} = \dfrac{z-0}{-1}$.

21. Let $\mathbf{v} = (0,1,0)$ and $P(0,0,1)$. Then we must have $Ax+By+C(z-1) = 0$ and $B - C = 0$. Combining these two equations we get that the plane $\Pi : Ax + By + B(z-1) = 0$ contains both $\mathbf{v}$ and P. Since A, B can be any two numbers, this gives an infinite number of planes.

Exercise Set 6.

1. $\dfrac{x+1}{2} = \dfrac{y-0}{-1} = \dfrac{z-1}{1}$, or $x = -1+2t, \quad y = -t, \quad z = 1+t.$

2. $\dfrac{x-2}{-2} = \dfrac{y-3}{1} = \dfrac{z-0}{3}$, or $x = 2-2t, \quad y = 3+t, \quad z = 3t.$

3. $\dfrac{x-2}{6} = \dfrac{y+1}{-2} = \dfrac{z-5}{9}$, or $x = 2+6t, \quad y = -1-2t, \quad z = 5+9t.$

4. $\dfrac{x+3}{9} = \dfrac{y-4}{11} = \dfrac{z+9}{1}$, or $x = -3+9t, \quad y = 4+11t, \quad z = -9+t.$

5. $x = 1, \dfrac{y-5}{1} = \dfrac{z+3}{-1}$, or $x = 1, \quad y = 5+t, \quad z = -3-t.$

6. $\dfrac{x-0}{2} = \dfrac{z-1}{2}, y = 0$, or $x = 2t, \quad y = 0, \quad z = 1+2t.$

7. $\dfrac{x+2}{-1} = t, y = 1, z = 0$, or $x = -2-t, \quad y = 0, \quad z = 0.$

8. $\dfrac{x-1}{-2} = \dfrac{y-0}{-1}, z = 1$, or $x = 1-2t, \quad y = -t, \quad z = 1, \quad 0 \le t \le 1.$

9. $\dfrac{x-0}{-1} = \dfrac{y-0}{2} = \dfrac{z-1}{1}$, or $x = -t, \quad y = 2t, \quad z = 1+t, \quad 0 \le t \le 1.$

10. $x = -1, \dfrac{y-0}{2} = \dfrac{z-2}{1}$, or $x = -1, \quad y = 2t, \quad z = 2+t, \quad 0 \le t \le 1.$

11. $\dfrac{x-1}{-1} = t, y = 1, z = -1$, or $x = 1-t, \quad y = 1, \quad z = -1, \quad 0 \le t \le 1.$

12. $\dfrac{x+2}{5/2} = \dfrac{y-1/2}{-3/2}, z = 0$, or $x = -2+\dfrac{5}{2}t, \quad y = \dfrac{1}{2} - \dfrac{3}{2}t, \quad z = 0, \ 0 \le t \le 1.$

13. $x = -1+t, \quad y = 1, \quad z = 0.$

14. $x = t, \quad y = -t, \quad z = 1.$

15. $x = 1+t, \quad y = 3+2t, \quad z = -3-t.$

16. $x = 1, \quad y = 1-t, \quad z = 1+t.$

17. $\dfrac{x-0}{-4} = \dfrac{y-2}{1} = \dfrac{z+1}{5}$

18. $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{-5}$

19. $\dfrac{x-0}{-1} = \dfrac{y+1}{10} = \dfrac{z-0}{7}$

20. $\dfrac{x-1/2}{3} = \dfrac{y-0}{-1} = \dfrac{z-2}{-4}$

21. $x = -1, \quad y = 2, \quad z = 3 + t.$

22. $x = 1 + t, \quad y = -3 + t, \quad z = 2 + t.$

23. $x = +t, \quad y = +t, \quad z = +t.$

24. $x = -t, \quad y = 2 - t, \quad z = -1.$

25. $\cos\theta = 0$ so $\theta = \pi/2$ or $\theta = 90^o$.

26. $\cos\theta = -\dfrac{3}{\sqrt{6}\sqrt{2}} < 0$. So, $\theta = 2.618$ and the angle between the planes is $\pi - 2.618 = 0.5236$ rads, or 30^o.

27. $\cos\theta = \dfrac{1}{\sqrt{2}} > 0$. So, $\theta = \pi/4$ rads, or 45^o.

28. $\cos\theta = \dfrac{1}{\sqrt{84}} > 0$. So, $\theta = 1.4615$ rads, or 83.74^o.

29. $\cos\theta = -\dfrac{12}{25} < 0$. So, $\theta = 2.0714$ and the required angle is $\pi - 2.0714 = 1.0701$ rads, or 61.31^o.

30. $\cos\theta = -\dfrac{7}{5\sqrt{2}} < 0$. So, $\theta = 3$ and the required angle is $\pi - 3 = 0.141$ rads, or 8.13^o.

31. $x = -1 + t, \quad y = 1 - 2t, \quad z = 3t.$

32. $x = 1 - y/2$, $z = y/2 \Longrightarrow x + y - z = (1 - y/2) + y - y/2 = 1 - y + y = 1$ for any value of y. So, every point $P(x, y, z)$ on the line also lies on the plane and the result is clear.

33. $\mathscr{L}_1$ has direction vector $\mathbf{v}_1 = (1, 2, 3)$. $\mathscr{L}_2$ has direction vector $\mathbf{v}_2 = (2, -1, 1)$. So, $\cos\theta = 3/\sqrt{84} > 0$, i.e., $\theta = 1.237$ rads or 70.89^o.

34. The direction $\mathbf{v}$ of the line $\mathscr{L}$ of intersection of the planes is given by finding the cross product of their normal vectors. So, $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (3, -1, 3)$. Next, we need a point on $\mathscr{L}$ to completely determine $\mathscr{L}$. It suffices to choose a point that lies on both the planes. For example, setting $x = 0$ and solving for y, z in the equations for Π_1, Π_2 we get, $y = 3, z = 3$. Thus $(0, 3, 3)$ in on $\mathscr{L}$. The parametric equation of the line is then $x = 0 + 3t, \quad y = 3 - t, \quad z = 3 + 3t$. Incidentally, we now have three points namely, $(0, 2, 6)$, $(0, 3, 3)$ and $(3, 2, 6)$ (set $t = 1$), that can be used to determine the desired plane. Its normal is then given by $(0, -9, -3)$ or, more simply by $(0, 3, 1)$. Its equation is then $0(x-0)+3(y-2)+1(z-6) = 0$ or $3y + z = 12$.

35. This is because we have seen that every unit vector $\mathbf{v} = (\cos\alpha, \cos\beta, \cos\gamma)$ where the cosines are the direction cosines of $\mathbf{v}$. Since the line has direction $\mathbf{v}$, these terms must appear in the denominator of the expression for the symmetric equations.

Exercise Set 7.

1. $(\sqrt{2}, 0)$
2. $(1, -1)$
3. $(1, 0)$
4. $(\sqrt{2}/2, \sqrt{2}/2)$
5. $(3, 2)$
6. $(3.268, 1.522)$
7. $(\sqrt{2}/2, -3\sqrt{2}/2)$
8. $(1, 0)$
9. $(1/2, \sqrt{2}/2)$
10. $(-3.933, -3.087)$
11. $(4.843, 1.241)$
12. $(-6, 3)$

Exercise Set 8.

1. $(\sqrt{2}/2, -3\sqrt{2}/2)$
2. $(0, \sqrt{2}/2)$
3. $(-\sqrt{3} - 1/2, 1 - \sqrt{3}/2)$
4. $(-3.933, -3.087)$
5. $(3\sqrt{3}, 3)$
6. $(2.6\sqrt{2}, -1.4\sqrt{2})$
7. $(1, 0)$
8. $(-\sqrt{2}/8, 3\sqrt{2}/8)$
9. $(\sqrt{3} + 3/2, 1 - 3\sqrt{3}/2)$
10. $(-2.455, 3.313)$
11. $(3.236, 2.351)$
12. $(5, -2)$
13. $(\sqrt{2}, 0)$
14. $(\sqrt{3} + 1/2, 1 - \sqrt{3}/2)$
15. $(3, 1)$

16. $(\sqrt{2}/2, \sqrt{2}/2)$

Exercise Set 9.

1. $(3\sqrt{2}/2, -3\sqrt{2}/2)$
2. $(1/2, \sqrt{3}/2)$
3. $(-1, 0)$
4. $(-1.763, 2.427)$ or $(-3\sin(\pi/5), 3\cos(\pi/5))$.
5. $(\sqrt{2}/2, \sqrt{2}/2)$
6. $(-3.098, -0.634)$ or $(-3\sqrt{3}/2 - 1/2, -3/2 + \sqrt{3}/2)$
7. $(1, -5)$
8. $(7, -2)$
9. $\mathbf{O} = (3, 2)$, $x' = x + 3, y' = y + 2$, $x' - y' = 0$, and $x' + y' = 0$.
10. $\mathbf{O} = (1, 0)$, and use (2.62)-(2.63), with $\theta = \pi/2$. Then $y' = -{x'}^2$

Exercise Set 10.

1. $x = t, y = (t-1)/2$, $0 \le t \le 3$.
2. $x = t, y = t^2 - 1$, $-1 \le t \le 4$
3. $x = t^3 + t + 1, y = t$, $|t| \le 2$.
4. $x = t, y = t^2 - 6$, $-6 \le t < \infty$
5. $x = -t/2, y = t$, $1 \le t \le 2$
6. $x = t, y = \sqrt[3]{2 - t^2}$, $|t| \le \sqrt{2}$
7. $x = t, y = (2t^2 + 1)/3$, $-3 \le t \le 5$
8. $x = (t^2 - 3)/2, y = t$, $0 \le t \le 1$
9. $x = \sqrt[5]{1 - 2t^2}, y = t$, $0 \le t \le 2$
10. $x = t, y = \sqrt{(1 - t^2)/2}$, $-1 \le t \le 1$

Exercise Set 11.

1. $x = 2\cos t, y = 2\sin t, t \in [0, 2\pi]$
2. $x = 1 + \cos t, y = -\sin t, t \in [0, 2\pi]$

3. $x = \sqrt{2}\cos t, y = -1 + \sqrt{2}\sin t, t \in [0, 2\pi]$
4. $x = 1 + \sqrt{12}\cos t, y = -1 + 4\sin t, t \in [0, 2\pi]$
5. $x = -1 + \sqrt{5}\cos t, y = 2 - \sqrt{5}\sin t, t \in [0, 2\pi]$
6. $x = \sqrt{7}\cos t, y = 3 - \sqrt{7}\sin t, t \in [0, 2\pi]$
7. $x = \sqrt{3}\cos t, y = -2\sin t, t \in [0, 2\pi]$
8. $x = 2\cos t, y = 3\sin t, t \in [0, 2\pi]$
9. $x = \sqrt{3}\cos t, y = 1 + \sqrt{5}\sin t, t \in [0, 2\pi]$
10. $x = -2 + \sqrt{2}\cos t, y = -2\sin t, t \in [0, 2\pi]$
11. $x = 1 + \sqrt{12}\cos t, y = -1 + 4\sin t, t \in [0, 2\pi]$
12. $x = -2 + \sqrt{1/2}\cos t, y = 1 + (1/2)\sin t, t \in [0, 2\pi]$

Exercise Set 12.

1. $x = \sqrt{2}\cos t, \quad y = \sqrt{3}\sin t, \quad t \in [0, 2\pi]$.
2. $x = 3\cos t, \quad y = 1 - 2\sin t, \quad t \in [0, 2\pi]$.
3. $x = -1 + \sqrt{2}\cos t, \quad y = 2 + \sqrt{3}\sin t, \quad t \in [0, 2\pi]$.
4. $x = 3 + \sqrt{5}\cos t, \quad y = -1 + \sqrt{2}\sin t, \quad t \in [0, 2\pi]$.
5. $x = -3 + 2\cos t, \quad y = -4 - 3\sin t, \quad t \in [0, 2\pi]$.
6. $x = \frac{1}{2}(2\cos t - \sqrt{6}\sin t), \quad y = \frac{1}{2}(2\cos t + \sqrt{6}\sin t), \quad t \in [0, 2\pi]$. c.c. orientation
7. $x = \frac{3}{2}\sqrt{3}\cos(t) - \sin(t), y = \sqrt{3}\sin(t) + \frac{3}{2}\cos(t) + 1, \quad t \in [0, 2\pi]$. c.c orientation
8. $x = \frac{3}{2}\sin(t) + \frac{\sqrt{2}}{2}\cos(t) - 1, \quad y = -\frac{\sqrt{6}}{2}\cos(t) + \frac{\sqrt{3}}{2}\sin(t) + 2, \quad t \in [0, 2\pi]$. c.c. orientation
9. $x = \frac{\sqrt{2}}{2}(\sqrt{5}\cos(t) - \sqrt{2}\sin(t) + 3\sqrt{2}), y = \frac{\sqrt{2}}{2}(\sqrt{5}\cos(t) + \sqrt{2}\sin(t) - \sqrt{2}), \quad t \in [0, 2\pi]$. c.c. orientation
10. $x = -3 + 3\sin(t), \quad y = -4 - 2\cos(t), \quad t \in [0, 2\pi]$. c.c. orientation

Exercise Set 13.

1. $x = t, y = \frac{1}{2}(t+1)^2 - 1$
2. $x = t^2 - t, y = t$
3. $x = 3 - 4(t-2)^2, y = t$
4. $x = t, y = 3 - 4(t-2)^2$

5. $x = -19t^2 + (19/2)\sqrt{2}t - 229/152, \quad y = 19t^2 - 113/152.$
6. $x = -6t^2 + 6t - 9/8, \quad y = 6t^2 + 5/8.$

Exercise Set 14.

1. Hyperbola
2. Hyperbola
3. Ellipse
4. Parabola
5. Hyperbola
6. Parabola
7. Hyperbola
8. Ellipse
9. $x = \pm 2\cosh t, \quad y = \pm\sqrt{2}\sinh t.$
10. $x = 1 \pm \sqrt{2}\cosh t, \quad y = -1 \pm \sqrt{2}\sinh t.$
11. $x = \pm\sinh t, y = \pm\cosh t.$
12. $x = (1/2)(\cosh t + \sinh t), \quad y = (1/2)(\cosh t - \sinh t)$, $t \in \mathbf{R}$

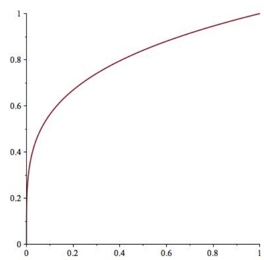
Exercise Set 15.

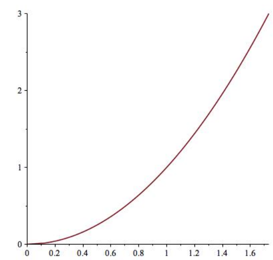
1. $y = \sqrt[3]{x/2}$, $x \in \mathbf{R}$.
2. $y = x^2$, $x \geq 0$.
3. $x^2 + y^2 = 9$, $x, y \in [-3, 3]$.
4. $y = \sin(\cos x)$, $x \in [0, \pi/2]$.
5. $x^2 + (y/2)^2 = 1$, $|x| \leq 1$, $|y| \leq 2$.
6. $y = (x-1)^{3/2} + \sqrt{x-1} - 1$, $1 \leq x \leq 2$.
7. $(x-1)^2 - y^2/9 = 1$, $x \leq 0$, or $x \geq 2$.
8. $y = \cos(2\ln(x-1))$, $x > 1$.

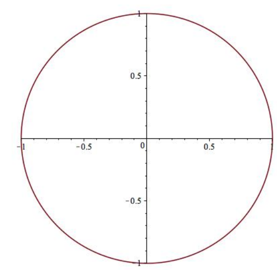
Exercise Set 16.

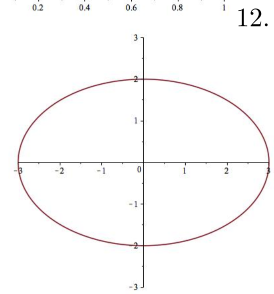
1. $y'(x) = 1/(4t^{3/2})$, $y''(x) = -3/(16t^{7/2})$

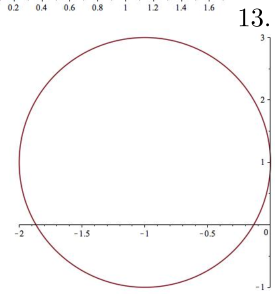
2. $y'(x) = 2\sqrt{t+1}$, $y''(x) = 2$
3. $y'(x) = -\tan t$, $y''(x) = -\sec^3 t$
4. $y'(x) = -(2/3)\tan t$, $y''(x) = -(2/9)\sec^3 t$
5. $y'(x) = 2\tan t$, $y''(x) = 2\sec^3 t$
6. $y'(x) = (1/2)\text{tanh}\, t$, $y''(x) = (1/4)\text{sech}^3\, t$
7. $y'(x) = 3\text{tanh}\, t$, $y''(x) = 3\text{sech}^3\, t$
8. $y'(x) = 6t/(2t-1)$, $y''(x) = -6/(2t-1)^3$
9. $y'(x) = -2/(t-1)^2$, $y''(x) = 4/(t-1)^3$
10. $y'(x) = -e^{-2t}$, $y''(x) = 2e^{-3t}$

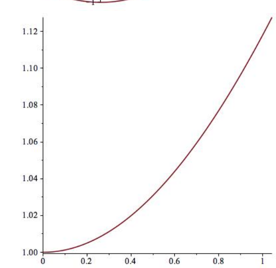
11. 

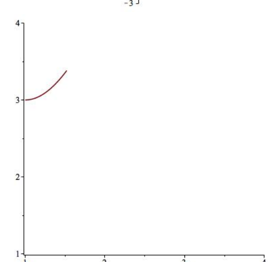
12. 

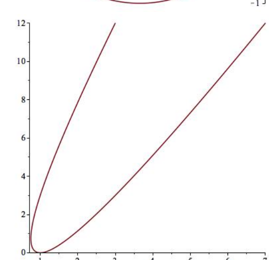
13. 

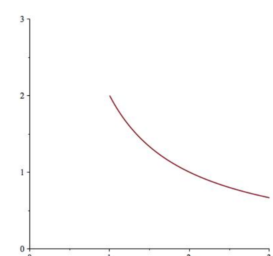
14. 

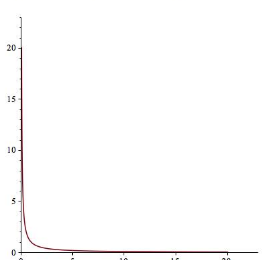
15. 

16. 

17. 

18. 

19. 

20. 

Exercise Set 17.

1. 4π
2. 9π
3. 4π
4. 2π
5. $1/4$
6. $2\sqrt{2}/15$
7. $\pi^2/4$
8. $81/20$
9. $1096/105$
10. $1/60$

Exercise Set 18.

1. $52/3$
2. $2\sqrt{2}$
3. $4\sqrt{2}$
4. $2\sqrt{2}$
5. $2\ln(2)\sqrt{5}$
6. 2π
7. $-(1/2)\sqrt{2} - (1/2)\ln(1+\sqrt{2}) + \sqrt{5} - (1/2)\ln(-2+\sqrt{5})$
8. $(1/2)\sqrt{5} - (1/4)\ln(-2+\sqrt{5})$
9. $16\int_0^{\pi/2}\sqrt{1-(3/4)\sin^2 t}\,dt = 19.38$
10. $3/2$

Exercise Set 19.

1. $(-3, 0)$
2. $(\sqrt{3}, -1)$
3. $(-4\sqrt{2}, 4\sqrt{2})$
4. $(2, -2\sqrt{3})$

5. $(1, \sqrt{3})$

6. $(0, 4)$

7. $(0.97, 0.7)$

8. $(-0.9, -0.43)$

9. $(-2.5, 4.3)$

10. $(-1.73, 1)$

11. $(\sqrt{2}, \pi/4)$ or $(-\sqrt{2}, 5\pi/4)$

12. $(3\sqrt{2}, 5\pi/4)$ or $(-3\sqrt{2}, \pi/4)$

13. $(\sqrt{5}, -0.46)$ or $(-\sqrt{5}, 2.68)$

14. $(4, \pi/2)$ or $(-4, -\pi/2)$

15. $(4, \pi)$ or $(-4, 0)$

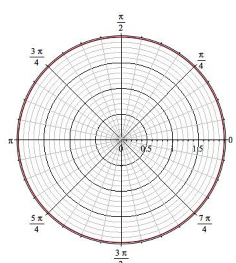
16. $(\sqrt{5}, \text{Arctan}\,(-2) + 2n\pi)$ or $(-\sqrt{5}, \text{Arctan}\,(-2) - \pi + 2n\pi)$, $n = \pm 1 \pm 2, \ldots$

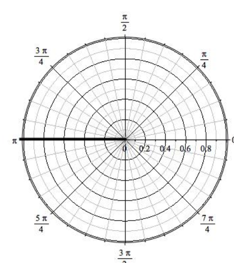
17. $(\sqrt{13}, \text{Arctan}\,(-2/3) + 2n\pi)$, or $(-\sqrt{13}, \text{Arctan}\,(-2/3) - \pi + 2n\pi)$

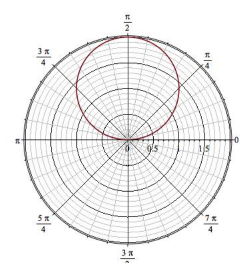
18. $(5, \text{Arctan}\,(4/3) + 2n\pi)$, or $(-5, \text{Arctan}\,(4/3) - \pi + 2n\pi)$, $n = \pm 1 \pm 2, \ldots$

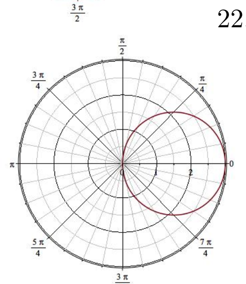
19. $(5, -\pi/2 + 2n\pi)$ or $(-5, \pi/2 + 2n\pi)$, $n = \pm 1 \pm 2, \ldots$

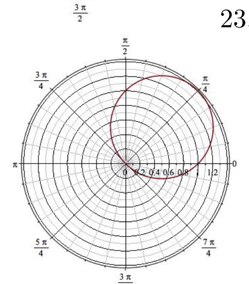
20. $(\sqrt{2}, 5\pi/4 + 2n\pi)$ or $(-\sqrt{2}, 5\pi/4 - \pi + 2n\pi)$, $n = \pm 1 \pm 2, \ldots$

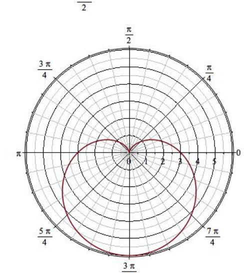
21. 

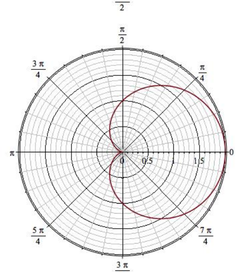
22. 

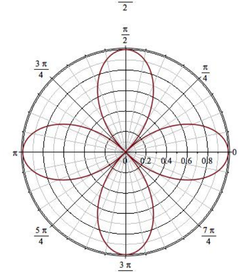
23. 

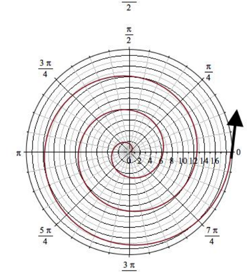
24. 

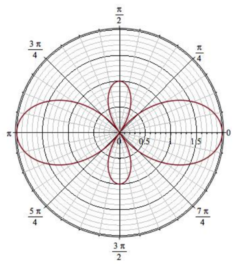
25. 

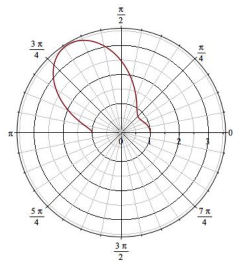
26. 

27. 

28. 

29. 

30. 

31. 

32. $x^2 + y^2 = 5x$
33. $x^2 + y^2 = 2y - x$
34. $r = (1/2)(\cos\theta - \sin\theta)$.
35. $r^2 \sin 2\theta = 6$
36. $y = (\tan\alpha)x$
37. $y = -2$
38. $\tan\theta = -1$
39. $r = -4\sec 2\theta$.

Exercise Set 20.

1. $\{(r,\theta) : 0 \le r \le 1, 0 \le \theta \le \pi/2\}$
2. $\{(r,\theta) : 0 \le r \le 2, 0 \le \theta \le \pi\}$
3. $\{(r,\theta) : 0 \le r \le \sin 2\theta, 0 \le \theta \le \pi/2\}$
4. $\{(r,\theta) : \sqrt{3} \le r \le 2\sin\theta, \pi/3 \le \theta \le 2\pi/3\}$
5. $\{(r,\theta) : 3/2 \le r \le 1 + \cos\theta, -\pi/3 \le \theta \le \pi/3\}$
6. $\{(r,\theta) : (4/3)\cos\theta \le r \le 1/(1+\cos\theta), -\pi/3 \le \theta \le \pi/3\}$
7. $\{(r,\theta) : 4\csc\theta \le r \le 8, \pi/6 \le \theta \le 5\pi/6\}$
8. $\{(r,\theta) : 3 + 2\sin\theta \le r \le 2, 7\pi/6 \le \theta \le 11\pi/6\}$
9. $\{(r,\theta) : 0 \le r \le \sin\theta, 0 \le \theta \le \pi/4\} \cup \{(r,\theta) : 0 \le r \le \cos\theta, \pi/4 \le \theta \le \pi/2\}$
10. $\{(r,\theta) : 0 \le r \le \sec\theta, 0 \le \theta \le \pi/4\} \cup \{(r,\theta) : 1 \le r \le \sec\theta, \pi/4 \le \theta \le \pi/2\}$

Exercise Set 21.

1. $0.37 \cdot 3^2 = 3.33$
2. $2\tan(\pi/3) = 2\sqrt{3}$

3. 24π
4. $\pi/3$
5. $4\pi/3 + 2\sqrt{3}$
6. $\pi/16$
7. $9\pi/2$
8. $\pi/2 - 1$
9. $1/2 + 3\pi/8 + \ln 2$
10. $25\sqrt{3}/2 + 25\pi/3$
11. 1
12. $4\pi^3/3$

Exercise Set 22.

1. $\pi/2$
2. $\sqrt{2}(1 - e^{-4})$
3. $11\pi/20$
4. $\dfrac{3\pi\sqrt{2}}{2}$
5. $\pi\sqrt{13}$
6. $\frac{2\sqrt{10}}{3}(e^9 - 1)$
7. $\frac{\sqrt{1+(\ln 3)^2}}{\ln 3}(9^\pi - 1)$
8. 4
9. $\frac{8}{3}(\pi^2\sqrt{\pi^2+1} + \sqrt{\pi^2+1} - 1)$
10. $\pi/2 + 3\sqrt{3}/8$
11. ≈ 0.642
12. $16/3$

Exercise Set 23.

1. Continuous as it is a polynomial.
2. Not continuous as limit doesn't exist. Try directions $x = 0$ and $y = x$. Both give different limits.
3. Continuous.

4. Continuous.
5. Not continuous as limit doesn't exist. Try the directions $y = 0$ to see this as it gives a value of $0 \neq f(1, 0)$.
6. Continuous.
7. Not continuous. Limits exist and are 0 but $f(0, -1) \neq 0$.
8. Not continuous. Limits don't exist.
9. Continuous.
10. Continuous
11. Discontinuous
12. Continuous
13. Discontinuous
14. Continuous
15. Continuous
16. $\sqrt{3}$
17. 36
18. π
19. 27
20. All points on or to the right of $x = y^2$
21. 5
22. 8
23. 0
24. Does not exist
25. 1
26. 2
27. 1
28. 2
29. 0
30. Does not exist
31. Does not exist

Exercise Set 24.

1. $f_x = 3x^2 + 2y,\ f_y = 2x - 2y.$

2. $f_x = -y^2/x^2,\ f_y = 2y/x.$

3. $f_x = 2xe^{xy}(2+xy),\ f_y = 2x^3e^{xy}$

4. $f_x = e^{3y},\ f_y = 3e^{3y}(x+\ln y) + e^{3y}/y$

5. $f_x = 2(x-2y+3),\ f_y = -4(x-2y+3)$

6. $f_x(-3,2) = 2,\ f_y(-3,2) = 3.$

7. $f_x(4,-1) = 1/3,\ f_y(4,-1) = -1/3.$

8. $f_x(1,2) = 6,\ f_y(1,2) = -1.$

9. That f is not continuous at $(0,0)$ is easy because $f(x,0) = 0$ yet $f(x,x) = 2$ for all x. Taking a limit as $x \to 0$ gives the result. Use the definition of partial derivatives to show that $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

10. We use the definition of continuity of f at $(0,1)$: Let $\varepsilon > 0$ be a given number. We need to find a number $\delta > 0$ (usually depending on ε) such that if $\sqrt{x^2+(y-1)^2} < \delta$, then $|f(x,y) - f(0,1)| < \varepsilon$.

    First note that $\sqrt{x^2+(y-1)^2} < \delta$ implies that $|x| = \sqrt{x^2} \le \sqrt{x^2+(y-1)^2} < \delta$. In addition, $x^2+(y-1)^2 \ge (y-1)^2$. From the theory of inequalities we get that

    $$|f(x,y) - f(0,1)| = \frac{x^2(y-1)^2}{x^2+(y-1)^2} \le \frac{x^2(y-1)^2}{(y-1)^2} = x^2 = |x|^2 < \delta^2.$$

    So, if we choose δ so that $\delta^2 = \varepsilon$ then we will indeed have $|f(x,y) - f(0,1)| < \varepsilon$. In other words, $\delta = \sqrt{\varepsilon}$ satisfies the definition, that is, given any number $\varepsilon > 0$ there is a δ (namely $\delta = \sqrt{\varepsilon}$) such that whenever the points (x,y) lie in the circle $\sqrt{x^2+(y-1)^2} < \delta$, of radius δ, then $|f(x,y) - f(0,1)| < \varepsilon$.

11. Recall that $f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$ and similarly for f_y. Setting $a = b = 0$ and using the given fact that $f(0,0) = 0$ we get that $\frac{f(h,0)-f(0,0)}{h} = \frac{f(h,0)}{h} = 0$ for any $h \ne 0$. Taking the limit gives 0. This proves that $f_x(0,0) = 0$. A similar proof applies in the case of f_y and $f_y(0,0) = 0$.

Exercise Set 25.

1. $\dfrac{x^y}{x}(y\ln(x)+1)$

2. $xy2^{xy}\ln(2)^2 + 2^{xy}\ln(2)$

3. $\dfrac{y^x}{y}(x\ln(y)+1)$

4. $-\dfrac{2(-y^3+x^2)}{(y^3+x^2)^2}$

5. $2x\left(yx^2+1\right)e^{yx^2}$

6. $e^x \cos y$
7. $12xyz$
8. $-\sin(zx\cos(y))z^2\cos^2(y)$
9. 0
10. $y(-xyz\sin(xyz)+\cos(xyz))$
11. 0
12. $-\dfrac{zx}{(x^2+y^2+z^2)^{3/2}}$
13. $f_x(\pi/4,-1,2)=0$
14. $f_{xy}=3x\cos(x^2y)-2x^3y\ln(xyz)\sin(x^2y)+2x\ln(xyz)\cos(x^2y)$.
15. $f_{xyz}=6\,3^{xy^3z^2}y^2z\ln(3)\left((\ln 3)^2x^2y^6z^4+3xy^3z^2\ln 3+1\right)$.
16. $f_{xxz}=12xyz$.
17. $f_{yxz}=\dfrac{16xyz}{(x^2+z^2)^3}$.
18. $f_{xx}=6xy,\ f_{yy}=-4x,\ f_{xy}=f_{yx}=3x^2-4y$.
19. $f_{xx}=f_{yy}=-\frac{2(x^2+y^2)}{(x^2-y^2)^2},\ f_{xy}=f_{yx}=\frac{4xy}{(x^2-y^2)^2}$
20. $f_{xx}=(x-2)e^{y-x},\ f_{yy}=xe^{y-x},\ f_{xy}=f_{yx}=(1-x)e^{y-x}$.
21. $f_{xx}=8,\ f_{yy}=-6x+6y,\ f_{xy}=f_{yx}=-6y$.
22. -2
23. 4

Exercise Set 26.

1. Not differentiable because $f_y(0,0)$ does not exist, even though $f_x(0,0)=1$.
2. Differentiable as it is a polynomial (so both partial derivatives exist and are continuous everywhere).
3. Not differentiable because $f_x(0,0)$ and $f_y(0,0)$ do not exist.
4. Differentiable, as each term is differentiable then so is their product.
5. Not differentiable by definition
6. Not differentiable, even though partials exist and are 0 at the point, the definition of differentiability fails.
7. Differentiable as f is a polynomial.
8. Not differentiable, as the partial derivatives do not exist at O.
9. Not differentiable as the partial derivative wrt x doesn't exist at the point, even the other partial derivatives *do* exist there.

10. Not differentiable as the partial derivative wrt x doesn't exist at the point.
11. $3\,dx + 2\,dy$
12. $3\,dx + 2y\,dy$

Exercise Set 27.

1. $14/5$
2. $7\sqrt{2}/50$
3. $6/\sqrt{5}$
4. $2\sqrt{3} - 4$
5. $4/5$
6. $10/3$
7. $2/\sqrt{6}$
8. $11/3$
9. $\sqrt{2}/2$
10. 1
11. $\sqrt{80}\ln 2$.
12. $\sqrt{13}/37$
13. $-1/12$
14. $(-1, -2, -2)$; rate of decrease $= 3$
15. $(1/6, 2/3, -1/2)$, rate of increase $= \sqrt{26}/6$
16. $(0, -4, 8)$, rate of increase $= \sqrt{80} = 4\sqrt{5}$
17. $(4\sin 6 + 36\cos 6, +6\sin 6 + 24\cos 6) = (-33.4, -21.4)$.
18. $(-4, -2)$. Maximum rate $= 2\sqrt{5} = 4.47$.
19. B, $y^2\mathbf{i} + 2xy\mathbf{j}$
20. C, $\mathbf{i} + 2y\mathbf{j}$
21. A, $e\mathbf{i} + 2e\mathbf{j}$
22. B, $\sin y\mathbf{i} + x\cos y\mathbf{j}$
23. B, $2\mathbf{i} + 3\mathbf{j}$
24. C, $-0.08\mathbf{i} - 0.16\mathbf{j}$
25. B, $2\sqrt{3}$
26. D, $3i + 2j$
27. A, $\frac{1}{6}$

28. A, $\frac{17}{3}$
29. B, $(1, 2)$

Exercise Set 28.

1. Use the Mean Value Theorem of this section. Let $\mathbf{u}$ be any vector, Now, writing $\mathbf{0} = (0, 0)$ we know that $f(\mathbf{u}) - f(\mathbf{0}) = \nabla f(P^*) \cdot (\mathbf{u} - \mathbf{0}) = \nabla f(P^*) \cdot \mathbf{u}$, where P^* is some point in $\mathscr{D}$.

   In addition, the assumption $D_{\mathbf{u}} f(P) = 0$ for all points $P \in \mathscr{D}$ is equivalent to saying that $\nabla f(P) \cdot \mathbf{u} = 0$ (even though $\mathbf{u}$ is not necessarily a unit vector, this is still true, see?) It follows that, in particular, $D_{\mathbf{u}} f(P^*) = 0$ too, that is, $\nabla f(P^*) \cdot \mathbf{u} = 0$, too. Therefore, $f(\mathbf{u}) - f(\mathbf{0}) = 0$ or $f(\mathbf{u}) = f(\mathbf{0})$. Since $\mathbf{u}$ is arbitrary, this means that the function f takes on the value $f(\mathbf{0})$ at any vector/ point $\mathbf{u}$, or that f or a constant function.

Exercise Set 29.

1. 0
2. $y_r = 3(t^2 + r - s + 1)^2$, $y_t = 6t\,(t^2 + r - s + 1)^2$.
3. $f_s = 3s^2t^3 + 4st^2 - 3t$, $f_t = 3s^3t^2 + 4s^2t - 3s$
4. $f_s = -s/\sqrt{r^2 - s^2}$, $f_r = r/\sqrt{r^2 - s^2}$
5. $df/dt = 4(2t^3 + 3t^2 - 6t + 1)$
6. $f_x = \frac{yz \cos(\sqrt{xyz+1})}{2\sqrt{xyz+1}}$, $f_y = \frac{xz \cos(\sqrt{xyz+1})}{2\sqrt{xyz+1}}$, $f_z = \frac{xy \cos(\sqrt{xyz+1})}{2\sqrt{xyz+1}}$. Finally, when $x = 1, y = -1, z = -1$ we get $f_x = -\cos(\sqrt{2})/(2\sqrt{2})$
7. $f_s = 2^{2+2s} \ln 2.$, $f_t = 4t$
8. $f_s = 2(s + t - 1)e^{(s+t-1)^2}$, $f_t = 2(s + t - 1)e^{(s+t-1)^2}$.
9. $f_s = 10s, f_t = 10t$
10. $f_x = 4x(x^2 + y^2), f_y = 4y(x^2 + y^2)$
11. $f_s = 4s(s^2 + t^2)\cos((s^2 + t^2)^2)$, $f_t = 4t(s^2 + t^2)\cos((s^2 + t^2)^2)$
12. $df/dt = (2\cos^2 t + \sin t \cos t - 1)e^t$, $d^2f/dt^2 = -(2 - 4\cos^2 t + 3\sin t \cos t)e^t$
13. $f_s = 22s + 2t$, $f_t = 2s + 10t$
14. $1/\pi$
15. -1
16. 1
17. 6

18. $z_r = 0, z_\theta = -4$
19. $z_\rho = 2\rho,\ z_\theta = 0$
20. $f_s = 0,\ f_t = 0$

Exercise Set 30.

1. $\mathbf{n} = (1, 0),\ \mathbf{T} = (0, 1)$.
2. $\mathbf{n} = (2, -1)\sqrt{5},\ \mathbf{T} = (1, -2)/\sqrt{5}$.
3. $\mathbf{n} = (2, 0, -1)/\sqrt{5}$, or any other vector (a, b, c) whose scalar product with $\mathbf{T}$ gives zero. $\mathbf{T} = (2, -3, 4)/\sqrt{29}$.
4. $\mathbf{n} = (0, -1, 0),\ \mathbf{T} = (1, 0, -1)/\sqrt{2}$.
5. $\mathbf{n} = (2, 1)/\sqrt{5},\ \mathbf{T} = (1, -2)/sqrt5$.
6. $\mathbf{n} = (\sqrt{3}, 1)/2,\ \mathbf{T} = (-1, \sqrt{3})/2$.
7. Let $\mathbf{r}(t_0) = P(x_0, y_0, z_0)$ on $\mathscr{C}$. Then $\frac{x-x_0}{t_0^3} = \frac{y-y_0}{t_0^2} = \frac{z-z_0}{t_0}$ is the tangent line and $t_0^3(x - x_0) + t_0^2(y - y_0) + t_0(z - z_0) = 0$ is the tangent plane.
8. Note that $t = 6$. So, we get $\frac{x-6a}{a} = \frac{y-18a}{6a} = \frac{z-72a}{36a}$, and $a(x-6a)+6a(y-18a) + 36a(z - 72a) = 0$.
9. $x + 1 - \pi/2 = y - 1 = \frac{z-2\sqrt{2}}{\sqrt{2}}$ and $x + y + \sqrt{2}\,z = 4 + \pi/2$.
10. $x = e + e\,t, y = \frac{1}{e} - \frac{t}{e}, z = \sqrt{2} + \sqrt{2}\,t$ and $ex - \frac{y}{e} + \sqrt{2}\,z = 2 - \frac{1}{e^2} + e^2$.
11. $x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,t, y = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,t, z = 1 + 2t$, and $x - y + 2\sqrt{2}\,z = \sqrt{2}$.
12. $x = 1 + 2t, y = -t, z = 1 + 3t$ and $2x - y + 3z = 5$.
13. $x = -1 + 8t, y = 13 + 12t, z = 24t$ and $2x + 3y + 6z = 7$.
14. $\frac{x-1}{1} = \frac{y-3}{-1/3} = \frac{z-4}{1/4}$ or $\frac{x-1}{12} = \frac{y-3}{-4} = \frac{z-4}{3}$, and $12x - 4y + 3z = 12$
15. Let $x_0 \geq 0$. Then $x = y = t$ and $z = t\sqrt{2}$. So, we get $\frac{x-x_0}{1} = \frac{y-y_0}{1} = \frac{z-z_0}{\sqrt{2}}$, and $x + y + \sqrt{2}\,z = 4x_0$.
    If $x_0 < 0$, then let $x = y = t, z = -t\sqrt{2}$, where $t < 0$. In this case we get $\frac{x-x_0}{1} = \frac{y-y_0}{1} = \frac{z-z_0}{-\sqrt{2}}$, and $x + y - \sqrt{2}\,z = 4x_0$
16. Let $y = t$! Then $\mathscr{C} = (t^2, t, t^4)$. We get $\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-1}{4}$, and $2x + y + 4z = 7$.
17. $\frac{x-1}{1} = \frac{y-2}{1/2} = \frac{z-2}{-1/2}$ and $2x + y - z = 2$
18. $2x + 4y + z = 15,\ x = 1 - 2t,\ y = 2 - 4t,\ z = 5 - t$.
19. $4x - 4y - z = 6,\ x = 2 + 4t,\ y = -1 - 4t,\ z = 6 - t$.
20. $3x + 4y - 3z = 14$, and $x = 1 + 3t,\ y = 2 + 4t,\ z = -1 - 3t$.
21. $z = 1$ is the tangent plane. $x = y = 0, z = 1 + 2t$ is the normal line.

22. $8x - 8y - z = 4$, and $\frac{x-2}{8} + \frac{y-1}{-8} + \frac{z-4}{-1}$.

23. $x + y - z = 1$ and $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{-1}$.

24. $z = -1$ is the tangent plane while $x = 1, y = 1, z = -1 + t$ is the normal line.

25. $x - y + 2z - \frac{\pi}{2} = 0$, and $\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-\frac{\pi}{2}}{2}$

26. $17x + 11y + 5z = 60$ and $\frac{x-3}{17} = \frac{y-4}{11} = \frac{z+7}{5}$.

27. $x + 11y + 5z = 18$ and $\frac{x-1}{1} = \frac{y-2}{11} = \frac{z+1}{5}$.

28. $2x + y + 11z = 25$ and $\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-2}{11}$.

29. $5x + 4y + z = 28$ and $\frac{x-2}{5} = \frac{y-3}{4} = \frac{z-6}{1}$.

30. $-3(x-2) - 2(y-1) + (z-4) = 0$ or $3x + 2y - z = 4$.

31. $2x - 2y - z = 6$

32. For example, $z = \sqrt{x^2 + y^2}$ at $(0, 0)$. It cannot have a normal line as one or more of the partial derivatives may not exist at the point in question.

33. The equation is equivalent to the ray $x + y/2 = 0$ or $y = -2x$, through the origin. Thus $(1, -2)$ is a tangent vector and $(2, 1)$ is a normal vector everywhere, including $(0, 0)$.

34. The tangent plane is parallel to the xy-plane at the points $(0, 3, 3)$ and $(0, 3, -7)$. It is parallel to the yz-plane at the points $(5, 3, -2)$ and $(-5, 3, -2)$. It is parallel to the xz-plane at the points $(0, -2, -2)$ and $(0, 8, -2)$.

35. $4x - 2y - 3z = 3$.

Apologies for the identically numbered questions 35.

35. $4i + 2j + k$.

36. $x + 2y + 2z = 9$

37. $z = 2x - 1$

38. $z = 3x - 2$

39. $z = 2x + 2y - 2$

40. $z = 2x + 4y - 3$

41. $z = x + y + 1$

42. $z = 2x + 2y + 1$

43. 4

44. 7

45. 26

46. 4

47. -1

48. 4

49. $2\sqrt{2}$
50. $(0,1,1)$
51. $3\mathbf{i}+e\mathbf{j}+\frac{\pi}{4}\mathbf{k}$
52. $(1,-1,e)$
53. $\mathbf{i}+\mathbf{j}$
54. $(1,2,3)$
55. $\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}$
56. $(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}},0)$
57. $\mathbf{i}$

Exercise Set 31.

1. No
2. Yes. A potential function is given by $f(x,y,z)=y+x^3y+3z^3+C$, where C is a constant.
3. Yes. A potential function is given by $f(x,y,z)=x^2y+yz+C$, where C is a constant.
4. No
5. No
6. Yes. A potential function is given by $f(x,y,z)=xy+\sin xz+C$, where C is a constant.
7. Yes. A potential function is given by $f(x,y,z)=xyz+C$, where C is a constant.
8. Yes. A potential function is given by $f(x,y)=x+x^2y^3-y^2+C$, where C is a constant.
9. No
10. Yes. A potential function is given by $f(x,y)=x+\frac{x^2\sin y}{2}+C$, where C is a constant.
11. $\operatorname{div}\mathbf{F}=3$, curl $\mathbf{F}=\mathbf{0}$.
12. $\operatorname{div}\mathbf{F}=x+z$, curl $\mathbf{F}=(2x^2y,x-2xy^2,y)$.
13. $\operatorname{div}\mathbf{F}=e^x\cos y+e^y\cos z+e^z\cos y$; curl $\mathbf{F}=(e^y\sin z,e^z\sin x,e^x\sin y)$.
14. $\operatorname{div}\mathbf{F}=3\sin^2 x\cos x+3\sin^2 y\cos y+3\sin^2 z\cos z$; curl $\mathbf{F}=\mathbf{0}$.
15. $\operatorname{div}\mathbf{F}=1/x+1/y+1/z$; curl $\mathbf{F}=\mathbf{0}$.
16. $\operatorname{div}\mathbf{F}=\frac{2x}{x^2+y^2}+\frac{2y}{y^2+z^2}+\frac{2z}{z^2+x^2}$; curl $\mathbf{F}=\left(-\frac{2z}{y^2+z^2},-\frac{2x}{z^2+x^2},-\frac{2y}{x^2+y^2}\right)$.

17. $2(x+y+z)$

18. curl $\mathbf{F} = (2x^2yz + y^2 + 2z^3 + C$.

19. 0

20. 5/9

21. 16.

22. Find the partial derivatives, add them, and simplify. For example, $f_x = \frac{2x}{x^2+y^2}$, $f_y = \frac{2y}{x^2+y^2}$. Then $f_{xx} = \frac{-2(x^2-y^2)}{(x^2+y^2)^2}$ and $f_{yy} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$. So, their sum is zero.

23. Similar to the preceding one, just one dimension higher: Here $f_{xx} = -\frac{2(x^2-y^2-z^2)}{(x^2+y^2+z^2)^2}$, $f_{yy} = \frac{2(x^2-y^2+z^2)}{(x^2+y^2+z^2)^2}$, $f_{zz} = \frac{2(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}$. Adding them up and simplifying we get the result.

    Writing $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ we get $f_{xx} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$, $f_{yy} = -\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$, $f_{zz} = -\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$. Adding them and simplifying we get the result.

24. Use the definitions, expand and then simplify the result.

25. Use the definitions, expand and then simplify the result.

26. $f \triangle g - g \triangle f$

27. Assume $\mathbf{F}, \mathbf{G}$ are each twice differentiable. Then we know that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ and similarly, $\nabla \cdot (\nabla \times \mathbf{G}) = 0$. But $\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot (c\mathbf{G}) = c \nabla \cdot \mathbf{G}$. Since $c \neq 0$, $\therefore \nabla \cdot \mathbf{G} = 0$. A similar proof shows that $\nabla \cdot \mathbf{F} = 0$.

    Since $\nabla \cdot \mathbf{F} = 0$ use of Exercise 25 above gives us that $\nabla \times (\nabla \times \mathbf{F}) = -\triangle \mathbf{F}$. But $\nabla \times \mathbf{F} = c\mathbf{G}$. $\therefore -\triangle \mathbf{F} = \nabla \times (c\mathbf{G}) = c(\nabla \times \mathbf{G}) = c^2 \mathbf{F}$, and the result is now clear. A similar proof applies to the other quantity.

28. $\nabla \times \mathbf{F} = (2x^2y - 2x^2z, 2y^2z - 2xy^2, 2xz^2 - 2yz^2)$. It follows that $\mathbf{F} \cdot (\nabla \times \mathbf{F}) = 0$ and so the two vector fields must be orthogonal.

29. No

30. Yes. A potential function is given by $f(x,y,z) = y + x^3y + 3z^3 + C$, where C is a constant.

31. Yes. A potential function is given by $f(x,y,z) = x^2y + yz + C$, where C is a constant.

32. No

33. No

34. Yes. A potential function is given by $f(x,y,z) = xy + \sin xz + C$, where C is a constant.

35. Yes. A potential function is given by $f(x,y,z) = xyz + C$, where C is a constant.

36. Yes. A potential function is given by $f(x,y) = x + x^2y^3 - y^2 + C$, where C is a constant.

37. No

38. Yes. A potential function is given by $f(x,y) = x + \frac{x^2 \sin y}{2} + C$, where C is a constant.

Exercise Set 32.

1. $-3\sqrt{14}/2$
2. 1386.63
3. $\frac{1}{3}(5\sqrt{5} - 1)$
4. 24.
5. $\sqrt{5}\log 2$
6. $m = \sqrt{2}\left(\frac{8\pi^3}{3} + 2\pi\right)$; $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{3\pi(2\pi^2+1)}{4\pi^2+3}, 0, 0\right)$.
7. $m = 2r^2$; $(\bar{x}, \bar{y}) = \left(\frac{r(\pi+2)}{8}, \frac{r(\pi+2)}{8}\right)$.
8. $\sqrt{2}/4$.
9. 1
10. 18
11. $\frac{20\sqrt{5}}{3} + \frac{4}{15}$
12. $\frac{5+2\sqrt{2}}{3}$, the same answer!

Exercise Set 33.

1. $-\dfrac{11}{12}$
2. $\dfrac{8e-5}{8}$
3. $-\dfrac{29}{30}$
4. $\dfrac{\pi^6}{192}$
5. $\dfrac{15 + \cos 1 - \cos 4}{2}$
6. 8
7. a)$\dfrac{1}{3}$, b)$\dfrac{1}{12}$, c)$\dfrac{17}{30}$, d) $-\dfrac{1}{20}$.
8. 0
9. $-2\pi ab$

10. $-\dfrac{4R}{3}$
11. 13
12. 0
13. 1
14. 4/3
15. 4/15
16. 8/15
17. $e + 7/20$
18. 11/3
19. 2π
20. 119/30

Exercise Set 34.

1. $f(x, y) = y\sin x + x\cos y + C$
2. $f(x, y) = xye^{xy} + x + y + C$
3. $f(x, y) = x^y + C$
4. $f(x, y) = \text{Arctan}\,(e^{xy}) + C$
5. $f(x, y) = \sin(\ln(xy)) + C$
6. $f(x, y, z) = \ln(xyz) + C$
7. $f(x, y, z) = \ln(x^2 + y^2 + z^2) + C$
8. $f(x, y, z) = \sin(xyz) + C$.
9. $f(x, y, z) = \ln(\sin(xyz)) + C$
10. $f(x, y, z) = \sin(xy) + \sin(yz) + \sin(xz) + C$.
11. $-\pi/2$.
12. 1
13. 3
14. 0
15. -1

Exercise Set 35.

1. a) Using vertical slices:

$$\mathscr{S} = \{(x, y) : 0 \leqslant y \leqslant x^2, 0 \leqslant x \leqslant 2\},$$

or,
b) Using horizontal slices:

$$\mathscr{S} = \{(x, y) : \sqrt{y} \leqslant x \leqslant 2, 0 \leqslant y \leqslant 4\}.$$

Both descriptions are correct.

2. a) Using vertical slices:

$$\mathscr{T} = \{(x, y) : 2x \leqslant y \leqslant 2, 0 \leqslant x \leqslant 1\},$$

or,
b) Using horizontal slices:

$$\mathscr{T} = \left\{(x, y) : 0 \leqslant x \leqslant \frac{y}{2}, 0 \leqslant y \leqslant 2\right\}.$$

Both descriptions are correct.

3. Using either vertical or horizontal slices we find:

$$\mathscr{R} = \{(x, y) : -2 \leqslant x \leqslant 3, -1 \leqslant y \leqslant 4\}.$$

Both descriptions are given by the same set of inequalities as the region is a rectangle.

4. a) Using vertical slices:

$$\mathscr{S} = \{(x, y) : x^2 - 4 \leqslant y \leqslant 4 - x^2, -2 \leqslant x \leqslant 2\}$$

or,
b) Using horizontal slices: $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2$ where

$$\mathscr{S}_1 = \left\{(x, y) : -\sqrt{4-y} \leqslant x \leqslant \sqrt{4-y}, 0 \leqslant y \leqslant 4\right\},$$

and

$$\mathscr{S}_2 = \left\{(x, y) : -\sqrt{4+y} \leqslant x \leqslant \sqrt{4+y}, -4 \leqslant y \leqslant 0\right\},$$

5. a) Using vertical slices:

$$\mathscr{T} = \{(x, y) : -1 \leqslant y \leqslant x, -1 \leqslant x \leqslant 3\},$$

or,
b) Using horizontal slices:

$$\mathscr{T} = \{(x, y) : y \leqslant x \leqslant 3, -1 \leqslant y \leqslant 3\}.$$

6. a) Using vertical slices: $\mathscr{K} = \mathscr{K}_1 \cup \mathscr{K}_2$ where

$$\mathscr{K}_1 = \{(x, y) : 2x - 2 \leqslant y \leqslant 1 - x, 0 \leqslant x \leqslant 1\},$$

and

$$\mathscr{K}_2 = \{(x, y) : -2x - 2 \leqslant y \leqslant x + 1, -1 \leqslant x \leqslant 0\},$$

or, using horizontal slices,
b) $\mathscr{K} = \mathscr{K}_3 \cup \mathscr{K}_4$ where

$$\mathscr{K}_3 = \{(x, y) : y - 1 \leqslant x \leqslant 1 - y, 0 \leqslant y \leqslant 1\},$$

and

$$\mathscr{K}_4 = \left\{(x, y) : -\frac{y+2}{2} \leqslant y \leqslant \frac{y+2}{2}, -2 \leqslant y \leqslant 0\right\}.$$

7. a) Using vertical slices:

$$\mathscr{W} = \{(x, y) : x^2 \leqslant y \leqslant \sqrt{x}, 0 \leqslant x \leqslant 1\},$$

b) or, using horizontal slices:

$$\mathscr{W} = \{(x, y) : y^2 \leqslant x \leqslant \sqrt{y}, 0 \leqslant y \leqslant 1\}.$$

8. a) Using vertical slices:

$$\mathscr{E} = \left\{(x, y) : -\sqrt{\frac{\pi^2}{4} - \left(x - \frac{\pi}{2}\right)^2} \leqslant y \leqslant \sin x, 0 \leqslant x \leqslant \pi\right\},$$

or (this part is hard),
b) Using horizontal slices: $\mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2$ where

$$\mathscr{E}_1 = \{(x, y) : \arcsin y \leqslant x \leqslant \pi - \arcsin y, 0 \leqslant y \leqslant 1\},$$

and

$$\mathscr{E}_2 = \left\{(x, y) : -\frac{\pi}{2} - \sqrt{\frac{\pi^2}{4} - y^2} \leqslant x \leqslant \frac{\pi}{2} + \sqrt{\frac{\pi^2}{4} - y^2}, -\frac{\pi}{2} \leqslant y \leqslant 0\right\}.$$

9. a) Using vertical slices: $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$ where:

$$\mathscr{T}_1 = \{(x, y) : 0 \leqslant y \leqslant x + 2, -2 \leqslant x \leqslant -1\},$$
$$\mathscr{T}_2 = \{(x, y) : 0 \leqslant y \leqslant 1, -1 \leqslant x \leqslant 1\},$$
$$\mathscr{T}_3 = \{(x, y) : 0 \leqslant y \leqslant 2 - x, 1 \leqslant x \leqslant 2\},$$

or, using horizontal slices,
b)

$$\mathscr{T} = \{(x, y) : y - 2 \leqslant x \leqslant 2 - y, 0 \leqslant y \leqslant 1\}.$$

10. a) Using vertical slices:

$$\mathscr{G} = \left\{(x, y) : 3 - \sqrt{1 - x^2} \leqslant y \leqslant 3 + \sqrt{1 - x^2}, -1 \leqslant x \leqslant 1\right\},$$

or, using horizontal slices,
b) $\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2$ where

$$\mathscr{G}_1 = \{(x, y) : -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 3\},$$

$$\mathscr{G}_2 = \left\{(x, y) : -\sqrt{1 - (y - 3)^2} \leqslant x \leqslant \sqrt{1 - (y - 3)^2}, 3 \leqslant y \leqslant 4\right\},$$

11.

$$\mathscr{S} = \{(r, \theta) : 0 \leqslant r \leqslant 3, \frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{3}\}.$$

12.

$$\mathscr{W} = \{(r, \theta) : 2 \leqslant r \leqslant 3, 0 \leqslant \theta < 2\pi\}.$$

13. $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2$ where

$$\mathscr{L}_1 = \left\{(r, \theta) : 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \frac{\pi}{2}\right\},$$
$$\mathscr{L}_2 = \left\{(r, \theta) : 0 \leqslant r \leqslant 1, \pi \leqslant \theta \leqslant \frac{3\pi}{2}\right\}.$$

14.
$$\mathscr{M} = \left\{(r,\theta) : 0 \leqslant r \leqslant \frac{4\cos\theta - \sqrt{16\cos^2\theta - 12}}{2}, -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}\right\}.$$

15.
$$\mathscr{C} = \{(r,\theta) : 0 \leqslant r \leqslant 1 + \cos\theta, 0 \leqslant \theta < \pi\}.$$

16.
$$\mathscr{L} = \{(r,\theta) : 0 \leqslant r \leqslant \cos 3\theta, -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}\}.$$

17. $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3$ where:
$$\mathscr{C}_1 = \left\{(r,\theta) : 0 \leqslant r \leqslant 4, -\arcsin\frac{1}{4} \leqslant \theta \leqslant \arcsin\frac{1}{4}\right\},$$
$$\mathscr{C}_2 = \left\{(r,\theta) : 0 \leqslant r \leqslant \csc\theta, \arcsin\frac{1}{4} \leqslant \theta \leqslant \frac{\pi}{2}\right\},$$
$$\mathscr{C}_3 = \left\{(r,\theta) : 0 \leqslant r \leqslant -\csc\theta, -\frac{\pi}{2} \leqslant \theta \leqslant -\arcsin\frac{1}{4}\right\}.$$

18. In polar coordinates: $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3$ where:
$$\mathscr{R}_1 = \left\{(r,\theta) : 0 \leqslant r \leqslant 2\sin\theta, 0 \leqslant \theta \leqslant \frac{\pi}{6}\right\},$$
$$\mathscr{R}_2 = \left\{(r,\theta) : 0 \leqslant r \leqslant 1, \frac{\pi}{6} \leqslant \theta \leqslant \frac{5\pi}{6}\right\},$$
$$\mathscr{R}_3 = \left\{(r,\theta) : 0 \leqslant r \leqslant 2\sin\theta, \frac{5\pi}{6} \leqslant \theta \leqslant \pi\right\},$$

or, in Cartesian coordinates (vertical slices):
$$\mathscr{R} = \left\{(x,y) : 1 - \sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2}, -\frac{\sqrt{3}}{2} \leqslant x \leqslant \frac{\sqrt{3}}{2}\right\},$$

or, using horizontal slices: $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2$ where
$$\mathscr{R}_1 = \left\{(x,y) : -\sqrt{1-y^2} \leqslant x \leqslant \sqrt{1-y^2}, -\frac{1}{2} \leqslant y \leqslant 1\right\},$$
$$\mathscr{R}_2 = \left\{(x,y) : -\sqrt{2y-y^2} \leqslant x \leqslant \sqrt{2y-y^2}, 0 \leqslant y \leqslant \frac{1}{2}\right\},$$

19. In polar coordinates: $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3$ where:
$$\mathscr{R}_1 = \left\{(r,\theta) : 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \frac{\pi}{2}\right\},$$
$$\mathscr{R}_2 = \left\{(r,\theta) : \csc\theta \leqslant r \leqslant 1, \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{4}\right\},$$
$$\mathscr{R}_3 = \left\{(r,\theta) : 1 \leqslant r \leqslant -\sec\theta, \frac{3\pi}{4} \leqslant \theta \leqslant \pi\right\}$$

or, in Cartesian coordinates (vertical slices): $\mathscr{R} = \mathscr{R}_1 \cup (\mathscr{R}_2 \cup \mathscr{R}_3)$ where
$$\mathscr{R}_1 = \left\{(x,y) : 0 \leqslant y \leqslant \sqrt{1-x^2}, 0 \leqslant x \leqslant 1\right\},$$
$$\mathscr{R}_2 \cup \mathscr{R}_3 = \{(x,y) : 0 \leqslant y \leqslant 1, -1 \leqslant x \leqslant 0\},$$

or, using horizontal slices (simplest of all):
$$\mathscr{R} = \left\{(x,y) : -1 \leqslant x \leqslant \sqrt{1-y^2}, 0 \leqslant y \leqslant 1\right\},$$

20. In polar coordinates:

$$\mathscr{S} = \left\{(r,\theta) : 0 \leqslant r \leqslant 2\sqrt{\csc\theta}, \frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{3}\right\}$$

or, in Cartesian coordinates (vertical slices): $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2$ where:

$$\mathscr{S}_1 = \left\{(x,y) : \frac{\sqrt{3}x}{3} \leqslant y \leqslant \sqrt{3}x, 0 \leqslant x \leqslant \frac{\sqrt{2}}{\sqrt[4]{3}}\right\},$$
$$\mathscr{S}_2 = \left\{(x,y) : \frac{\sqrt{3}x}{3} \leqslant y \leqslant \frac{2}{x}, \frac{\sqrt{2}}{\sqrt[4]{3}} \leqslant x \leqslant \sqrt{2}\sqrt[4]{3}\right\}.$$

21. $\mathscr{R}_{yx} = \{(x,y) : x^2 \leq y \leq x, 0 \leq x \leq 1\}$
22. $\mathscr{R}_{xy} = \{(x,y) : y/a \leq x \leq y/b, 0 \leq y \leq 1\}$
23. $\mathscr{R}_{yx} = \{(x,y) : 0 \leq y \leq \sqrt{1-x}, 0 \leq x \leq 1\}$
24. $\mathscr{R}_{xy} = \{(x,y) : 0 \leq x \leq y/2, 0 \leq y \leq 2\}$
25. $\mathscr{R}_{xy} = \{(x,y) : -\sqrt{y} \leq x \leq \sqrt{y}, 0 \leq y \leq 1\}$

Exercise Set 36.

1. 1
2. 16/3
3. 7/6
4. 9
5. $3\pi/2$
6. $\sqrt{\frac{1}{8}}$
7. $\sqrt{\frac{5}{16}}$
8. $\frac{y}{3}$
9. $\frac{x^2}{2}$
10. $\frac{9}{2}$

Exercise Set 37.

1. 1/2
2. 1/2
3. 1/6

4. $1/3$
5. 2
6. $e + e^{-1} - 2$
7. 2
8. $\frac{9}{2}\ln(2)$
9. $4 + 2\pi$
10. $\frac{5}{6} + \ln\left(\sqrt{\frac{2}{3}}\right) = \frac{5}{6} + \frac{1}{2}\ln(2/3)$
11. $\frac{1}{6}$
12. $\frac{6}{2}$
13. $\frac{5}{6}$
14. 1
15. $4 + 2\pi$
16. 12
17. $\frac{4\pi}{3}$
18. $\frac{3}{2}$
19. $\frac{5}{6}$
20. $\frac{3}{4}$

Exercise Set 38.

1. $I = \int_0^4 \int_0^{\sqrt{y}} (xy^2 + x)\, dx\, dy$
2. $\frac{43}{210}$. $\mathscr{R}$ is bounded below by $y = 0$, above by $y = x^3$, on the left by $x = 0$.
3. $I = \int_0^3 \int_0^{x^2} \sin(\pi x^3)\, dy\, dx = \dfrac{2}{3\pi}$
4. $\dfrac{e^{16} - 1}{4}$
5. $I = \dfrac{1}{2}(1 - \cos 1)$ (Use Fubini!)
6. $\dfrac{2a^3}{3}$
7. $\int_0^4 \int_{y^2}^{16} f(x, y)\, dx\, dy$

8. a) $\dfrac{3}{4}$

   b) $I = \displaystyle\int_0^1 \int_{\sqrt{1-y}}^1 x\,dx\,dy + \int_1^e \int_{\ln y}^1 x\,dx\,dy = \frac{3}{4}.$

9. $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 (y+x^2)\,dy\,dx + \int_0^9 \int_{-3}^{-\sqrt{x}} (y+x^2)\,dy\,dx = \frac{212}{21} + \frac{2349}{28} = \frac{7895}{84}.$

10. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy.$

11. $2/3 = \displaystyle\int_0^1 \int_{-x}^x (y+x)^2\,dy\,dx.$

12. $49\pi/32$

13. $\frac{109}{4}$

14. $\displaystyle\int_0^1 \int_{y^2}^y \frac{e^y}{y}\,dx\,dy = e-2.$

15. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{y}{\sqrt{1-x^2-y^2}}\,dy\,dx$

16. $\pi/4$

17. $2 - 2\cos 2$

18. $e^{2/3} - 1$

19. $\sqrt{2}/3 - 1/6$

20. $1/2$

Exercise Set 39.

1. 3
2. 2
3. $(9\ln 2)/2$
4. $1/3$
5. $1/6$
6. $2\sqrt{5} - \ln(\sqrt{5}-2)$
7. 28

Exercise Set 40.

1. $\pi/24 + \sqrt{3}/16.$

2. $3\pi/2$. Use polar coordinates.
3. $\pi\,(9 - \frac{5^{3/2}}{3})$.
4. $\pi(e-1)$
5. $\frac{1}{4}(\frac{\pi}{2}+1)$
6. 39
7. $\frac{\pi}{24}(1-\cos 1)$
8. $3/4$
9. 4
10. $\pi/3$
11. $\frac{2}{3}\ln(\sec 1 + \tan 1)$
12. $4\pi/\sqrt{3}$
13. $4v$
14. $-2s$
15. $\pi/4$
16. 2π
17. $6uv^2$
18. $\pi/3$
19. 4
20. -1/8

Exercise Set 41.

1. A vertical elliptic cylinder
2. A right-circular cylinder whose central axis is the y-axis.
3. A vertical hyperbolic cylinder
4. A parabolic cylinder whose vertex is the x-axis.
5. A hyperbolic paraboloid
6. Hyperboloid of one sheet
7. Hyperboloid of two sheets
8. Right-circular cylinder whose central axis is the x-axis
9. An elliptic cone
10. An ellipsoid
11. A hyperbolic paraboloid

12. An elliptic cylinder whose central axis is the y-axis.
13. Elliptic cylinder
14. Hyperbolic cylinder
15. Hyperbolic paraboloid
16. Right-circular cylinder
17. Parabolic cylinder
18. Hyperbolic cylinder
19. Elliptic cone
20. Hyperboloid of two sheets
21. Ellipsoid
22. Hyperboloid of one sheet
23. Elliptic paraboloid

Exercise Set 42.

1. $x = u, y = v, z = 3v^2 - 2u^3 + 6,\ u, v \in \mathbf{R}$
2. $y = u, z = v, x = ve^u/3,\ u, v \in \mathbf{R}$
3. $x = u, z = v, y = \frac{1}{2}\,(\sin(uv) - u^2),\ u, v \in \mathbf{R}$
4. $x = 3\sin u \cos v,\ \ y = 3\sin u \sin v,\ \ z = 3\cos v,\ u, v \in [0, 2\pi]$
5. $x = 2\cos u,\ \ y = v,\ \ z = 2\sin u,\ u \in [0, 2\pi],\ v \in \mathbf{R}$.
6. $x = u\cos v,\ \ y = (1/2)u\sin v,\ \ z = u/3,\ v \in [0, 2\pi],\ u \in \mathbf{R}$.
7. $x = u,\ \ y = \sqrt{u}\cos v,\ \ z = \sqrt{u}\sin v,\ u \geq 0,\ v \in [0, 2\pi]$.
8. $x = 1 + u\cos v,\ \ y = 2 + u\sin v,\ \ z = v,\ u \in \mathbf{R}$ and $v \in [0, 2\pi]$
9. $x = \cos u\ \sin v,\quad y = (1/2)\ \sin u\ \sin v,\ z = (1/4)\ \cos v$, where $u \in [0, 2\pi]$, $v \in [0, \pi]$.
10. $x = v\cos u,\ \ y = v\sin u,\ \ z = v,\ u \in [0, 2\pi],\ v \in \mathbf{R}$
11. $x = (1/\sqrt{2})\sinh u,\ \ y = (1/2)\cosh u,\ \ z = v.\ \ u, v \in \mathbf{R}$
12. $x = (2 + \cos v)\cos u,\quad y = (2 + \cos v)\sin u,\quad z = \sin v$. where $u, v \in [0, 2\pi)$, and $c > a > 0$
13. $\mathbf{r}(u, v) = u\,\mathbf{i} + 2\cos v\,\mathbf{j} + 2\sin v\,\mathbf{k}$
14. $\mathbf{r}(u, v) = 2\cos u\ \sin v\,\mathbf{i} + 2\sin u\ \sin v\,\mathbf{j} + 2\cos v\,\mathbf{k},\qquad u \in \mathbf{R},\ v \in [0, 2\pi]$
15. $\mathbf{r}(u, v) = \dfrac{1}{3}\sinh u\,\mathbf{i} + \cosh u\,\mathbf{j} + v\,\mathbf{k}$
16. $\mathbf{r}(u, v) = v\ \cos u\,\mathbf{i} + v\,\mathbf{j} + 2v\sin u\,\mathbf{k}$

17. $9x^2 + 4y^2 = 36z$
18. $x + y + z = 2$
19. $y^2 - z^2 = 1$
20. $x^2 + y^2 + z^2 + 12 - 8\sqrt{x^2 + y^2} = 0$

Exercise Set 43.

1. $4\sqrt{61}$
2. $\sqrt{3}/120$
3. $\pi R^3/4$
4. 0
5. $2\pi \arctan(H/a)$
6. 2
7. 2
8. -4
9. $1.33\mu C$. The answer gives the total charge on the plate.
10. $a\,\delta/\varepsilon$
11. $\pi/4$
12. $1/2$
13. $\sqrt{3}/6$
14. 0
15. 0
16. $\dfrac{3\sqrt{14}}{2}$
17. 1
18. $\dfrac{13\pi}{3}$
19. $149\pi = 30\displaystyle\int_0^{2\pi}\int_0^{\sqrt{2}} r^3\sqrt{1+4r^2}\,dr\,d\theta$
20. $\dfrac{17\pi}{96}$. Note the answer here!

Exercise Set 44.

1. 0

2. -6π
3. 0
4. $-14\pi\sqrt{2}$
5. 2/3
6. 1/2
7. 12π
8. 0
9. -2π
10. -1
11. -24π
12. 27/4
13. 2
14. $625\pi/2$
15. 3π
16. 1/30
17. 6π
18. 120π
19. 0
20. 12

Exercise Set 45.

1. $4\pi = \int_{\mathscr{C}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$ where $\mathscr{C}$ is given parametrically by $\mathbf{r}(t) = (2\cos t, 2\sin t, 2),\ 0 \le t \le 2\pi$ and $\mathbf{r}'(t) = (-2\sin t, 2\cos t, 0)$. Furthermore, $\mathbf{F}(\mathbf{r}(t)) = (2\cos t - 2\sin t, 4, 4\cos^2 t)$. It follows that

   $$\int_{\mathscr{C}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = 4\int_0^{2\pi} (\cos^2 t - \cos t \sin t + 8\cos t)\,dt = 4\pi,$$

   as before.
2. 36π
3. 0
4. π
5. -2
6. 80π

7. 256π
8. -18π
9. $\pi/4$
10. $-\sqrt{3}/6$
11. $\nabla \times \mathbf{r} = \mathbf{0}$ and the result is clear.
12. Break up the line integral into two parts and use the previous result.
13. 8π
14. 0
15. $-\pi$
16. $(-2, 3, 0)/\sqrt{13}$
17. $(xy + z, -1, yz)$
18. 0
19. $-\pi$
20. 0

Exercise Set 46.

1. 2
2. 2
3. 9/2
4. 1/2
5. 95/1512
6. $\displaystyle\int_0^3 \int_0^1 \int_0^2 x\, e^{xy}\, dy\, dz\, dx = \frac{e^6 - 7}{2}$
7. $\displaystyle\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{\frac{1-2x^2}{3}}}^{\sqrt{\frac{1-2x^2}{3}}} \int_{-\sqrt{1-2x^2-3y^2}}^{\sqrt{1-2x^2-3y^2}} y\, dz\, dy\, dx.$
8. Project the solid $\mathscr{T}$ on to the xy-plane and use y-slices for the description of $\mathscr{R}_{yx}$.
9. $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{1-x^2-y^2} z\, dz\, dx\, dy.$
10. Projecting $\mathscr{T}$ on the xz-plane we substitute $x = 0$ into $z = 1 - x^2 - y^2$ to get the projection $\mathscr{R}$. Using horizontal slices in the yz-plane we find

$$\mathscr{R}_{yz} = \{(0, y, z) : 0 \le y \le \sqrt{1 - z}, \quad 0 \le z \le 1.\}$$

Next, for a fixed point $(0, y, z) \in \mathscr{R}_{yz}$ an x-slice goes from 0 to a point on the surface where now x is written as a function of y, z, i.e., $x^2 = 1 - y^2 - z$

which gives $x = \sqrt{1-y^2-z}$ (since we are in the first octant). It follows that

$$I = \iint_{\mathscr{R}_{yz}} \int_0^{\sqrt{1-y^2-z}} z\,dx\,dA_{yz} = \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{\sqrt{1-y^2-z}} z\,dx\,dy\,dz.$$

11. See the previous Exercise and look at the same region $\mathscr{R}$, but now change the slice to a vertical slice to get $\mathscr{R}_{zy}$. This gives the integral

$$I = \iint_{\mathscr{R}_{zy}} \int_0^{\sqrt{1-y^2-z}} z\,dx\,dA_{zy} = \int_0^1 \int_0^{1-y^2} \int_0^{\sqrt{1-y^2-z}} z\,dx\,dz\,dy.$$

12. This time we project $\mathscr{T}$ onto the xz-plane. In this case the projected region $\mathscr{R}$ is enclosed by the parabola $z = 1-x^2$ (since $y = 0$ on the xz-plane) and the coordinate axes. Since we want $\mathscr{R}_{xz}$ we need to take a horizontal slice of $\mathscr{R}$ in the xz-plane. This gives us the description,

$$\mathscr{R}_{xz} = \{(x,0,z) : 0 \le x \le \sqrt{1-z},\quad 0 \le z \le 1\}.$$

Now, for a fixed point $(x,0,z) \in \mathscr{R}_{xz}$ a y-slice goes from 0 to a point on the surface where now y is written as a function of x, z, i.e., $y^2 = 1-x^2-z$ which gives $y = \sqrt{1-x^2-z}$ (since we are in the first octant). So, as before, we get

$$I = \iint_{\mathscr{R}_{xz}} \int_0^{\sqrt{1-y^2-z}} z\,dy\,dA_{xz} = \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{\sqrt{1-x^2-z}} z\,dy\,dx\,dz.$$

13. Once again we refer to the previous example but where we change the slice of $\mathscr{R}$ from horizontal to vertical to get $\mathscr{R}_{zx}$. We get

$$\mathscr{R}_{zx} = \{(x,0,z) : 0 \le z \le 1-x^2,\quad 0 \le x \le 1\}.$$

The iterated integral now becomes,

$$I = \iint_{\mathscr{R}_{zx}} \int_0^{\sqrt{1-y^2-z}} z\,dy\,dA_{zx} = \int_0^1 \int_0^{1-x^2} \int_0^{\sqrt{1-x^2-z}} z\,dy\,dz\,dx.$$

14. Evaluating the innermost integral in Exercise 8 we get,

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} z\,dz\,dy\,dx.$$

In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2+y^2$ and the Jacobian, $J = r$. Since we are integrating over a quarter circle we get $\theta \in [0, \pi/2]$ not the full circle, i.e.,

$$I = \frac{1}{2}\int_0^{\pi/2} \int_0^1 r\,(1-r^2)^2\,dr\,d\theta = \frac{\pi}{4}\int_0^1 r\,(1-r^2)^2\,dr = \frac{\pi}{24}.$$

Of course, all the previous integrals, # 9, 10, 11, 12 and 13 give the same answer!

15. Mass $= \displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x^2+y^2+z^2)\,dz\,dy\,dx = \frac{1}{20}.$

16. $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{5}{18}, \frac{5}{18}, \frac{5}{18}\right)$. (Don't forget to divide by the Mass!)

17. The total charge in $\mathscr{T}$ is simply the triple integral of the density function (just like in c.m. problems). In this case the total charge in $\mathscr{T}$ is the iterated integral,

$$\text{Total charge} = I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} x\,y\,z\;dz\,dy\,dx.$$

Evaluating the inner integral first we get

$$I = \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} x\,y\,(a^2-x^2-y^2)^2 dy\,dx.$$

Changing to polar coordinates we find, (don't forget the Jacobian!)

$$I = \frac{1}{2}\int_0^{\pi/2} \int_0^a r^3\,(a^2-r^2)\,\cos\theta\,\sin\theta\,dr\,d\theta = \frac{a^6}{48}\text{ Coul.}$$

18. a) Project this eighth part of $\mathscr{T}$ onto the xy-plane and describe the region, $\mathscr{R}_{yx}$. You'll get

$$\{(x,y,z) : 0 \le z \le \sqrt{\frac{1-ax^2-by^2}{c}},\quad 0 \le y \le \sqrt{\frac{1-ax^2}{b}},\quad 0 \le x \le \frac{1}{\sqrt{a}}\}.$$

From this to the integral is an easy matter.

b) Use the elliptic coordinate transformation in the hint. The Jacobian is found using the usual determinant. Integrating out the innermost integral we get a double integral of the form

$$8\int_0^{\frac{1}{\sqrt{a}}} \int_0^{\sqrt{\frac{1-ax^2}{b}}} \sqrt{\frac{1-ax^2-by^2}{c}}\,dy\,dx.$$

In this case note that $r^2 = ax^2+by^2$ and the region of projection $\mathscr{R}_{r\theta}$ is a quarter-ellipse with $ax^2+by^2=1$ on the ellipse itself and so $r_{\max}=1$. From this we find,

$$\mathscr{R}_{r\theta} = \{(r,\theta,0) : 0 \le r \le 1,\quad 0 \le \theta \le \pi/2\}.$$

Combining this information we get the double integral in polar coordinates in (c).

c) Next,

$$\begin{aligned}\text{Volume of }\mathscr{T} &= \frac{8}{\sqrt{abc}}\int_0^{\pi/2}\int_0^1 r\,\sqrt{1-r^2}\,dr\,d\theta.\\ &= \frac{8\pi}{2\sqrt{abc}}\int_0^1 r\,\sqrt{1-r^2}\,dr\\ &= \frac{8\pi}{2\sqrt{abc}}\cdot\frac{1}{3} = \frac{4\pi}{3\sqrt{abc}},\end{aligned}$$

as required.

d) Finally,

$$ax^2+by^2+cz^2 = R^2, \Longleftrightarrow \left(\frac{a}{R^2}\right)x^2 + \left(\frac{b}{R^2}\right)y^2 + \left(\frac{c}{R^2}\right)z^2 = 1.$$

These coefficients are now the "new" a, b, c in (c). So replacing a, b, c in (c) by $a/R^2, b/R^2, c/R^2$ in the volume expression in (c) we get,

$$\frac{4\pi}{3\sqrt{\frac{a}{R^2}\frac{b}{R^2}\frac{c}{R^2}}} = \frac{4\pi}{3\sqrt{\frac{abc}{R^6}}} = \frac{4\pi\, R^3}{3\sqrt{abc}}.$$

as required.

Exercise Set 47.

1. $\dfrac{1024}{9}$
2. $\dfrac{1}{8}$
3. $\dfrac{32}{21}$
4. $\dfrac{23}{24}$
5. $\dfrac{16384}{75}$
6. $\dfrac{93}{35}$
7. $\dfrac{27}{8}$. Project onto the xy-plane. Using vertical slices there we get $I = \int_0^3 \int_0^{3-x} \int_0^{3-x-y} x\, dz\, dy\, dx = \int_0^3 \int_0^{3-x} x(3-x-y)\, dz\, dy\, dx.$
8. 16
9. 3
10. 6
11. $\dfrac{2}{5}$
12.
13. $\dfrac{4}{27}$
14. $\dfrac{261}{32}$
15. $4\pi/5$. Use spherical coordinates. Then $I = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} \rho^4 \sin\varphi\, d\rho\, d\varphi\, d\theta.$
16. $(\pi a^4 \sin^2 \beta)/2$

17.

$$\begin{aligned}\frac{\pi}{12} &= \int_0^{2\pi}\int_0^1\int_{r^2}^{r} z\,r\,dz\,dr\,d\theta \quad \text{(cylindrical coordinates)}\\ &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} z\,dz\,dy\,dx, \quad \text{(rectangular coordinates)}\\ &= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\cot\varphi\,\csc\varphi}(\rho\,\cos\varphi)\,(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta.\end{aligned}$$

(spherical coordinates).

18. $1/2$
19. $74\pi/3$
20. $25\pi/2$
21. $4\pi/15$
22. $\frac{1}{6}$
23. $\frac{1}{12}$
24. $\frac{1}{12}$
25. $\frac{1}{60}$
26. $\frac{3}{2}$
27. $\frac{4}{3}$
28. $\frac{1}{6}$
29. $\frac{1}{12}$
30. $\frac{1}{2}$

Exercise Set 48.

1. 39
2. $4\pi a^3$
3. 3
4. -8π
5. $5\pi a^4/4$
6. 8π

7. Let $\mathbf{F} = \mathbf{r} = (x, y, z)$. Then $\nabla \cdot \mathbf{F} = 3$. Applying the Divergence Theorem with $\mathbf{F} = \mathbf{r}$ we get

$$\iint_{\mathscr{S}} \mathbf{r} \cdot \mathbf{n}\, dS = \iiint_{\mathscr{T}} \nabla \cdot \mathbf{r}\, dV = \iiint_{\mathscr{T}} 3\, dV = 3\, V.$$

Solving for V we get the result.

8. An easy calculation shows that

$$\nabla |\mathbf{r}|^2 = \nabla(x^2 + y^2 + z^2) = (2x, 2y, 2z) = 2\mathbf{r}.$$

So,

$$\iint_{\mathscr{S}} (\nabla |\mathbf{r}|^2) \cdot \mathbf{n}\, dS = 2 \iint_{\mathscr{S}} \mathbf{r} \cdot \mathbf{n}\, dS = 2 \cdot 3 \cdot V,$$

by the preceding exercise. Solving for V once again gives the answer.

9. 180

10. 3π

11. $11664\pi/5$

12. Stokes' Theorem tells us that

$$\iint_{\mathscr{S}} (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, dS = \int_{\mathscr{C}} \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \iint_{\mathscr{S}} \mathbf{B} \cdot \mathbf{n}\, dS,$$

by hypothesis. Combining the first and last terms of the previous display we get

$$\iint_{\mathscr{S}} \left\{ (\nabla \times \mathbf{E}) + \frac{\partial \mathbf{B}}{\partial t} \right\} \cdot \mathbf{n}\, dS = 0,$$

where we can justify taking the derivative of the integral as the integral of the derivative. Now it must be the case that $(\nabla \times \mathbf{E}) + \frac{\partial \mathbf{B}}{\partial t} = 0$, or else an argument similar to the one at the top of p. 483, but applied to surfaces instead of solid regions, gives a contradiction.

13. 8π

14. $16\pi/3$

15. 144π

16. a^2bc

17. 12π

Exercise Set 49.

1. $(0, 0)$, global max., value 1.

2. $(1, -2)$, global min., value 1

3. Saddle points at $(1, 2/3)$, $(-1, -4/3)$, No local max/min

4. Absolute min at $(0, 0)$, even though $D = 0$. Absolute min. value, 0

5. Saddle point at $(0, 0)$.
6. Absolute minimum at every point (x, y) on the line $y = 1 - x$.
7. Absolute min at $(0, 0)$, value $= 0$ and absolute max at every point on the unit circle $x^2 + y^2 = 1$
8. Saddle point at $(0, 0)$; Loc. min at $(1, 1)$, value $= 2$
9. Saddle point at $(0, 0)$
10. No interior critical points. Absolute max at $(1, 1)$, value $= 2$ and absolute min at $(0, 0)$, value $= -1$
11. Absolute min at every point $(x, 0)$ and $(0, y)$ with value $= 0$. Absolute max at $(1, 1)$, value $= 1$
12. Absolute max at $(\pm\sqrt{3}/2, 1/2)$ (value $= 14$) and absolute min at $(0, -1/2)$ value $= 4$
13. Saddle points at $(\pi/2, 0)$ and $(3pi/2, 0)$. Absolute max at $(\pi/2, \pm 1)$, value $= \cosh(1)$. Absolute min at $(3\pi/2, \pm 1)$, value $= -\cosh 1$. Note that $\cosh(-1) = \cosh(1)$
14. (x, x), for any $x \in \mathbf{R}$. global max., value 2
15. $(0, 0)$, saddle point, even though $D = 0$. Note that $f(x, x/\sqrt{3}) > 0$ for $x > 0$ and $f(x, x/\sqrt{3}) < 0$ for $x < 0$
16. No interior critical points. (The apparent critical point at $(1, 0)$ is not in $\mathscr{D}$.) A local and so global minimum at $(-1/\sqrt{3}, 0)$ (value $= (1 - 2\sqrt{3})/3$) and a local and so global maximum at $(1/\sqrt{3}, 0)$ (value $= (1 + 2\sqrt{3})/3$) on perimeter of ellipse
17. $(6, 3, 4)$, Max value $= 288$
18. $7/\sqrt{3}$
19. Absolute max at $(1/2, 1/2)$ value $= 1/4$. No absolute minimum
20. Global max at $(\pm\sqrt{2}, 0)$, value $= 2/e$. Global min at $(0, \pm\sqrt{2})$, value $= -2/e$. Saddle point at $(0, 0)$
21. Dimensions are a cubical box having a side length $= \sqrt{2/3}$. The maximum volume is $2\sqrt{2}/(3\sqrt{3}) = 0.54m^3$
22. Yes, it's true. Find the first partials in both cases and set them equal to zero. The proof is too long to put here. (Maybe on the website.)

Exercise Set 50.

1. $(3, 1)$, min. value $= 22$.
2. $(8, -3)$, max. value $= 13$.
3. $(3, -1)$, min. value $= 5$.
4. $(1/2, 1/2)$, min value $= 3/4$.

5. $x = y = z = 500/3$
6. $(1,1,1)/\sqrt{3}$, a max with value $\sqrt{3}$; $-(1,1,1)/\sqrt{3}$ a min with value $-\sqrt{3}$
7. 36.
8. $(5,3,5/3)$, min value $= 45$.
9. $x = y = z = 5$
10. $1/e^3$ occurs at $(1,1,1)$.
11. ± 27
12. $\sqrt{12} - 1$
13. $(\pm 1, 0)$ max value $= 2$; $(0,0)$ min value $= 0$.
14. $(\pm\sqrt{3}, 1)/2$ max value $= 14$; $(0, -1/2)$ min value $= 4$.
15. $(0,5)$ max value 275; $(0,0)$ min value 0.
16. $(1/2, 1/2)$ min value $= 1/2$
17. Min value $= \pm 1/\sqrt{2}$.

Exercise Set 51.

1. $m = 9$, and $(\bar{x}, \bar{y}) = (1,2)$.
2. $m = \frac{10}{3}, (\bar{x}, \bar{y}) = (2.1, 0.3)$
3. $\frac{3\pi}{2}$
4. $(\frac{6}{7}, \frac{1}{2})$
5. 144π
6. $2\pi/3$
7. 6π
8. 9π
9. $\pi^2/2$
10. $2\pi^2$
11. $\sqrt{3}/2$
12. $3\sqrt{14}/2$
13. $19\sqrt{5}/15$
14. $\sqrt{15}/6$.
15. $2\sqrt{5} - \ln(\sqrt{5} - 2)$
16. $\dfrac{9\sqrt{37}}{2} - \dfrac{3\ln(\sqrt{37} - 6)}{4}$
17. $\left(\dfrac{64}{35}, \dfrac{5}{7}\right)$
18. $\dfrac{256\pi}{15}$