University of Guelph
CIS 2910 F18 – Midterm (Oct. 18)
Instructor: Joe Sawada

First Name: ______________________________

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Problem 1: (5 marks)		Problem 5: (6 marks)	
Problem 2: (4 marks)		Problem 6: (5 marks)	
Problem 3: (4 marks)		Problem 7: (5 marks)	
Problem 4: (6 marks)			
		Total (35 marks)	

This test is closed book and lasts 75 minutes.
You may not use any electronic/mechanical computation devices.
There are 9 pages including the cover page.

Problem 1: [5 marks]

(a) TRUE or FALSE: A recent mayoral election had 4 candidates: Jennifer, Kyle, Ling, and Mohamed. If there was a total of 81 votes cast then at least one candidate received more than 21 votes.

Solution: False The votes could have been 21, 20, 20, 20.

(b) TRUE or FALSE: The number of binary strings of length $2n$ with exactly two 0s equals $2\binom{n}{2}+n^2$.

Solution: True Consider three cases depending on where the two 0s are: The two 0s are both in the first half, the two zeros are both in the second half, or there is one in the first half and one in the second half.

(c) TRUE or FALSE: If a random sequence of integers has length 101, then it must contain a decreasing subsequence of length 10.

Solution: False It could be strictly increasing.

(d) TRUE or FALSE: Let S be an infinite subset of the positive integers. The Well Ordering Property implies S has a least element.

Solution: True

(e) TRUE or FALSE: A proposition $P(n)$ is true for all integers $n \geq 4$ if (a) $P(4)$ is true and (b) the implication $P(k) \rightarrow P(k+1)$ evaluates to true for any $k \geq 4$.

Solution: True From the principle of mathematical induction.

Problem 2: [4 marks]
The Toronto Raptors are gearing up for a new season. Their practice roster has 18 players. In the final practice before the regular season, the coach divides them into three teams: **B**lack, **W**hite, and **R**ed.

(a) **[1 mark]** How many different ways can the 18 players be assigned to teams if there are no restrictions (some team could have no players)?

Solution: 3^{18}

(b) **[1 mark]** How many different ways can the 18 players be assigned to teams if each team has exactly 6 players?

Solution: $\frac{18!}{6!\ 6!\ 6!}$

(c) **[1 mark]** How many ways can the 18 players be lined up to get a lecture from the coach?

Solution: 18!

(d) **[1 mark]** The three teams play a game where the total points obtained by the three teams is 50. For instance, one outcome would have team White with 30 points, team Black with 14 points and team Red with 6 points. How many different such outcomes are possible? Assume the number of points for each team is an integer.

Solution: Consider the three teams to be $n = 3$ distinguishable boxes, and you have $r = 50$ indistinguishable points to place in the boxes. Apply the formula $\binom{n+r-1}{r} = \binom{52}{50} = \binom{52}{2}$

Problem 3: [4 marks]
Consider the following recurrence for $n \geq 0$:

$$T(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ T(n-2) + n & \text{if } n > 1. \end{cases}$$

(a) **[1 mark]** What is $T(2), T(3), T(4)$, and $T(5)$?

Solution: $T(2) = T(0) + 2 = 2$. $T(3) = T(1) + 3 = 4$, $T(4) = T(2) + 4 = 6$ and $T(5) = T(3) + 5 = 9$

(b) **[1 mark]** Perform two substitutions to express $T(n)$ in terms of $T(n-6)$.

Solution:

$$\begin{aligned} T(n) &= T(n-2) + n \\ &= T(n-4) + (n-2) + n = T(n-4) + 2n - 2 \\ &= (T(n-6) + (n-4)) + 2n - 2 = T(n-6) + 3n - 6 \end{aligned}$$

(c) **[2 marks]** A closed form solution for $T(n)$ when n is odd is given by:

$$T(n) = \sum_{j=1}^{\lceil n/2 \rceil} (2j - 1)$$

What is $T(99)$ as a decimal number?

Solution:

$$\begin{aligned} T(n) &= \sum_{j=1}^{\lceil 99/2 \rceil} (2j - 1) \\ &= (\sum_{j=1}^{50} 2j) - 50 \\ &= 2(\sum_{j=1}^{50} j) - 50 \\ &= (2)(50)(51)/2 - 50 = 2500 \end{aligned}$$

Problem 4: [6 marks]

(a) **[1 mark]** Express $C(33, 31) = \binom{33}{31}$ as a decimal number.

Solution: $33/(31! \cdot 2!) = 33 \cdot 32/2 = 33 \cdot 16 = 528$

(b)**[1 mark]** How many binary strings of length 10 begin and end with the same bit?

Solution: 2^9

(c)**[1 mark]** How many binary strings of length 10 begin with 000 or contain exactly 4 zeros?

Solution: Inclusion/Exclusion: $2^7 + \binom{10}{4} - 7 = 331$.

(d)**[1 mark]** Consider all DNA sequences of length 10 constructed from the letters C,G,A,T. How many such sequences of length 10 contain at least 2 G's ?

Solution: Take all strings and subtract those with exactly 1 G, or 0 G. Thus: $4^{10} - \binom{10}{1}3^9 - 3^{10}$.

(e)**[2 marks]** How many ternary strings (strings with symbols 0,1,2) of length 10 include at least one of each symbol?

Solution: This is like your homework assignment. Take all 3^{10} strings and subtract those that contain at most 2 symbols. First choose a symbol to remove, then count all strings of length 10 over 2 symbols. Then apply inclusion/exclusion to account for the non-empty intersection: those strings that use only one symbol. Thus $3^{10} - (3 \cdot 2^{10} - 3)$.

Problem 5: [6 marks] Multiple Choice.

(a) Which of the following corresponds to the *Generalized Pigeonhole Principle*?

- (a) If $k+1$ pigeons are placed into k holes, then there is a hole containing two pigeons.
- (b) If more than k pigeons are placed into k holes, then then at least one holes contains two pigeons.
- (c) If N pigeons are placed into k holes, then there is exactly one hole containing at least $\lceil N/k \rceil$ pigeons.
- (d) If N pigeons are placed into k holes, then there is at least one hole containing exactly $\lceil N/k \rceil$ pigeons.
- (e) (none of the above)

Solution: (e)

(b) How many binary strings of length $n = 20$ are there with exactly 7 ones?

- (a) $2^{20} - 2^7$
- (b) 2^7
- (c) $\binom{20}{13}$
- (d) 17
- (e) 2^{13}
- (f) (none of the above)

Solution: (c) $\binom{20}{13} = \binom{20}{7}$

(c) How many one-to-one functions are there from a set A with n elements to a set B with m elements, assuming $n \leq m$?

- (a) 0
- (b) $n!$
- (c) $m!$
- (d) n^m
- (e) m^n
- (f) $P(m, n)$
- (g) $C(m, n)$
- (h) (none of the above)

Solution: (f)

(d) How many bijective functions are there from a set A with n elements to a set B with n elements?

(a) $n!$
(b) n^n
(c) 2^n
(d) n^2
(e) (none of the above)

Solution: (a)

(e) Consider a 3-dimensional grid. How many ways can you travel from (0,0,0) to (5,10,10) if each step is of unit length along one of the axis and gets you closer to your destination?

(a) $\frac{25!}{5! \cdot 10! \cdot 10!}$
(b) $5! \cdot 10! \cdot 10!$
(c) $2^5 \cdot 2^{10} \cdot 2^{10}$
(d) 3^{25}
(e) 2^{25}
(f) (none of the above)

Solution: (a) Same as number of ways to order 5 Xs, 10 Ys and 10 Zs.

(f) The expression $\sum_{j=5}^{20} 2^j$ can be simplified to:

(a) $2^{21} - 2^5$
(b) $2^{21} - 2^4$
(c) $2^{21} - 17$
(d) $2^{20} + 2^5$
(e) (none of the above)

Solution: (a)

Problem 6: [5 marks]

Consider a set $S = \{1, 2, 3, \ldots, n\}$ with n elements.

(a) **[3 marks]** Describe a bijection between the set of all binary strings of length n and the **power set of** S. In addition, illustrate your bijection for the case when $n = 3$.

Solution: Consider a binary string $b_1 b_2 \cdots b_n$. Map it to the subset of S that has element i if and only if $b_i = 1$.

Binary String	**Subset of** $\{1, 2, 3\}$
000	$\emptyset$
001	$\{3\}$
010	$\{2\}$
011	$\{2, 3\}$
100	$\{1\}$
101	$\{1, 3\}$
110	$\{1, 2\}$
111	$\{1, 2, 3\}$

(b) **[1 mark]** Consider a sequence of n coin flips where each outcome is either Heads or Tails. Describe a correspondence between the number of such sequences where there are exactly k Heads and the number of subsets of S of size $n-k$?

Solution: Consider the positions of the $n - k$ Tails in the sequence. The indexes of the positions of these Tails will correspond to a subset of S with size $n - k$. This forms a bijection.

(c) **[1 mark]** How many subsets of $S = \{1, 2, \ldots, 10\}$ contain 6 consecutive integers? One such subset would be $\{2, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$.

Solution: This the same as the number of binary strings with 6 consecutive 1s. Similar to the assignment, we consider the starting position of a streak of 6 1s or more. If it starts with 6 1s, there are 2^4 such strings. Otherwise, the run could start in positions 2, 3, 4 or 5. But the bit before the 1s must be 0. Thus there are $4 \cdot 2^3$ to cover these cases. Total $= 2^4 + 4 \cdot 2^3 = 16 + 32 = 48$.

Problem 7: [5 marks]

(a) **[2 marks]** How many permutations of the 26 capital letters **A-Z** contain the substring **POT** or the substring **HERB** or the substring **JOINT**?

Solution: Three set Inclusion/Exclusion: There are 2^{21} permutations with both POT and HERB (each thought of as one letter). There are 2^{19} permutations with both HERB and JOINT. There are 0 with both POT and JOINT. The intersection of all 3 is also empty. Thus:

$$(24! + 23! + 22!) - 21! - 19! - 0 + 0$$

(b) **[1 mark]** Trick or treat. Five children show up looking for candy at Halloween. You have 25 identical candy bars to give to the 5 children. How many ways can you distribute the candy bars if each child gets at least 2 candy bars?

Solution: Give each kid 2 candy bars (they get at least 2). That leaves $r = 15$ identical bars to give to $n = 5$ distinct kids. Thus: $\binom{19}{15} = \binom{19}{4}$.

(c) **[2 marks]** In the 2020 Olympic winter games, Team Canada has 100 athletes but Team Jamaica has only 10 athletes. As a show of solidarity, the two teams decide to enter the opening ceremony together. If the players march out one at a time, how many different ways can they be ordered so **no athlete** from Jamaica follows another athlete from Jamaica?

Solution: Consider a sequence of 100 C's and 10 J's with no consecutive J's. Start with J C J C J C J C J C J C J C J C J C J. Consider all places now to add an extra C. There are 11 such places, and there are 91 more Cs to add. Thus there are $\binom{11+91-1}{91}$ ways to do this. For each such string we must consider all permutations of the Canadian and Jamaican athletes:

$$\binom{101}{91} \cdot 100! \cdot 10!$$

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