1. For the translational mechanical system shown below where $x_3(t)$ and $x_4(t)$ are known displacements, sketch the free body diagram for **A:** M_1 **B:** M_2 and write the corresponding D'Alembert's law.

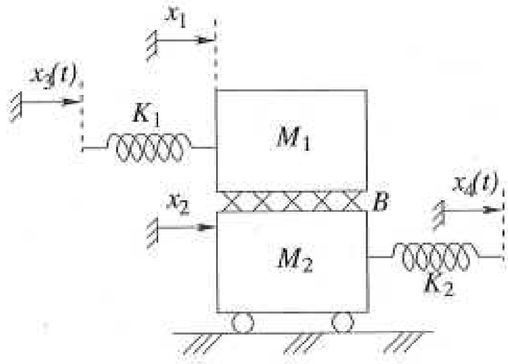
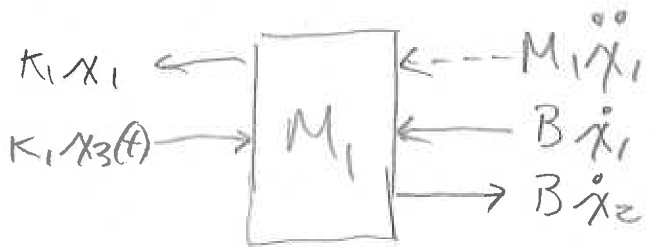


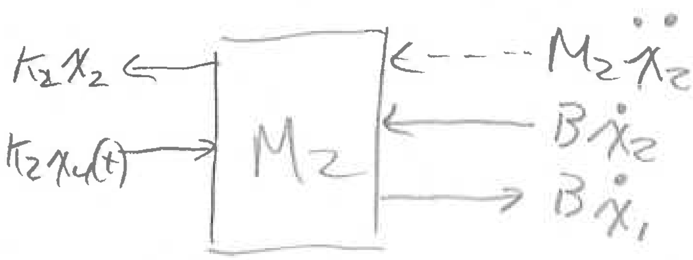
Figure 1: Translational mechanical system for question 1.

A:



$$M_1\ddot{x}_1 + B\dot{x}_1 + K_1x_1 - B\dot{x}_2 = K_1x_3(t)$$

B:



$$M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2 - B\dot{x}_1 = K_2x_4(t)$$

2. For the translational mechanical system shown below, sketch the free body diagram for **A:** M_1 **B:** M_2 and write the corresponding D'Alembert's law.

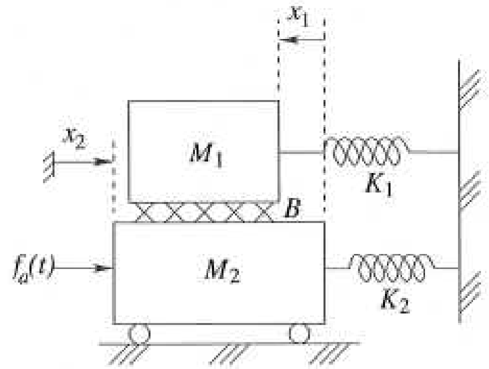


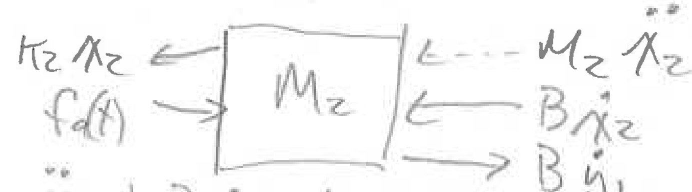
Figure 2: Translational mechanical system for question 2.

A: Define y_1 as absolute motion of M_1 +ve → s.t. $y_1 = x_2 - x_1$

$$M_1\ddot{y}_1 + B\dot{y}_1 + K_1 y_1 - B\dot{x}_2 = 0$$

$$M_1(\ddot{x}_2 - \ddot{x}_1) - B\dot{x}_1 + K_1(x_2 - x_1) = 0$$

B: Define y_1 as absolute motion of M_1 +ve → s.t. $y_1 = x_2 - x_1$



$$M_2\ddot{x}_2 + B\dot{x}_2 + K_2 x_2 - B\dot{y}_1 = f_a(t)$$

$$M_2\ddot{x}_2 + B\dot{x}_1 + K_2 x_2 = f_a(t)$$

3. For the rotational mechanical system shown below, where $\tau_a(t)$ is a known torque, sketch the free body diagram for **A:** J_1 **B:** J_2 and write the corresponding D'Alembert's law.

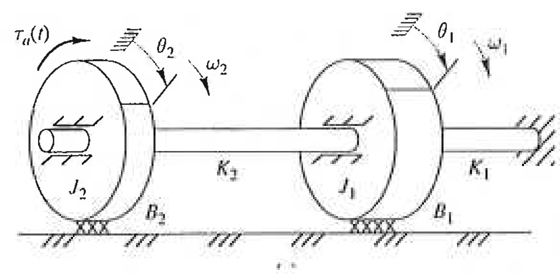
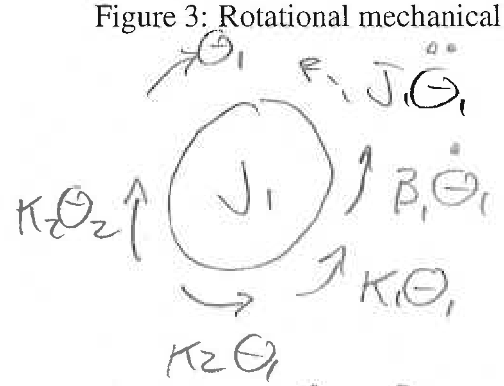


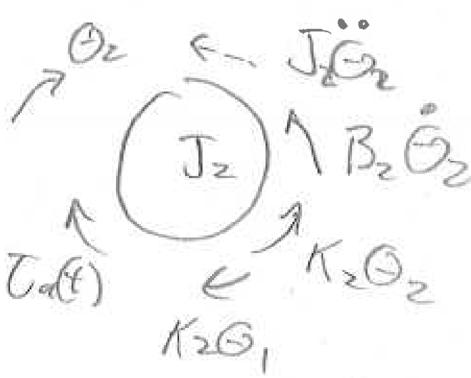
Figure 3: Rotational mechanical system for question 3.

A:



$$J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + (K_1+K_2)\theta_1 - K_2\theta_2 = 0$$

B:



$$J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + K_2\theta_2 - K_2\theta_1 = \tau_a(t)$$

4. For the mechanical system shown below, where $f_a(t)$ is a known force, sketch the free body diagram for **A:** M_1 **B:** M_2 and write the corresponding D'Alembert's law using only the variables x_1, x_2, and θ as indicated in the figure. Assume that the lever is ideal, i.e., no mass or friction.

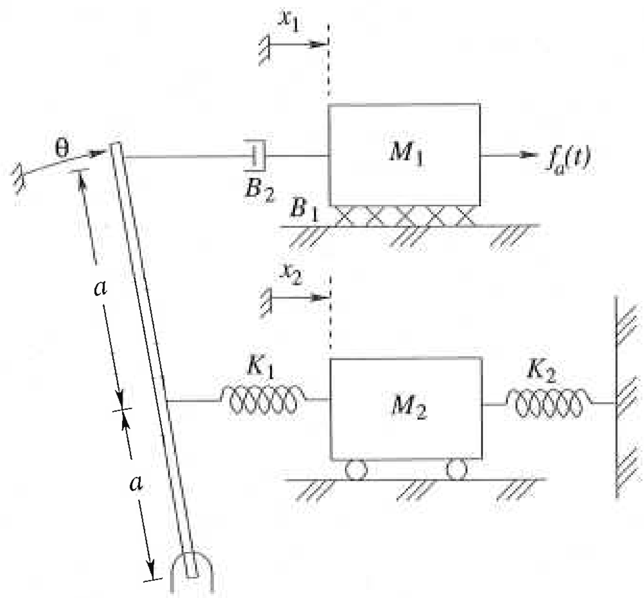
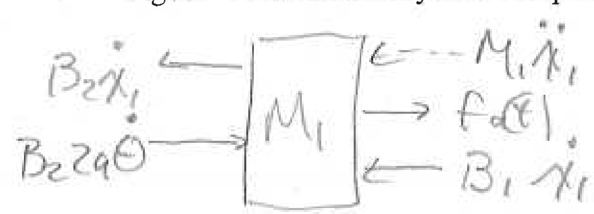


Figure 4: Mechanical system for question 4.

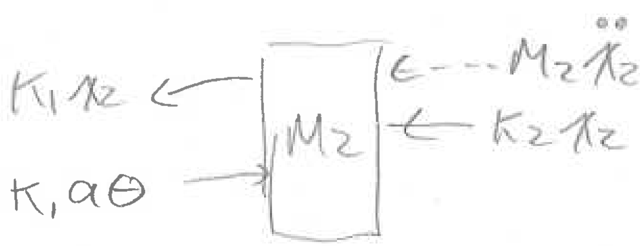
assuming small θ

A:



$$M_1\ddot{x}_1 + (B_1+B_2)\dot{x}_1 - B_2 2a\dot{\theta} = f_a(t)$$

B:



$$M_2\ddot{x}_2 + (K_1+K_2)x_2 - K_1a\theta = 0$$

5. For the mechanical system shown below, sketch the free body diagram for J and write the corresponding D'Alembert's law. Do take into account gravitational force(s).

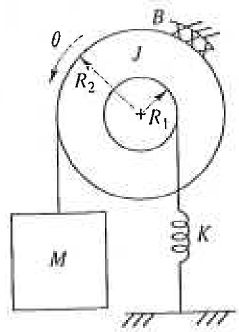
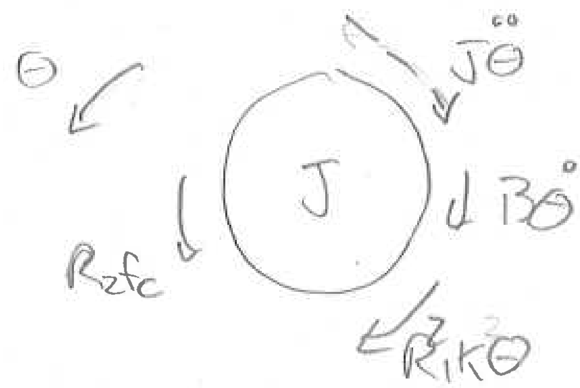


Figure 5: Mechanical system for question 5.



$\downarrow f_K = Kx$
$= KR_1\theta$

$\tau_K = R_1 K R_1 \theta$

contact force $f_c\downarrow$

$\tau_c = R_2 f_c$

$$J\ddot{\theta} + B\dot{\theta} + R_1^2 K\theta - R_2 f_c = 0$$

(can define $f_c\uparrow$)

5. For the mechanical system shown below, sketch the free body diagram for J and write the corresponding D'Alembert's law. Do take into account gravitational force(s).

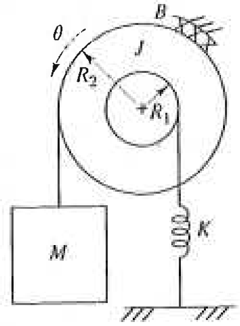
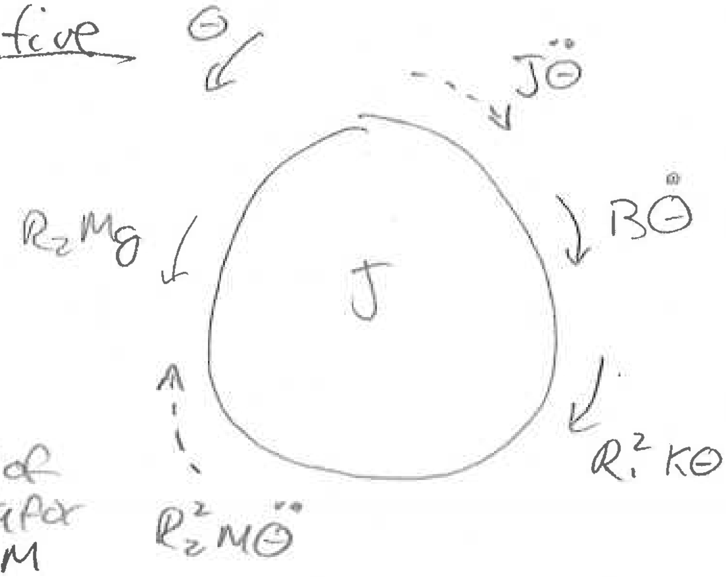


Figure 5: Mechanical system for question 5.

Alternative



$f_K = Kx$
$= KR_1\Theta$
$\tau_K = R_1KR_1\Theta$

Moment of inertia for mass M

R_2^2M

$$(J+R_2^2M)\ddot{\Theta} + B\dot{\Theta} + R_1^2K\Theta = R_2Mg$$

6. If applying D'Alembert's law to the two free body diagrams of some **A:** translational **B:** rotational mechanical system yields the following two differential equations, write the input-output differential equation where

   A: $f_a(t)$ is the input and x_1 is the output.

$$\ddot{x}_1 + 5\dot{x}_1 + 2x_1 - \dot{x}_2 - x_2 = f_a(t)$$
$$\ddot{x}_2 + \dot{x}_2 + 2x_2 - \dot{x}_1 - x_1 = 0$$

   B: $\tau_a(t)$ is the input and θ_1 is the output.

$$\ddot{\theta}_1 + 5\dot{\theta}_1 + 2\theta_1 - \dot{\theta}_2 - \theta_2 = \tau_a(t)$$
$$\ddot{\theta}_2 + \dot{\theta}_2 + 2\theta_2 - \dot{\theta}_1 - \theta_1 = 0$$

A:) $(D^2+5D+2)x_1 + (-D-1)x_2 = f_a(t)$ ①

$(-D-1)x_1 + (D^2+D+2)x_2 = 0$ ②

to eliminate x_2: ① $\times (D^2+D+2)$ − ② $\times (-D-1)$

$$(D^2+D+2)(D^2+5D+2)x_1 = (D^2+D+2)f_a(t)$$
$$+(D+1)(-D-1)x_1 = 0$$

$$[D^4+5D^3+2D^2+D^3+5D^2+2D+2D^2+10D+4$$
$$-D^2-2D-1]x_1 = (D^2+D+2)f_a(t)$$

$$(D^4+6D^3+8D^2+6D+3)x_1 = (D^2+D+2)f_a(t)$$

$$D^4x_1+6D^3x_1+8D^2x_1+6Dx_1+3x_1 = D^2f_a(t)+Df_a(t)+2f_a(t)$$

B) Similarly for Θ_1

$$D^4\Theta_1+6D^3\Theta_1+8D^2\Theta_1+6D\Theta_1+3\Theta_1 = D^2\tau_a(t)+D\tau_a(t)+2\tau_a(t)$$

7. If applying D'Alembert's law to the two free body diagrams of the **A:** rotational **B:** translational mechanical system as shown below yields the following two differential equations, express the system in state variable form by
   (a) Defining an appropriate set of state variables
   (b) Writing the state variable equations in matrix form.

A:

$$J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + (K_1+K_2)\theta_1 - K_2\theta_2 = 0$$
$$J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + K_2\theta_2 - K_2\theta_1 = \tau_a(t)$$

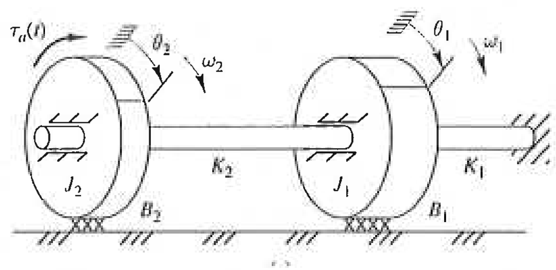


Figure 6: Rotational mechanical system for question 7.

a) Define θ_1 ω_1 θ_2 ω_2 as state variables

b) $\dot{\theta}_1 = \omega_1$

$\dot{\theta}_2 = \omega_2$

$\dot{\omega}_1$: from J_1 FBD

$$J_1\dot{\omega}_1 + B_1\omega_1 + (K_1+K_2)\theta_1 - K_2\theta_2 = 0$$

$$\dot{\omega}_1 = -\frac{(K_1+K_2)}{J_1}\theta_1 - \frac{B_1}{J_1}\omega_1 + \frac{K_2}{J_1}\theta_2 + 0\,\omega_2$$

$\dot{\omega}_2$: from J_2 FBD

$$J_2\dot{\omega}_2 + B_2\omega_2 + K_2\theta_2 - K_2\theta_1 = \tau_a(t)$$

$$\dot{\omega}_2 = \frac{K_2}{J_2}\theta_1 + 0\,\omega_1 - \frac{K_2}{J_2}\theta_2 - \frac{B_2}{J_2}\omega_2 + \frac{1}{J_2}\tau_a(t)$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\omega}_1 \\ \dot{\theta}_2 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(K_1+K_2)}{J_1} & \frac{-B_1}{J_1} & \frac{K_2}{J_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_2}{J_2} & 0 & \frac{-K_2}{J_2} & \frac{-B_2}{J_2} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \omega_1 \\ \theta_2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_2} \end{bmatrix} \tau_d(t)$$

1

B:

$$M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 = f_a(t)$$
$$M_2\ddot{x}_2 + B_2\dot{x}_2 + (K_1 + K_2)x_2 - K_1x_1 = 0$$

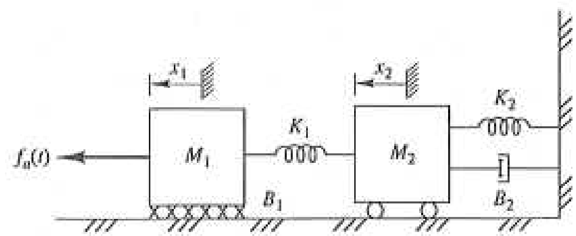


Figure 7: Translational mechanical system for question 7.

Similarly

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(K_1+K_2)}{M_1} & \frac{-B_1}{M_1} & \frac{K_2}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_2}{M_2} & 0 & \frac{-K_2}{M_2} & \frac{-B_2}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \end{bmatrix} f_a(t)$$