University of Ottawa
Department of Mathematics and Statistics
Calculus III for Engineers
MAT 2322 X00 - Spring-Summer 2018
Midterm II - V.1 – July 10
Professor: Abdelkrim El basraoui

Solution

Name:________________________ ID Number:____________________

Instructions: (Please read carefully.)

- This exam has 10 pages and 6 questions, and you have 80 minutes to complete it.
- This is a closed book exam.
- **The only calculators which are allowed are those approved by the faculty of science such as Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Questions 1 and 2 are worth 5 marks each, Questions 3 and 4 are worth 4 marks each, and Questions 5 and 6 are worth 6 marks, so organize your time accordingly.
- Answer each question in the space provided or using backs of pages if necessary.
- Page 8 is an extra page for Question 5.
- **A correct answer requires a full, clearly-written and detailed solution.**
- Do not unstaple the test.
- Good luck!
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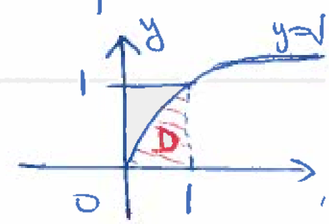
Signature: ________________________________

1. Consider the solid that lies below the plane $z = 1 + x$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$. This solid has a mass density given by the function $\delta(x, y, z) = y$. Find the total mass of this solid.

- Solid region: $E = \{(x,y,z) \mid (x,y) \in D,\ 0 \leq z \leq 1+x\}$,

where $D = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{x}\}$

(or $D = \{(x,y) \mid 0 \leq y \leq 1,\ y^2 \leq x \leq 1\}$)



- Total mass:

$$m = \iiint_E \delta(x,y,z)\, dV$$

$$= \iint_D \left(\int_0^{1+x} y\, dz \right) dA$$

$$= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x} y\, dz\, dy\, dx \quad \left(\overset{\text{or}}{=} \int_0^1 \int_{y^2}^1 \int_0^{1+x} y \cdot dz\, dx\, dy \right)$$

$$= \int_0^1 \int_0^{\sqrt{x}} (1+x)\, y\, dy\, dx$$

$$= \int_0^1 \frac{x}{2}(1+x)\, dx$$

$$= \boxed{\frac{5}{12}}$$

2. Compute the following double integral

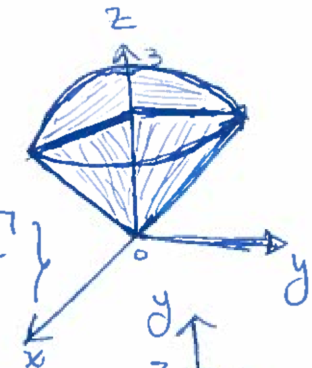
$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{|y|}^{\sqrt{4-y^2}} (x^2 + y^2)\, dx\, dy$$

Hint: Sketch the region of integration in the xy-plane, and then express this region and the integral in a different coordinate system.

(See Version 2)

3. Consider the solid in the first octant which is bounded by the cone $z = \frac{\sqrt{3}}{3}\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 9$. This solid has a mass density given by the function $\delta(x,y,z) = 27 - 4x^2 - 4y^2 - 4z^2$. Set up a triple integral in **spherical coordinates** which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**

- First, note the angle the cone makes with the positive z-axis. It's $\frac{\pi}{3} = \arctan\left(\frac{1}{(\sqrt{3}/3)}\right)$.

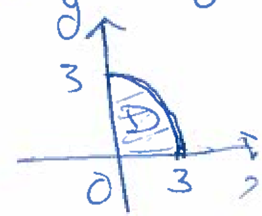


- Solid region:

$E = \left\{(x,y,z) \mid (x,y)\in D,\ \frac{\sqrt{3}}{3}\sqrt{x^2+y^2} \le z \le \sqrt{9-x^2-y^2}\right\}$

where D is the projection of E in the xy-plane.

So, $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le \sqrt{9-x^2}\}$.



- In spherical coords

$E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 3,\ \overbrace{0 \le \theta \le \tfrac{\pi}{2}}^{\text{because of 1st octant}},\ 0 \le \phi \le \tfrac{\pi}{3}\right\}$

& $\delta(\rho,\theta,\phi) = 27 - 4\rho^2$,

so that the total mass is

$$\boxed{m = \int_0^3 \int_0^{\pi/2} \int_0^{\pi/3} (27-4\rho^2)\, \overbrace{\rho^2 \sin\phi \, d\phi\, d\theta\, d\rho}^{dV}}$$

4. Find the total arclength of the curve which is parametrized by the following vector function

$$\vec{r}(t) = \sin(t)\,\vec{\mathbf{i}} + \cos(t)\,\vec{\mathbf{j}} + 4t\,\vec{\mathbf{k}}, \quad 0 \le t \le \pi/4.$$

- Tangent vector: $\vec{r}\,'(t) = \cos t\,\vec{i} - \sin t\,\vec{j} + 4\vec{k}$

$\Longrightarrow \quad \|\vec{r}\,'(t)\| = \sqrt{17}$.

- Arc length:

$$L = \int_0^{\pi/4} \|\vec{r}\,'(t)\|\,dt = \boxed{\frac{\pi\sqrt{17}}{4}}$$

5. Among the following three vector fields there is **exactly one** conservative vector field.

$$\vec{F} = (2x + 2xe^{-y})\vec{\mathbf{i}} - x^2e^{-y}\vec{\mathbf{j}}$$

$$\vec{G} = x^2y\vec{\mathbf{i}} - xy^2\vec{\mathbf{j}}$$

$$\vec{H} = e^{xy}\cos(x)\vec{\mathbf{i}} + e^{xy}\sin(y)\vec{\mathbf{j}}$$

(a) Determine which **one** of these vector fields is conservative. If a vector field is not conservative, justify why it is not conservative.

(b) Find a potential function f **for the conservative vector field** in (a).

(c) Evaluate the line integral of **the conservative vector field** in (a) along the curve C, where C is any smooth curve from $(0,0)$ to $(1,1)$.

(a) i/ $\vec{F}$: There $P(x,y) = 2x + 2xe^{-y}$ & $Q(x,y) = -x^2e^{-y}$

So that $P_y = -2xe^{-y} = Q_x$ & so $\boxed{\vec{F} \text{ is conservative}}$

ii/ $\vec{G}$: There $P(x,y) = x^2y$ & $Q(x,y) = -xy^2$

So that $P_y = x^2 \neq Q_x = -y^2$ & so $\vec{G}$ is not conservative

iii/ $\vec{H}$: There $P(x,y) = e^{xy}\cos x$ & $Q(x,y) = e^{xy}\sin y$

So that $P_y = xe^{xy}\cos x \neq Q_x = ye^{xy}\sin y$,

& so $\vec{H}$ is not conservative.

In conclusion: $\vec{F}$ is the only conservative vector-field.

(b) If $f(x,y)$ is such that $\vec{F} = \nabla f$, then

$$\begin{cases} f_x = 2x + 2xe^{-y} \\ \& \\ f_y = -x^2e^{-y} \end{cases}$$

(Integrate either f_x or f_y)

For instance, we have

Extra page for Question 5. v.1

$f_y = -x^2 e^{-y} \implies f(x,y) = \int f_y \, dy = x^2 e^{-y} + g(x)$

But $f_x = 2x e^{-y} + g'(x) = 2x + 2x e^{-y}$

$\implies g'(x) = 2x \implies g(x) = x^2 + C$, take $C = 0$.

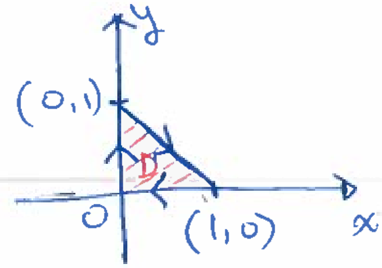
Thus, a potential fn is

$$\boxed{f(x,y) = x^2 e^{-y} + x^2}$$

(c) By the F.T.L.I, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1,1) - f(0,0) = \boxed{e^{-1} + 1}$$

6. Use Green's Theorem to evaluate the line integral $\oint_C \underbrace{x^4}_{P}\,dx + \underbrace{xy}_{Q}\,dy$, where C is the triangular curve consisting of the line segments from (0,0) to (0,1), from (0,1) to (1,0), and from (1,0) to (0,0).



- The curve C is oriented clockwise. (So, it's not a positive orientation).
- The region D is therefore

$$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}.$$

- We apply Green's Thm to $-C$. We have

$$\oint_{-C} x^4\,dx + xy\,dy = \iint_D (y - 0)\,dA$$

$$= \int_0^1 \int_0^{1-x} y\,dy\,dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2\,dx$$

$$= \boxed{\frac{1}{6}}$$

Thus $$\oint_C x^4\,dx + xy\,dy = -\oint_{-C} x^4\,dx + xy\,dy = \boxed{\boxed{-\frac{1}{6}}}$$

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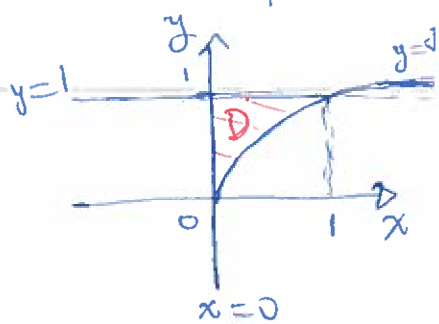
- Solid region: $E = \{(x,y,z) \mid (x,y) \in D,\ 0 \leq z \leq 1+y\}$,

where $D = \{(x,y) \mid 0 \leq x \leq 1,\ \sqrt{x} \leq y \leq 1\}$

(or $D = \{(x,y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y^2\}$)



- Total mass:

$$m = \iiint_E \delta(x,y,z)\, dV$$

$$= \iint_D \left(\int_0^{1+y} x \, dz \right) dA$$

$$= \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1+y} x \, dz\, dy\, dx \quad \left(\overset{\text{or}}{=} \int_0^1 \int_0^{y^2} \int_0^{1+y} x\, dz\, dx\, dy \right)$$

$$= \int_0^1 \int_{\sqrt{x}}^1 (1+y)\, x\, dy\, dx$$

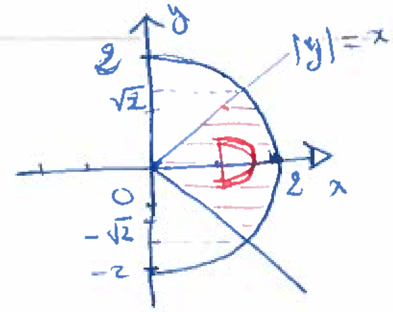
$$= \int_0^1 x\left(\frac{3}{2} - \sqrt{x} - \frac{x}{2} \right) dx$$

$$= \boxed{\frac{11}{60}}$$

2. Compute the following double integral

$$I = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{|y|}^{\sqrt{4-y^2}} (x^2 + y^2)\, dx\, dy$$

Hint: Sketch the region of integration in the xy-plane, and then express this region and the integral in a different coordinate system.



- Region of integration:

$$D = \{(x,y) \mid -\sqrt{2} \leq y \leq \sqrt{2},\ |y| \leq x \leq \sqrt{4-y^2}\}$$

- In polar coords.:

$$D = \{(r,\theta) \mid 0 \leq r \leq 2,\ -\pi/4 \leq \theta \leq \pi/4\}$$

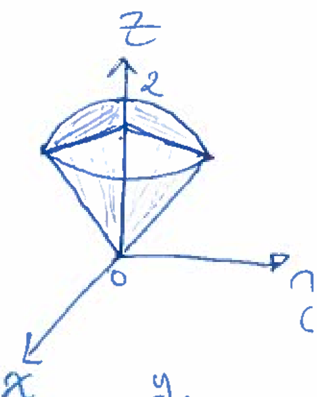
& $x^2 + y^2 = r^2$.

So that $I = \int_0^2 \int_{-\pi/4}^{\pi/4} r^2 \cdot r\, d\theta\, dr$ (as $dx\, dy = r\, d\theta\, dr$

$$I = \int_0^2 \frac{\pi}{2} r^3\, dr = \boxed{2\pi}$$

3. Consider the solid in the first octant which is bounded by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=4$. This solid has a mass density given by the function $\delta(x,y,z) = 18-3x^2-3y^2-3z^2$. Set up a triple integral in **spherical coordinates** which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**

- Fist, note the angle the cone makes with the positive z-axis. It's $\frac{\pi}{6} = \arctan(1/\sqrt{3})$.



- Solid region:

$$E = \{(x,y,z) \mid (x,y)\in D,\ \sqrt{3}\sqrt{x^2+y^2} \le z \le \sqrt{4-x^2-y^2}\},$$

where D is the projection of E in the xy-plane.

So, $D = \{(x,y) \mid 0\le x\le 2,\ 0\le y\le \sqrt{4-x^2}\}$

- In spherical coords.

$$E = \{(\rho,\theta,\phi) \mid 0\le\rho\le 2,\ \underbrace{0\le\theta\le\tfrac{\pi}{2}}_{\text{because of 1st octant}},\ 0\le\phi\le\tfrac{\pi}{6}\}$$

& $\delta(\rho,\theta,\phi) = 18-3\rho^2$, so that the total mass is

$$\boxed{m = \int_0^2\int_0^{\pi/2}\int_0^{\pi/6}(18-3\rho^2)\,\underbrace{\rho^2\sin\phi\,d\phi\,d\theta\,d\rho}_{dV}}$$

4. Find the total arclength of the curve which is parametrized by the following vector function

$$\vec{r}(t) = \cos(t)\,\vec{\mathbf{i}} + \sin(t)\,\vec{\mathbf{j}} + 3t\,\vec{\mathbf{k}}, \quad 0 \le t \le \pi/4.$$

- Tangent vector: $\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 3\,\vec{k}$

$$\Longrightarrow \quad \|\vec{r}'(t)\| = \sqrt{10}$$

- Arc length:

$$L = \int_0^{\pi/4} \|\vec{r}'(t)\|\,dt = \boxed{\frac{\pi\sqrt{10}}{4}}$$

5. Among the following three vector fields there is **exactly one** conservative vector field.

$$\vec{F} = x^2 y\vec{\mathrm{i}} - xy^2\vec{\mathrm{j}}$$

$$\vec{G} = 2xe^{-y}\vec{\mathrm{i}} + (2y - x^2e^{-y})\vec{\mathrm{j}}$$

$$\vec{H} = e^x\cos(xy)\vec{\mathrm{i}} + e^x\sin(xy)\vec{\mathrm{j}}$$

(a) Determine which **one** of these vector fields is conservative. If a vector field is not conservative, justify why it is not conservative.

(b) Find a potential function f **for the conservative vector field** in (a).

(c) Evaluate the line integral of **the conservative vector field** in (a) along the curve C, where C is any smooth curve from $(0,0)$ to $(1,1)$.

(a) i/ $\vec{F}$: Here $P(x,y) = x^2y$ & $Q(x,y) = -xy^2$.
So that $P_y = x^2 \neq Q_x = -y^2$ & so $\vec{F}$ is not conservative.

ii/ $\vec{G}$: Here $P(x,y) = 2xe^{-y}$ & $Q(x,y) = 2y - x^2e^{-y}$
So that $P_y = -2xe^{-y} = Q_x$ & so $\boxed{\vec{G} \text{ is conservative.}}$

iii/ $\vec{H}$: here $P(x,y) = e^x\cos(xy)$ & $Q(x,y) = e^x\sin(xy)$.
So that $P_y = -xe^x\sin(xy) \neq Q_x = e^x\sin(xy) + ye^x\cos(xy)$.
& so $\vec{H}$ is not conservative.

In conclusion: $\vec{G}$ is the only conservative vector-field.

(b) If $f(x,y)$ is such that $\vec{G} = \nabla f$, then

$$\begin{cases} f_x = 2xe^{-y} \\ \& \\ f_y = 2y - x^2e^{-y} \end{cases}$$

(Integrate either f_x or f_y)

For instance, we have

Extra page for Question 5. v.2

$f_x = 2xe^{-y} \implies f(x,y) = \int f_x \, dx = x^2 e^{-y} + h(y)$

But $f_y = -x^2 e^{-y} + h'(y) = 2y - x^2 e^{-y} \iff h'(y) = 2y$

$\implies h(y) = y^2 + C$. Take $C = 0$.

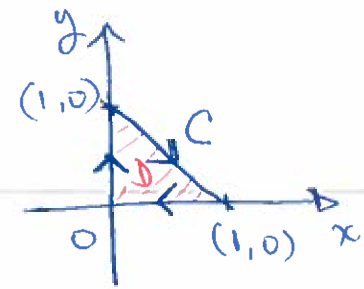
Thus, a potential fcn for $\vec{G}$ is

$$\boxed{f(x,y) = x^2 e^{-y} + y^2}$$

(c) By the F.T.L.I, we have

$$\int_C \vec{G} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1,1) - f(0,0) = \boxed{e^{-1} + 1}$$

6. Use Green's Theorem to evaluate the line integral $\oint_C \overset{P}{xy}\,dx + \overset{Q}{y^4}\,dy$, where C is the triangular curve consisting of the line segments from (0,0) to (0,1), from (0,1) to (1,0), and from (1,0) to (0,0).



• The curve C is oriented clockwise.
(So, it's not a positive orientation).

• The region D is therefore

$$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$$

• We apply Green's Thm to $-C$. We have

$$\oint_{-C} xy\,dx + y^4\,dy = \iint_D (0 - x)\,dA$$

$$= \int_0^1 \int_0^{1-x} -x\,dy\,dx$$

$$= \int_0^1 -x(1-x)\,dx$$

$$= \boxed{-\frac{1}{6}}$$

Thus $$\oint_C xy\,dx + y^4\,dy = -\oint_{-C} xy\,dx + y^4\,dy = \boxed{\frac{1}{6}}$$