University of Ottawa
Department of Mathematics and Statistics
Calculus III for Engineers
MAT 2322 X00 - Spring-Summer 2018
Midterm I - V.1
Professor: Abdelkrim El basraoui
Duration: 80 minutes.

Name:______________________________ ID Number:____________________

Instructions: (Please read carefully.)

- This exam has 8 pages and 6 questions, and you have 80 minutes to complete it.
- This is a closed book exam.
- **The only calculators which are allowed are those approved by the faculty of science such as Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Questions 1, 2, 4, 5 and 6 are worth 7 marks each, and question 3 is worth 5 marks, so organize your time accordingly.
- Answer each question in the space provided, using backs of pages or the extra page at the end if necessary.
- **A correct answer requires a full, clearly-written and detailed solution.**
- Do not unstaple the test.
- Good luck!
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Signature: ____________________________________

1. Find and classify the critical points of the function $f(x, y) = x^3 - 3x + 3xy^2 + 10$.

- Critical points: We have $\nabla f = \langle 3x^2 - 3 + 3y^2, 6xy \rangle$

So $\nabla f = \langle 0, 0 \rangle \iff \begin{cases} 3x^2 - 3 + 3y^2 = 0 \\ \& \\ 6xy = 0 \end{cases} \iff \begin{cases} x^2 + y^2 = 1 \quad (1) \\ \& \\ x = 0 \text{ or } y = 0 \end{cases}$.

If $x = 0$, equation (1) gives $y = \pm 1$ & we have the points $(0, \pm 1)$.

If $y = 0$, equation (1) gives $x = \pm 1$ & we have the points $(\pm 1, 0)$.

- Classification: We have $f_{xx} = 6x$, $f_{xy} = 6y = f_{yx}$, $f_{yy} = 6x$

So that $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 36x^2 - 36y^2$.

* At the points $(0, \pm 1)$, we have $D(0, \pm 1) = -36 < 0$ & thus $(0, \pm 1)$ are saddle points.

* At the points $(\pm 1, 0)$, we have $D(\pm 1, 0) = 36 > 0$ & $f_{xx}(\pm 1, 0) = \pm 6$. Therefore $(-1, 0)$ is a local maximum & $(1, 0)$ is a local minimum.

2. Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y) = x^2 + 2y^3$ on the unit circle $\underbrace{x^2 + y^2 = 1 = g(x,y)}_{\text{constraint}}$.

We have $\nabla f = \lambda \nabla g \iff \begin{cases} 2x = 2\lambda x \\ \& \\ 6y^2 = 2\lambda y \end{cases} \iff \begin{cases} 2x(1-\lambda) = 0 \\ \& \\ 2y(3y-\lambda) = 0 \end{cases}$

$$\iff \begin{cases} x = 0 \text{ or } \lambda = 1 \\ \& \\ y = 0 \text{ or } y = \frac{\lambda}{3}. \quad (2) \end{cases}$$

- If $x = 0$, the constraint implies that $y = \pm 1$ & we have the points $(0, \pm 1)$.
- If $\lambda = 1$, then (2) implies that either $y = 0$ or $y = \frac{1}{3}$.
  - * If $y = 0$, then the constraint gives $x = \pm 1$ & so we get the points $(\pm 1, 0)$
  - * If $y = \frac{1}{3}$, we get from the constraint that $x^2 = \frac{8}{9} \iff x = \pm \frac{2\sqrt{2}}{3}$ & we get the points $\left(\pm \frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$;

To conclude, we compare the following values of f.

$f(\pm 1, 0) = 1$; $f\left(\pm \frac{2\sqrt{2}}{3}, \frac{1}{3}\right) = \frac{26}{27}$; $f(0,-1) = -2$; $f(0,1) = 2$.

Thus a max value of 2 is attained at the point $(0,1)$ & a min value of -2 is attained at the point $(0,-1)$.

3. If $f(x,y) = x\,e^{-xy}$ and R is the rectangle $[0,2] \times [0,3]$, what is the value of the double integral of f over R, $\iint_R f\,dA$?

The longer way

$$\iint_R f\,dA = \int_0^2 \int_0^3 x\,e^{-xy}\,dy\,dx \quad \left(\underline{\text{or}}\ \int_0^3 \int_0^2 x\,e^{-xy}\,dx\,dy \right)$$

$$= \int_0^2 \left[-e^{-xy} \right]_0^3 dx$$

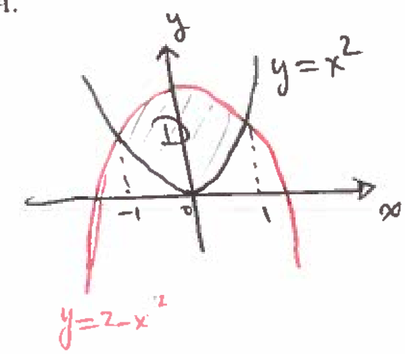
$$= \int_0^2 \left(1 - e^{-3x} \right) dx$$

$$= \left[x + \frac{1}{3} e^{-3x} \right]_0^2$$

$$= \boxed{\frac{5 + e^{-6}}{3}}$$

4. Let D be the region in the xy-plane bounded by the parabolas $y = 2 - x^2$ and $y = x^2$. **Sketch** the region D then **evaluate** the following double integral $\iint_D y\, dA$.

Region $\longrightarrow$



We see that $D = \{(x,y) \mid -1 \leq x \leq 1,\ x^2 \leq y \leq 2-x^2\}$

(In fact, $x^2 = 2 - x^2 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$).

Therefore $$\iint_D y\, dA = \int_{-1}^{1}\int_{x^2}^{2-x^2} y\, dy\, dx$$

$$= \int_{-1}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{2-x^2} dx$$

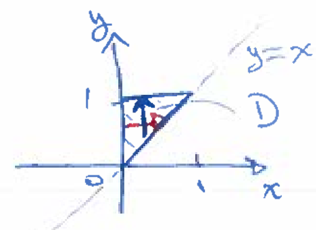
$$= \int_{-1}^{1} (2 - 2x^2)\, dx$$

$$= \left[2x - \frac{2}{3}x^3\right]_{-1}^{1}$$

$$= \boxed{\frac{8}{3}}$$

5. Sketch the region of integration then **evaluate** the following double integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = I$

[Hint: use the region to reverse the order of integration.]



Region : $D = \{(x,y) \mid 0 \leq x \leq 1,\ x \leq y \leq 1\}$

It's a type I & II region.

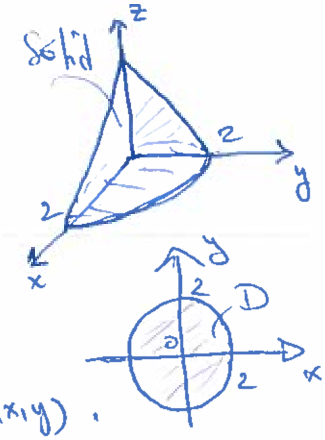
D as type II is in fact $D = \{(x,y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}$.

Therefore $I = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 y \sin(y^2)\, dy$

$= \left[-\frac{1}{2}\cos(y^2)\right]_0^1$

$= \boxed{\frac{1-\cos(1)}{2}}$

6. Set up a double integral in **polar coordinates** to compute the volume of the solid under the cone $z = 4 - 2\sqrt{x^2 + y^2}$ and above xy−plane, then **evaluate it**.



The cone meets the xy-plane (with equation $z=0$) at $0 = 4 - 2\sqrt{x^2+y^2} \Longleftrightarrow x^2+y^2 = 2^2$.

Therefore, the region of integration is the disk $D = \{(x,y) \mid x^2+y^2 \leq 4\}$.

The function to integrate is $\underbrace{4-2\sqrt{x^2+y^2}}_{\text{Top surface}} - \underbrace{0}_{\text{Bottom surface}} = f(x,y)$.

Now, in polar coordinates

$D = \{(r,\theta) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$.

& $f(r\cos\theta, r\sin\theta) = 4 - 2r$

Thus, the volume is $V = \int_0^2 \int_0^{2\pi} (4-2r) \underbrace{r\,d\theta\,dr}_{dA}$

$$V = \int_0^2 2\pi(4r-2r^2)\,dr = \left[2\pi\left(2r^2 - \frac{2}{3}r^3\right)\right]_0^2$$

$$= \boxed{\frac{16\pi}{3}}$$

Note: The volume of this solid in cartesian coordinates is

$$V = \iint_D (\underbrace{f(x,y)}_{\text{Top Surface}} - \underbrace{0}_{\text{Bottom Surface}})\,dA$$

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• Critical points: We have $\nabla f = \langle 3y^2 - 3 + 3x^2, 6xy \rangle$.

so $\nabla f = \langle 0,0 \rangle \iff \begin{cases} 3y^2 - 3 + 3x^2 = 0 \\ \& \\ 6xy = 0 \end{cases} \iff \begin{cases} x^2 + y^2 = 1 & (1) \\ \& \\ x = 0 \text{ or } y = 0 \end{cases}$

If $x=0$, equation (1) gives $y = \pm 1$ & we have the points $(0, \pm 1)$.

If $y = 0$, equation (1) gives $x = \pm 1$ & we have the points $(\pm 1, 0)$

• Classification: We have $f_{xx} = 6x$, $f_{xy} = 6y = f_{yx}$, $f_{yy} = 6x$

so that $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 36x^2 - 36y^2$.

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We solve $\nabla f = \lambda \nabla g \iff \begin{cases} 2x = 2\lambda x \\ \& \\ -6y^2 = 2\lambda y \end{cases} \iff \begin{cases} 2x(1-\lambda) = 0 \\ \& \\ 2y(3y+\lambda) = 0 \end{cases}$

$\iff \begin{cases} x = 0 \text{ or } \lambda = 1 \\ \& \\ y = 0 \text{ or } y = -\frac{\lambda}{3} \end{cases}$. (2)

- If $x = 0$, then the constraint gives $y = \pm 1$ & so the points $(0, \pm 1)$.
- If $\lambda = 1$, then (2) implies that either $y = 0$ or $y = -\frac{1}{3}$.

  * If $y = 0$, then the constraint gives $x = \pm 1$ & so the points $(\pm 1, 0)$
  * If $y = -\frac{1}{3}$, then constraint implies that $x^2 = \frac{8}{9} \iff x = \pm \frac{2\sqrt{2}}{3}$ & so the points $\left(\pm \frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$.

To conclude, we compare the following values of f:

$f(\pm 1, 0) = 1$; $f\left(\pm \frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = \frac{26}{27}$; $f(0,-1) = 2$; $f(0,1) = -2$

Thus a max value of 2 is obtained at the point $(0,-1)$ & a min value of -2 is obtained at the point $(0,1)$.

3. If $f(x, y) = y e^{-xy}$ and R is the rectangle $[0, 3] \times [0, 2]$, what is the value of the double integral of f over R, $\iint_R f \, dA$?

The Longer way

$$\iint_R f \, dA = \int_0^2 \int_0^3 y e^{-xy} \, dx \, dy \quad \left(\overset{\text{or}}{=} \int_0^3 \int_0^2 y e^{-xy} \, dy \, dx \right)$$

$$= \int_0^2 \left[-e^{-xy} \right]_0^3 dy$$

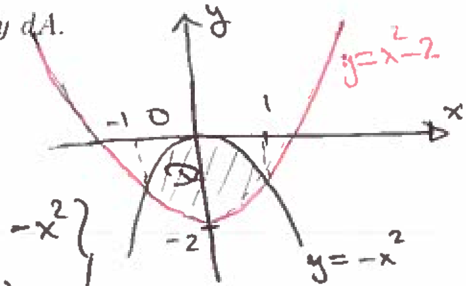
$$= \int_0^2 \left(1 - e^{-3y} \right) dy$$

$$= \left[y + \frac{1}{3} e^{-3y} \right]_0^2$$

$$= \boxed{\frac{5 - e^{-6}}{3}}$$

4. Let D be the region in the xy-plane bounded by the parabolas $y = x^2 - 2$ and $y = -x^2$. **Sketch** the region D then **evaluate** the following double integral $\iint_D y\, dA$.

Region : ⟶



We see that $D = \{(x,y) \mid -1 \le x \le 1,\ x^2-2 \le y \le -x^2\}$

(In fact, $x^2 - 2 = -x^2 \iff x^2 = 1 \iff x = \pm 1$).

Therefore, $\iint_D y\, dA = \int_{-1}^{1} \int_{x^2-2}^{-x^2} y\, dy\, dx$

$$= \int_{-1}^{1} \left[\frac{y^2}{2}\right]_{x^2-2}^{-x^2} dx$$

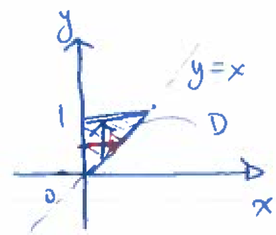
$$= \int_{-1}^{1} (2x^2 - 2)\, dx$$

$$= \left[\frac{2}{3}x^3 - 2x\right]_{-1}^{1}$$

$$= \boxed{\frac{-2}{3}}$$

5. Sketch the region of integration then **evaluate** the following double integral $\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx = I$

[Hint: use the region to reverse the order of integration.]



Region: $D = \{(x,y) \mid 0 \leq x \leq 1,\ x \leq y \leq 1\}$.

It's a type I & II region.

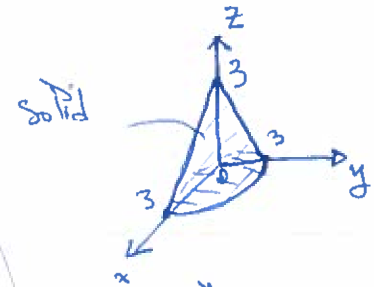
D as type II is $D = \{(x,y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}$.

Therefore $I = \int_0^1 \int_0^y \cos(y^2)\, dx\, dy = \int_0^1 y \cos(y^2)\, dy$

$= \left[\frac{1}{2} \sin(y^2) \right]_0^1$

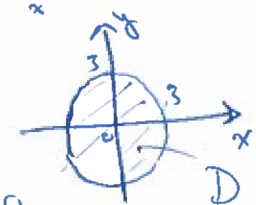
$= \boxed{\frac{\sin(1)}{2}}$

6. Set up a double integral in **polar coordinates** to compute the volume of the solid under the cone $z = 3 - \sqrt{x^2+y^2}$ and above $xy-$plane, then **evaluate it**.



The cone meets the xy-plane (with equation $z=0$) at $0 = 3 - \sqrt{x^2+y^2} \iff x^2+y^2 = 3^2$.

Therefore, the region of integration is the disk



$D = \{(x,y) \mid x^2+y^2 \le 9\}$.

The function to integrate is $\underbrace{3 - \sqrt{x^2+y^2}}_{\text{Top surface}} - \underbrace{0}_{\text{Bottom surface}} = f(x,y)$.

Now, in polar coordinates

$D = \{(r,\theta) \mid 0 \le r \le 3, \ 0 \le \theta \le 2\pi\}$

& $f(r\cos\theta, r\sin\theta) = 3 - r$.

Thus, the volume is $V = \int_0^3 \int_0^{2\pi} (3-r) \underbrace{r\,d\theta\,dr}_{dA}$

$$V = \int_0^3 2\pi(3r - r^2)\,dr = \left[2\pi\left(\frac{3}{2}r^2 - \frac{r^3}{3}\right)\right]_0^3$$

$$= \boxed{9\pi}$$

Note: the volume of this solid in cartesian coordinates is

$$V = \iint_D (\underbrace{f(x,y)}_{\text{Top surface}} - \underbrace{0}_{\text{Bottom surface.}})\,dA.$$