Name: ______________ **Last** (Family) **Name:** ______________

Section: ___

WILFRID LAURIER UNIVERSITY

Waterloo, Ontario

Mathematics 121 – Introduction to Mathematical Proofs

Midterm 1 – October 18, 2017

(SOLUTIONS)

Instructors:
Dr. R. Rundle: Section A - 10:00 a.m.
Dr. P. Zhang: Section B - 11:30 a.m.

Time Allowed: *80 minutes*

Total Value: *85 marks*

Number of Pages: *4 plus cover page*

Instructions:

No calculators are allowed. *No other aids are allowed.*

Check that your test paper has no missing, blank, or illegible pages.

Answer in the spaces provided. ***Please note that questions are printed on both sides of the page.***

Show all your work. ***Insufficient justification will result in a loss of marks.***

Student Number: ____________

[*5 marks*] 1. Let $U = \{a, b, c, d, e\}$ be the universal set. Let $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{c, d, e\}$. Determine:

(a) $(A \cup B) \cap C = \{c, d\}$

(b) $A \cap B \cap C = \{c\}$

(c) $A^c \cup (B^c \cap C^c) = \{a, d, e\}$

(d) $(A - (A \cap B)) \cup \{\emptyset\} = \{a, \emptyset\}$

(e) $(A \cap B) - (A \cup B) = \emptyset$

[*8 marks*] 2. (a) Complete the following truth tables.

p	q	$p\wedge \sim q$	$\sim p \vee p$	$\sim p \wedge q$	$p \to q$	$\sim (q \to p)$	$p \to \sim q$	$\sim q \to \sim p$	$\sim (q \wedge \sim q)$
T	T	F	T	**F**	T	F	F	**T**	T
T	F	T	**T**	F	F	**F**	T	F	T
F	T	F	T	T	T	T	T	T	**T**
F	F	**F**	T	F	**T**	F	**T**	T	T

[*4 marks*] (b) According to the truth tables above in part (a), determine whether each of the following statements is TRUE or FALSE. Circle your answer.

1) $(\sim (\sim q \wedge q)) \Leftrightarrow (p \vee \sim p)$ ($\boxed{TRUE}$ / FALSE) 2) $(\sim p \wedge q) \Leftrightarrow (p \wedge \sim q)$ (TRUE / $\boxed{FALSE}$)

3) $\sim p \wedge q \Leftrightarrow \sim (q \to p)$ ($\boxed{TRUE}$ / FALSE) 4) $(\sim q \to \sim p) \Leftrightarrow (p \to q)$ ($\boxed{TRUE}$ / FALSE)

[*4 marks*] 3. (a) Complete the truth table of the compound statement with the the disjunctive normal form

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r).$$

p	q	r	statement
T	T	T	**T**
T	T	F	T
T	F	T	**T**
T	F	F	T
F	T	T	**T**
F	T	F	T
F	F	T	**F**
F	F	F	T

[*3 marks*] (b) Simplify the statement in part (a) to an **equivalent** compound **conditional** statement that has the simplest expression in which each of p, q and r appears **only once**.

Answer: $\sim (\sim p \wedge \sim q \wedge r) \Leftrightarrow p \vee q \vee \sim r \Leftrightarrow r \to (p \vee q)$

[*7 marks*] 4. Let the set of positive integers $\mathbb{N}$ be the universal set (universe of discourse) for N. Let the set of real numbers $\mathbb{R}$ be the universal set for x. Let $p(N, x)$ denote the mathematical statement

$$f(x) > N.$$

(a) Convert the following quantified logical statement **S** into a **mathematical** statement.

$$\mathbf{S}: \quad \forall N \ [\ \exists x \ [p(N, x)] \ ]$$

The mathematical statement that it represnts is :

Answer: *For each positive integer N, there exists some real number x such that $f(x) > N$.*

(b) Write a logical statement that is equivalent to the **negation** of the quantified statement **S**, in which the expression $\sim p(N, x)$ appears.

Answer: $\sim \ \forall N \ [\ \exists x \ [p(N, x)] \ ] \ \Leftrightarrow \ \exists N \ [\sim \exists x \ [p(N, x)] \ ] \ \Leftrightarrow \ \exists N \ [\ \forall x \ [\ \sim p(N, x)] \ ]$

(c) Translate the final answer in part (b) into a **mathematical** statement.

Answer: *There exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}$, $f(x) \leq N$.*

[*10 marks*] 5. Let S be the following statement in set theory:

$$\textit{If } A \subset B, \textit{ then } (A \cup B) - (A \cap B) \neq \emptyset.$$

Complete the following sentences.

(a) S is a ___ $\boxed{\textit{TRUE}}$ ___ (TRUE or FALSE) statement.

(b) The **converse** of the statement S is:

_____ **Answer**: *If* $(A \cup B) - (A \cap B) \neq \emptyset$, *then* $A \subset B$. _______ .

(c) The **converse** of the statement S is a ___ $\boxed{\textit{FALSE}}$ ___ (TRUE or FALSE) statement.

(d) The **contrapositive** of S is a ___ $\boxed{\textit{TRUE}}$ ___ (TRUE or FALSE) statement, which is

_____ **Answer**: *If* $(A \cup B) - (A \cap B) = \emptyset$, *then* $A \not\subset B$. _______ .

(e) The **negation** of the statement S is a ___ $\boxed{\textit{FALSE}}$ ___ (TRUE or FALSE) statement, which ___ $\boxed{\textit{IS}}$ ___ ___ (IS or IS NOT) equivalent to the statement:

$$\textit{There are sets } A \textit{ and } B \textit{ such that } A \subset B \textit{ and } B - A = \emptyset.$$

(f) The fact that

$$\mathbb{N} \subset \mathbb{Z} \textit{ and so } \mathbb{Z} - \mathbb{N} \neq \emptyset$$

is _____ $\boxed{\textit{an example}}$ _____ (an **example** or a **counterexample**) for the statement of S.

(g) The example in part (f) alone _____ $\boxed{\textit{DOES NOT PROVE}}$ _____ (PROVES or DOES NOT PROVE) the statement S.

(h) The **converse** statement of the **contrapositive** of S ___ $\boxed{\textit{HAS}}$ ___ (HAS or HAS NO) **conterexamples**.

[*8 marks*] 6. (a) Write a **constructive proof** of the statement:

There exists an integer n *such that for any integer* m, *either* $mn \geq m-n$ *or* $mn \geq n-m$.

Proof 1. Let $n = 0$. It follows that, for any $m \in \mathbb{Z}$, $mn = m \cdot 0 = 0$.
If $m \geq 0$, then $-m \leq 0$, and hence $mn = 0 \geq -m = 0 - m = n - m$.
If $m < 0$, $mn = 0 \geq m = m - 0 = m - n$. $\square$

[An alternative argument: Since $0 - m = -(m - 0)$, one of the integers $0 - m$ and $m - 0$ must be ≥ 0 and the other ≤ 0. Therefore either $mn = 0 \geq m - n$ or $mn = 0 \geq n - m$.]

Proof 2. Let $n = 1$. It follows that $mn = m \cdot 1 = m$ for any $m \in \mathbb{Z}$.
Therefore $mn = m \geq m - 1 = m - n$ for any $m \in \mathbb{Z}$.
Thus it is true that, for any $m \in \mathbb{Z}$, either $mn \geq m - n$ or $mn \geq n - m$. $\square$

(b) Use the method of **proof by contradiction** to write a **nonconstructive proof** of the same statement in part (a).
[Hint: Assume first that for any integer n, there exists an integer m such the the statement "*either* $mn \geq m - n$ *or* $mn \geq n - m$" is false.]

Proof. Assume that for any integer n, there exists an integer m such that *both* $mn < m - n$ *and* $mn < n - m$.
For $n = 0$, in particular, this would mean that there exists $m \in \mathbb{Z}$ such that both $0 = mn < m - n = m$ and $0 = mn < n - m = -m$.
This is absurd (since it would mean that $m > 0 > m$).
The statement is true. $\square$

[An alternative argument: For $n = 1$, in particular, this would mean that there exists $m \in \mathbb{Z}$ such that both $m = mn < m - n = m - 1$ and $m = mn < n - m = 1 - m$.
It would follow from $m < m - 1$ that $0 < -1$, absurd.]

7. **Prove** or **disprove** the following statements. Write **all necessary steps**. In each case, indicate first **which method** from the following four methods you are using:

   1) *Direct proof of implication*

   2) *Proof of the contrapositive of implication*

   3) *Indirect proof of implication by contradiction*

   4) *Disproof of implication by counterexample*

[*5 marks*] (a) Let $n \in \mathbb{Z}$.

$$\textit{If } n \textit{ is odd, then } n^2 + n + 3 \textit{ is odd.}$$

The method: ___ **(1)** ___

Proof. Suppose that n is odd; that is, $n = 2k + 1$, where $k \in \mathbb{Z}$.
Then $n^2 + n + 3 = (2k + 1)^2 + (2k + 1) + 3 = 4k^2 + 4k + 1 + 2k + 4 = 2(2k^2 + 2k + 2) + 1$ is odd because $2k^2 + 2k + 2 \in \mathbb{Z}$.

[*4 marks*] (b) Let a and b be real numbers.

$$\textit{If } a > 0 \textit{ and } b > 0, \textit{ then } \sqrt{ab} \geq \frac{1}{2}(a + b).$$

The method: ___ **(4)** ___

Solution. Let $a = 2 > 0$ and $b = 8 > 0$.
$\sqrt{ab} = \sqrt{2 \times 8} = 4$ while $\frac{1}{2}(a + b) = \frac{1}{2}(2 + 8) = 5$.
In this case $\sqrt{ab} \not\geq \frac{1}{2}(a + b)$.

[*5 marks*] (c) Let a and b be real numbers.

$$\textit{If } \sqrt{(a-b)^2} = b - a, \textit{ then } a \leq b.$$

The method: __ **(1)** __

Proof. Let $\sqrt{(a-b)^2} = b - a$.
Since $\sqrt{(a-b)^2} = |a - b|$, we have $|a - b| = b - a = -(a - b)$.
This implies that $a - b \leq 0$. Therefore $a \leq b$. □

[*6 marks*] (d) Let A and B be subsets of a universal set U.

$$\textit{If } A^c \cap B = A \cup B^c, \textit{ then } A = \emptyset.$$

The method: __**3**__

Proof. Assume that $A^c \cap B = A \cup B^c$ and $A \neq \emptyset$.
Then there exists $x \in A$.
Note that $A \subseteq A \cup B^c = A^c \cap B \subseteq A^c$.
So $x \in A$ implies that $x \in A^c$ as well; that is $A \cap A^c \neq \emptyset$.
The absurdness shows that the statement is true. □

[*8 marks*] 8. Prove that for any sets A, B and C,

$$A \cap (B - C) \subseteq (A \cap B) - (A \cap C).$$

Proof.

Let $x \in A \cap (B - C)$.

$$\begin{aligned}
& x \in A \cap (B - C) \\
\Longrightarrow\ & x \in A \text{ and } x \in (B - C) \\
\Longrightarrow\ & x \in A \text{ and } (x \in B \text{ and } x \notin C) \\
\Longrightarrow\ & (x \in A \text{ and } x \in B) \text{ and } x \notin C \\
\Longrightarrow\ & x \in A \cap B \text{ and } x \notin A \cap C \qquad \text{(because } A \cap C \subseteq C\text{)} \\
\Longrightarrow\ & x \in (A \cap B) - (A \cap C)
\end{aligned}$$

Therefore, $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$.

[*8 marks*] 9. Use the Principle of **Mathematical Induction** to prove that, for all natural numbers $n \in \mathbb{N}$, the number $n(n+1)(n+2)$ is divisible by 3; that is, $n(n+1)(n+2) = 3m$ for some $m \in \mathbb{Z}$.

Proof.

It is true for $n = 1$ since $1(1+1)(1+2) = 6 = 3 \times 2$, a multiple of 3.

Assume that it is true for some $n = k$ with $k \geq 1$; that is,

$$k(k+1)(k+2) = 3m \quad \text{for some integer } m.$$

It then follows that if $n = k + 1$, we have

$$\begin{aligned}
& (k+1)[(k+1)+1][(k+1)+2] \\
&= (k+1)(k+2)(k+3) \\
&= (k+3)(k+1)(k+2) \\
&= k(k+1)(k+2) + 3(k+1)(k+2) \\
&= 3m + 3(k+1)(k+2) \\
&= 3[m + (k+1)(k+2)],
\end{aligned}$$

where $m + (k+1)(k+2)$ is an integer.

Therefore the number $n(n+1)(n+2)$ is a multiple of 3 for any positive integer n by induction.