Midterm Test 1

MAT1341E, Fall 2018

Total = 60 marks

$12+7+20+15$
$=54$

Part I. Multiple-choice Questions ($4 \times 4 = 16$ marks)

Write your answer to multiple-choice questions in the following boxes:

1.	2.	3.	4.
A	B	D	F

1. Find parametric equations of the line containing the point (2, 1, 0) and parallel to two planes $x + 3y - z = 0$ and $3x + 4y - 5z = 1$.

(A) $x = 2 + 11t, y = 1 + 2t, z = 5t.$
(B) $x = -2 + 5t, y = 1 - 10t, z = -5t.$
(C) $x = -2t, y = t, z = 0.$
(D) $x = 2 + 11t, y = 1 - 2t, z = 5t.$
(E) $x = -2 + 11t, y = 1 + 2t, z = -3t.$
(F) $x = 2 + 11t, y = 1 - 2t, z = 1 + 5t.$

2. If the coefficient matrix A of a homogeneous system with 18 variables and 16 equations is known to have rank 11, how many parameters are there in the general solution?

(A) 19; (B) 7; (C) 9; (D) 3; (E) 10; (F) 16.

18

16

$R = 11$

3. Given vectors $\mathbf{u} = (1, -2, 3)$, $\mathbf{v} = (0, -1, 2)$, and $\mathbf{w} = (1, 2, -1)$, find the cosine of the angle θ between $\mathbf{v} \times \mathbf{w}$ and $\mathbf{u} \times \mathbf{v}$.

(A) $\frac{2}{21}$; (B) $\sqrt{\frac{2}{21}}$; (C) $-\frac{1}{\sqrt{7}}$; (D) $-\frac{1}{\sqrt{21}}$; (E) $-\frac{1}{21}$; (F) $\frac{2}{\sqrt{7}}$.

$-1 \ 2 \ 0 \ -1$ $\quad$ $-2 \ 3 \ 1 \ -2$

$2 \ -1 \ 1 \ 2$ $\quad$ $-1 \ 2 \ 0 \ -1$

$v \times w = (-3, 2, 1)$ $\quad$ $u \times v = (-1, -2, -1)$

$\frac{3+(-4)-1}{\sqrt{6}} = \frac{-2}{\sqrt{6}}$

$\frac{-2}{\sqrt{14}\sqrt{6}} = \frac{-2}{\sqrt{84}} = \frac{-1}{\sqrt{21}}$

$\sqrt{9+4+1} = \sqrt{14}$

4. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, and B is a 3×5 matrix, then the second row of the matrix AB is

(A) the same as the first row of B;
(B) the sum of the first and the second rows of B;
(C) the sum of the first, the second and the third rows of B;
(D) the sum of the first and the third rows of B;
(E) the same as the third row of B;
(F) the sum of the second and the third rows of B.

Part II. True / False Questions ($4 \times 2 = 8$ marks)

5. Mark whether each of the following statements is TRUE (T) or FALSE (F) in the box.

(i) It is possible that a system of linear equations has exactly 4 solutions. F

(ii) A homogeneous system of linear equations can be inconsistent. F

(iii) There exists a linear system of three equations such that its coefficient matrix has rank 4. T

(iv) If a system has 5 equations and 3 variables, then this system cannot have a solution. F

6. Mark whether each of the following statements is TRUE (T) or FALSE (F) in the box.

(i) For any three 3×3 matrices A, B and C, we have $ABC = CBA$. F

(ii) For any scalar c and any two square matrices A and B, $c(AB) = (cA)(cB)$. T ✗

(iii) Multiplying a 1×3 matrix A by a 3×4 matrix B, the result AB is a 1×4 matrix. T

(iv) For any two 3×3 matrices A and B, $(A - B)^2 = A^2 - 2AB + B^2$. F

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Part III. Detailed-Answer Questions (36 marks)

You must show all necessary steps to convince the marker that you know how the final answer is reached.

7. (20 marks) Suppose e and f are real numbers. Consider the linear system with variables x, y and z:

$$2x - 2y - ez = f,$$
$$2x + y + z = 0,$$
$$x + z = -1.$$

(a) (10 marks) Let A be the coefficient matrix, and let B be the augmented matrix of this system. Then rank A and rank B depend on the values of e and f. Give all possible values of rank A and rank B, and the corresponding values of e and f. Justify your answer.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & -2 & -e & f \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & (-e-2) & (f+2) \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & (-e-4) & (f+6) \end{array}\right]$$

Rank $A = 2$ if $e = -4$ Rank $B = 2$ if $e = -4$ and $f = -6$

$= 3$ if $e \neq -4$ $= 3$ if $e \neq -4$ or $f \neq -6$

2 if bottom row is all 0's

3 otherwise.

$-e-4=0$ $f+6=0$

$e=-4$ $f=-6$

(b) (10 marks) Using the result of part (a), determine, for which value(s) of e and f, does this system have (i) a unique solution, (ii) infinitely many solutions, or (iii) no solution.

i) unique solution if $e \neq -4$ since Rank A = # of columns

ii) infinite solutions if $e = -4$ and $f = -6$ since Rank A < # of columns

iii) no solution if $e = -4$ and $f \neq -6$ since $0 = \alpha$ where $\alpha \neq 0$ inconsistent

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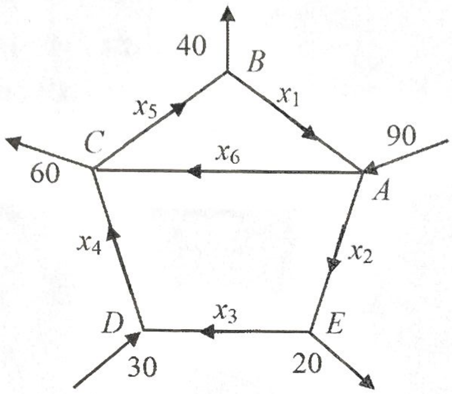
8. (16 marks) Consider a network of streets with intersections A, B, C, D, and E as shown below. The arrows indicate the direction of traffic flow along these one-way streets, and the numbers refer to the exact number of cars observed to enter or leave A, B, C, D, and E during one minute. Each variable x_i denotes the unknown number of cars which passes along the indicated streets during the same period.



(a) (6 marks) Write a system of linear equations that shows the balance condition for each intersection, together with all the constrains on the variables x_i, $i = 1, 2, \ldots, 6$.

A: $x_1 + 90 = x_2 + x_6$

B: $x_5 = x_1 + 40$

C: $x_4 + x_6 = x_5 + 60$

D: $x_3 + 30 = x_4$

E: $x_2 = x_3 + 20$

$x_i \geq 0$, $i = 1, 2, \ldots, 6$

x_i is an integer

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(continue on next page)

(b) (6 marks) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$A = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 50 \\ 0 & 0 & 1 & 0 & -1 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

Give the general solution of this system.

$x_1 = x_5 - 40$

$x_2 = x_5 - x_6 + 50$

$x_3 = x_5 - x_6 + 30$

$x_4 = x_5 - x_6 + 60$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x_5 + \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \end{bmatrix} x_6 + \begin{bmatrix} -40 \\ 50 \\ 30 \\ 60 \end{bmatrix} \middle| \; x_5, x_6 \in \mathbb{R} \right\}$$

solution is a vector in $\mathbb{R}^6$!

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(c) (4 marks) Use your result from part (b), if the road from E to D was closed due to roadwork, find the minimum flow along the road from A to C. Justify your answer!

if $x_3 = 0$,

then $0 = x_5 - x_6 + 30$

$\boxed{x_5 = x_6 - 30}$

↳ above formulas:

$x_1 = x_6 - 70$

$x_2 = 20$

$x_3 = 0$

$x_4 = 30$

since $x_1 \geq 0$

$\boxed{x_6 \geq 70}$

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