MAT2122 Multivariable Calculus (Fall 2018)

Assignment 1 solutions (95 points)

1 (7 points). Find an equation of the line passing through the points $P = (-5, 0, 4)$ and $Q = (6, -3, 2)$. What is the distance between P and Q?

Solution: By the general formula this equation is

$$\mathbf{v}(t) = \overrightarrow{OP} + t \cdot \overrightarrow{PQ} \ , \quad ✓$$

where

$$\overrightarrow{OP} = (-5, 0, 4) \ ,$$

and

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (6, -3, 2) - (-5, 0, 4) = (11, -3, -2) \ , \quad ✓$$

whence

$$\mathbf{v}(t) = (-5, 0, 4) + t \cdot (11, -3, -2) \ , \quad ✓$$

or, in the coordinate form,

$$\begin{cases} x(t) & = -5 + 11t \ , \\ y(t) & = -3t \ , \\ z(t) & = 4 - 2t \ , \end{cases} \quad ✓$$

where $\mathbf{v}(t) = (x(t), y(t), z(t))$.

The distance between P and Q is the length of the vector $\overrightarrow{PQ}$, ✓ i.e.,

$$\sqrt{11^2 + (-3)^2 + (-2)^2} = \sqrt{134} \ . \quad ✓$$

Absence of minor mistakes. ✓

2 (7 points). A triangle has vertices $A = (0, 0, 0)$, $B = (1, 1, 1)$, and $C = (0, 2, 3)$. Find its area.

Solution: The sides of the triangle issued from the vertex $(0, 0, 0)$ are the vectors $(1, 1, 1)$ and $(0, 2, 3)$. ✓ Therefore, the area A of this triangle is equal to one half of the area of the parallelogram spanned by $(1, 1, 1)$ and $(0, 2, 3)$, ✓ i.e.,

$$A = \frac{1}{2} \big\| (1, 1, 1) \times (0, 2, 3) \big\| \ . \quad ✓$$

The vector product in question is

$$(1, 1, 1) \times (0, 2, 3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} = (1, -3, 2) \ , \quad ✓$$

so that its length is

$$\sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{1 + 9 + 4} = \sqrt{14} \ , \quad ✓$$

whence

$$A = \frac{\sqrt{14}}{2} \ . \quad ✓$$

Absence of minor mistakes. ✓

3 (5 points). Find the volume of the parallelepiped spanned by the vectors $\mathbf{i}, 3\mathbf{j}-\mathbf{k}$, and $4\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

Solution: The volume of the parallelepiped spanned by 3 vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is the absolute value of the triple product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ (which does not depend on the order of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$), and coincides with the abolute value of the determinant of the matrix whose rows are the coordinates of the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. ✓ In our case $\mathbf{u}=\mathbf{i}, \mathbf{v}=3\mathbf{j}-\mathbf{k}$, and $\mathbf{w}=4\mathbf{i}+2\mathbf{j}-\mathbf{k}$, so that the above matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 4 & 2 & -1 \end{pmatrix} \, , \quad ✓$$

and its determinant is $3(-1)-(-1)2=-3+2=-1$, ✓ whence the volume is $|-1|=1$. ✓

Absence of minor mistakes. ✓

4 (10 points). Find an equation of the plane that passes through the points

(a) (0, 0, 0), (2, 0, -1), and (0, 4, -3);
(b) (1, 2, 0), (0, 1, -2), and (4, 0, 1);
(c) (2, -1, 3), (0, 0, 5), and (5, 7, -1).

Solution: **(a)** Let $A=(0,0,0), B=(2,0,-1), C=(0,4,-3)$. Then the plane (ABC) contains the vectors

$$\mathbf{u}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(2,0,-1)-(0,0,0)=(2,0,-1)$$

and

$$\mathbf{v}=\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(0,4,-3)-(0,0,0)=(0,4,-3)\ ,$$

so that it is orthogonal to the vector

$$\mathbf{w}=\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 4 & -3 \end{vmatrix}=\mathbf{i}\begin{vmatrix} 0 & -1 \\ 4 & -3 \end{vmatrix}-\mathbf{j}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix}+\mathbf{k}\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}=4\mathbf{i}+6\mathbf{j}+8\mathbf{k}=(4,6,8)\ . \quad ✓$$

Thus, a point $P=(x,y,z)$ belongs to the plane (ABC) if and only if the vector

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=(x,y,z)$$

is orthogonal to $\mathbf{w}$, or, equivalently, to $\mathbf{w}'=(2,3,4)$, i.e., if and only if $\overrightarrow{AP}\cdot\mathbf{w}'=0$. ✓ Therefore,

$$2x+3y+4z=0$$

is an equation of the plane (ABC). ✓

(b) For $A=(1,2,0), B=(0,1,-2)$, and $C=(4,0,1)$ the plane contains the vectors

$$\mathbf{u}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(0,1,-2)-(1,2,0)=(-1,-1,-2)$$

and

$$\mathbf{v}=\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(4,0,1)-(1,2,0)=(3,-2,1)\ ,$$

so that the plane is orthogonal to the vector

$$\mathbf{w}=\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -2 \\ 3 & -2 & 1 \end{vmatrix}=\mathbf{i}\begin{vmatrix} -1 & -2 \\ -2 & 1 \end{vmatrix}-\mathbf{j}\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix}+\mathbf{k}\begin{vmatrix} -1 & -1 \\ 3 & -2 \end{vmatrix}=-5\mathbf{i}-5\mathbf{j}+5\mathbf{k}=(-5,-5,5)\ . \quad ✓$$

Thus, a point $P=(x,y,z)$ belongs to the plane (ABC) if and only if the vector

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=(x-1,y-2,z)$$

is orthogonal to $\mathbf{w}$, or, equivalently, to $\mathbf{w}' = (1, 1, -1)$, i.e., if and only if $\overrightarrow{AP} \cdot \mathbf{w}' = 0$. ✓ Therefore,

$$(x-1) + (y-2) - 5z = x + y - z - 3 = 0$$

is an equation of the plane (ABC). ✓

(c) For $A = (2, -1, 3), B = (0, 0, 5)$, and $C = (5, 7, -1)$ the plane contains the vectors

$$\mathbf{u} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (0, 0, 5) - (2, -1, 3) = (-2, 1, 2)$$

and

$$\mathbf{v} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (5, 7, -1) - (2, -1, 3) = (3, 8, -4)\ ,$$

so that the plane is orthogonal to the vector

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ 3 & 8 & -4 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 2 \\ 8 & -4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & 2 \\ 3 & -4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 1 \\ 3 & 8 \end{vmatrix} = -20\mathbf{i} - 2\mathbf{j} - 19\mathbf{k} = (-20, -2, -19)\ .$$ ✓

Thus, a point $P = (x, y, z)$ belongs to the plane (ABC) if and only if the vector

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x-2, y+1, z-3)$$

is orthogonal to $\mathbf{w}$, or, equivalently, to $\mathbf{w}' = (20, 2, 19)$, i.e., if and only if $\overrightarrow{AP} \cdot \mathbf{w}' = 0$. ✓ Therefore,

$$20(x-2) + 2(y+1) + 19(z-3) = 20x + 2y + 19z - 95 = 0$$

is an equation of the plane (ABC). ✓

Absence of minor mistakes. ✓

5 (4 points). Find the first order partial derivatives of the function $z(x, y) = \sqrt{a^2 - x^2 - y^2}$ at the points $(0, 0)$ and $(a/2, a/2)$.

Solution:

$$\frac{\partial z}{\partial x}(x, y) = \frac{\partial \sqrt{a^2 - x^2 - y^2}}{\partial x} = -\frac{2x}{2\sqrt{a^2 - x^2 - y^2}} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}\ .$$ ✓

In precisely the same way

$$\frac{\partial z}{\partial y}(x, y) = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}\ ,$$ ✓

whence

$$\frac{\partial z}{\partial x}(0, 0) = \frac{\partial z}{\partial y}(0, 0) = 0\ ,$$

$$\frac{\partial z}{\partial x}(a/2, a/2) = \frac{\partial z}{\partial y}(a/2, a/2) = -\frac{a/2}{\sqrt{a^2 - (a/2)^2 - (a/2)^2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}\ .$$ ✓

Absence of minor mistakes. ✓

6 (4 points). Find an equation of the plane tangent to the graph of the function $f(x, y) = x \cos x \cos y$ at the point $(0, \pi)$.

Solution: The partial derivatives of the function f are

$$f'_x(x, y) = (x \cos x \cos y)'_x = \cos y (x \cos x)'_x = \cos y (\cos x - x \sin x)$$

and

$$f'_y(x,y) = (x\cos x\cos y)'_y = -x\cos x\sin y \ , \quad ✓$$

so that for the point $(x_0, y_0) = (0, \pi)$

$$f(x_0,y_0) = 0 \ , \qquad f'_x(x_0,y_0) = -1 \ , \qquad f'_y(x_0,y_0) = 0 \ . \quad ✓$$

Therefore, the tangent plane equation is

$$z = f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0) = -x \ . \quad ✓$$

Absence of minor mistakes. ✓

7 (6 points). A particle following the path $\mathbf{c}(t) = (e^t, e^{-t}, \cos t)$ flies off on a tangent at $t = t_0 = 1$. Compute the position of the particle at time $t_1 = 2$.

Solution: The equation of the considered tangent line $\mathbf{l}$ is

$$\mathbf{l}(t) - \mathbf{c}(t_0) = (t-t_0)\dot{\mathbf{c}}(t_0) \ ,$$

or

$$\mathbf{l}(t) = \mathbf{c}(t_0) + (t-t_0)\dot{\mathbf{c}}(t_0) \ . \quad ✓$$

In our case $t_0 = 1$, and

$$\mathbf{c}(t_0) = \mathbf{c}(1) = (e, e^{-1}, \cos 1) \ . \quad ✓$$

On the other hand,

$$\dot{\mathbf{c}}(t) = (e^t, -e^{-t}, -\sin t) \ , \quad ✓$$

so that

$$\dot{\mathbf{c}}(t_0) = \dot{\mathbf{c}}(1) = (e, -e^{-1}, -\sin 1) \ , \quad ✓$$

whence

$$\mathbf{l}(2) = (e, e^{-1}, \cos 1) + (e, -e^{-1}, -\sin 1) = (2e, 0, \cos 1 - \sin 1) \ . \quad ✓$$

Absence of minor mistakes. ✓

8 (5 points). Compute the matrix of partial derivatives of the functions

(a) $f(x,y) = (e^x, \sin xy)$,
(b) $f(x,y,z) = (x-y, y+z)$,
(c) $f(x,y) = (x+y, x-y, xy)$,
(d) $f(x,y,z) = (x+z, y-5z, x-y)$.

Solution: **(a)**

$$\mathbf{D}f(x,y) = \begin{pmatrix} \dfrac{\partial\, e^x}{\partial x} & \dfrac{\partial\, e^x}{\partial y} \\[2ex] \dfrac{\partial(\sin xy)}{\partial x} & \dfrac{\partial(\sin xy)}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ y\cos xy & x\cos xy \end{pmatrix} \quad ✓$$

(b)

$$\mathbf{D}f(x,y,z) = \begin{pmatrix} \dfrac{\partial(x-y)}{\partial x} & \dfrac{\partial(x-y)}{\partial y} & \dfrac{\partial(x-y)}{\partial z} \\ \\ \dfrac{\partial(y+z)}{\partial x} & \dfrac{\partial(y+z)}{\partial y} & \dfrac{\partial(y+z)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ \\ 0 & 1 & 1 \end{pmatrix}$$ ✓

(c)

$$\mathbf{D}f(x,y) = \begin{pmatrix} \dfrac{\partial(x+y)}{\partial x} & \dfrac{\partial(x+y)}{\partial y} \\ \\ \dfrac{\partial(x-y)}{\partial x} & \dfrac{\partial(x-y)}{\partial y} \\ \\ \dfrac{\partial(xy)}{\partial x} & \dfrac{\partial(xy)}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \\ 1 & -1 \\ \\ y & x \end{pmatrix}$$ ✓

(d)

$$\mathbf{D}f(x,y,z) = \begin{pmatrix} \dfrac{\partial(x+z)}{\partial x} & \dfrac{\partial(x+z)}{\partial y} & \dfrac{\partial(x+z)}{\partial z} \\ \\ \dfrac{\partial(y-5z)}{\partial x} & \dfrac{\partial(y-5z)}{\partial y} & \dfrac{\partial(y-5z)}{\partial z} \\ \\ \dfrac{\partial(x-y)}{\partial x} & \dfrac{\partial(x-y)}{\partial y} & \dfrac{\partial(x-y)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ \\ 0 & 1 & -5 \\ \\ 1 & -1 & 0 \end{pmatrix}$$ ✓

Absence of minor mistakes. ✓

9 (7 points). Let $g(u,v) = (e^u, u + sinv)$ and $f(x,y,z) = (xy, yz)$. Compute $\mathbf{D}(g \circ f)$ at $(0,1,0)$ (a) by using the chain rule; (b) by a direct calculation using an explicit form of the function $g \circ f$.

Solution: The derivatives of the functions g and f are, respectively

$$\mathbf{D}g(u,v) = \begin{pmatrix} \frac{\partial e^u}{\partial u} & \frac{\partial e^u}{\partial v} \\ \\ \frac{\partial(u+\sin v)}{\partial u} & \frac{\partial(u+\sin v)}{\partial v} \end{pmatrix} = \begin{pmatrix} e^u & 0 \\ \\ 1 & \cos v \end{pmatrix}$$ ✓

and

$$\mathbf{D}f(x,y,z) = \begin{pmatrix} \frac{\partial xy}{\partial x} & \frac{\partial xy}{\partial y} & \frac{\partial xy}{\partial z} \\ \\ \frac{\partial yz}{\partial x} & \frac{\partial yz}{\partial y} & \frac{\partial yz}{\partial z} \end{pmatrix} = \begin{pmatrix} y & x & 0 \\ \\ 0 & z & y \end{pmatrix}.$$ ✓

The evaluation of f at the point $(0,1,0)$ and of g at the point

$$f(0,1,0) = (0,0)$$ ✓

gives

$$\mathbf{D}f(0,1,0) = \begin{pmatrix} 1 & 0 & 0 \\ \\ 0 & 0 & 1 \end{pmatrix},$$ ✓

and

$$\mathbf{D}g(0,0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \quad \checkmark$$

whence

$$\mathbf{D}(g \circ f)(0,1,0) = \mathbf{D}g(0,0)\mathbf{D}f(0,1,0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} . \quad \checkmark$$

Absence of minor mistakes. ✓

10 (7 points). Compute the second partial derivatives and verify the theorem on equality of mixed derivatives for the function $f(x,y) = e^{-xy^2} + x^4y^3$.

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -y^2e^{-xy^2} + 4x^3y^3 , \quad \checkmark \\
\frac{\partial^2 f}{\partial x^2} &= y^4e^{-xy^2} + 12x^2y^3 , \quad \checkmark \\
\frac{\partial f}{\partial y} &= -2xye^{-xy^2} + 3x^4y^2 , \quad \checkmark \\
\frac{\partial^2 f}{\partial y^2} &= -2xe^{-xy^2} + 4x^2y^2e^{-xy^2} + 6x^4y , \quad \checkmark \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(-y^2e^{-xy^2} + 4x^3y^3\right) = -2ye^{-xy^2} + 2xy^3e^{-xy^2} + 12x^3y^2 , \quad \checkmark \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial x}\left(-2xye^{-xy^2} + 3x^4y^2\right) = -2ye^{-xy^2} + 2xy^3e^{-xy^2} + 12x^3y^2 \quad \checkmark \quad .
\end{aligned}$$

Absence of minor mistakes. ✓

11 (6 points). Find all second order partial derivatives of the function $z(x,y) = x^2y^2e^{2xy}$.

Solution: The first derivatives of f are

$$f'_x(x,y) = 2xy^2e^{2xy} + x^2y^2 \cdot 2ye^{2xy} = 2xy^2(1+xy)e^{2xy} , \quad \checkmark$$

and, in the same way (since f does not change when one switches x and y),

$$f'_y(x,y) = 2x^2y(1+xy)e^{2xy} . \quad \checkmark$$

Therefore,

$$\begin{aligned}
f''_{xx}(x,y) &= 2y^2(1+xy)e^{2xy} + 2xy^3e^{2xy} + 4xy^3(1+xy)e^{2xy} \\
&= 2y^2(1+4xy+2x^2y^2)e^{2xy} , \quad \checkmark
\end{aligned}$$

and in the same way (see above)

$$f''_{yy}(x,y) = 2x^2(1+4xy+2x^2y^2)e^{2xy} . \quad \checkmark$$

Finally, the mixed second derivative is

$$\begin{aligned} f''_{xy}(x,y) &= 4xy(1+xy)e^{2xy} + 2x^2y^2e^{2xy} + 4x^2y^2(1+xy)e^{2xy} \\ &= 2xy(2+5xy+2x^2y^2)e^{2xy}\ . \checkmark \end{aligned}$$

Absence of minor mistakes. ✓

12 (8 points). Find the second order Taylor approximation for the function $f(x,y) = e^{-x^2-y^2}\cos(xy)$ at the point $(0,0)$.

Solution: The partial derivatives of the function f are

$$\begin{aligned} \frac{\partial f}{\partial x} &= -2xe^{-x^2-y^2}\cos(xy) - ye^{-x^2-y^2}\sin(xy)\ , \checkmark \\ \frac{\partial f}{\partial y} &= -2ye^{-x^2-y^2}\cos(xy) - xe^{-x^2-y^2}\sin(xy)\ , \checkmark \\ \frac{\partial^2 f}{\partial x^2} &= -2e^{-x^2-y^2}\cos(xy) + 4x^2e^{-x^2-y^2}\cos(xy) + 2xye^{-x^2-y^2}\sin(xy) \\ &\quad + 2xye^{-x^2-y^2}\sin(xy) - y^2e^{-x^2-y^2}\cos(xy)\ , \checkmark \\ \frac{\partial^2 f}{\partial x\partial y} &= 4xye^{-x^2-y^2}\cos(xy) + 2x^2e^{-x^2-y^2}\sin(xy) - e^{-x^2-y^2}\sin(xy) \\ &\quad + 2y^2e^{-x^2-y^2}\sin(xy) - xye^{-x^2-y^2}\cos(xy)\ , \checkmark \\ \frac{\partial^2 f}{\partial y^2} &= -2e^{-x^2-y^2}\cos(xy) + 4y^2e^{-x^2-y^2}\cos(xy) + 2xye^{-x^2-y^2}\sin(xy) \\ &\quad + 2xye^{-x^2-y^2}\sin(xy) - x^2e^{-x^2-y^2}\cos(xy)\ , \checkmark \end{aligned}$$

so that for the point $(x_0,y_0) = (0,0)$

$$\begin{aligned} f(x_0,y_0) &= 1\ , \\ \frac{\partial f}{\partial x}(x_0,y_0) &= 0\ , \\ \frac{\partial f}{\partial y}(x_0,y_0) &= 0\ , \\ \frac{\partial^2 f}{\partial x^2}(x_0,y_0) &= -2\ , \checkmark \\ \frac{\partial^2 f}{\partial x\partial y}(x_0,y_0) &= 0\ , \\ \frac{\partial^2 f}{\partial y^2}(x_0,y_0) &= -2\ . \end{aligned}$$

Thus, the second order Taylor approximation of the function f at the point (x_0,y_0) is

$$f(x,y) \cong 1 - x^2 - y^2\ . \checkmark$$

Alternatively, this approximation can be found by using the 2nd order Taylor formulas for the functions exp and cos at the neighbourhood of 0:

$$\exp t \cong 1 + t + t^2/2\ , \qquad \cos t \cong 1 - t^2/2\ ,$$

whence by dropping the terms whose order is higher than 2

$$f(x,y) \cong (1-x^2-y^2)\left(1-\frac{x^2y^2}{2}\right) \cong 1-x^2-y^2 \ .$$

Absence of minor mistakes. ✓

13 (8 points). Find the second order Taylor approximation for the function $f(x,y) = x\cos(\pi y) - y\sin(\pi x)$ at the point $(1,2)$.

Solution: The partial derivatives of the function f are

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \cos(\pi y) - \pi y\cos(\pi x)\ , \quad \checkmark\\
\frac{\partial f}{\partial y} &= -\pi x\sin(\pi y) - \sin(\pi x)\ , \quad \checkmark\\
\frac{\partial^2 f}{\partial x^2} &= \pi^2 y\sin(\pi x)\ , \quad \checkmark\\
\frac{\partial^2 f}{\partial x\partial y} &= -\pi\sin(\pi y) - \pi\cos(\pi x)\ , \quad \checkmark\\
\frac{\partial^2 f}{\partial y^2} &= -\pi^2 x\cos(\pi y)\ , \quad \checkmark
\end{aligned}$$

so that for the point $(x_0,y_0) = (1,2)$

$$\begin{aligned}
f(x_0,y_0) &= \cos(2\pi) - 2\sin(\pi) = 1\ ,\\
\frac{\partial f}{\partial x}(x_0,y_0) &= \cos(2\pi) - 2\pi\cos(\pi) = 1+2\pi\ ,\\
\frac{\partial f}{\partial y}(x_0,y_0) &= -\pi\sin(2\pi) - \sin(\pi) = 0\ ,\\
\frac{\partial^2 f}{\partial x^2}(x_0,y_0) &= 2\pi^2\sin(\pi) = 0\ , \qquad \checkmark\\
\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0) &= -\pi\sin(2\pi) - \pi\cos(\pi) = \pi\ ,\\
\frac{\partial^2 f}{\partial y^2}(x_0,y_0) &= -\pi^2\cos(2\pi) = -\pi^2\ .
\end{aligned}$$

Thus, the second order Taylor approximation of the function f at the point (x_0,y_0) is

$$f(x,y) \cong 1 + (1+2\pi)(x-1) + \pi(x-1)(y-2) - \frac{\pi^2}{2}(y-2)^2\ . \quad \checkmark$$

Alternatively, this approximation can be found by using the 2nd order Taylor formulas for the functions sin and cos at the neighbourhood of 0:

$$\sin t \cong t\ , \qquad \cos t \cong 1 - t^2/2\ ,$$

whence by dropping the terms whose order is higher than 2

$$\begin{aligned} f(x,y) &= (1+(x-1))\cos(2\pi+\pi(y-2)) - (2+(y-2))\sin(\pi+\pi(x-1)) \\ &= (1+(x-1))\cos(\pi(y-2)) + (2+(y-2))\sin(\pi(x-1)) \\ &\cong (1+(x-1))\big(1-\frac{\pi^2(y-2)^2}{2}\big) + (2+(y-2))\cdot\pi(x-1) \\ &\cong 1+(1+2\pi)(x-1)+\pi(x-1)(y-2)-\frac{\pi^2}{2}(y-2)^2 \,. \end{aligned}$$

Absence of minor mistakes. ✓

14 (11 points). By using first and second order Taylor's approximations find approximate values of $(2.03+1)^2/(3.98-1)^2$.

Solution: Let

$$f(x,y) = \frac{(x+1)^2}{(y-1)^2} \,.$$

Then we have to find approximate values for $f(2.03, 3.98)$. ✓ Let us first find the values of the function f and its derivatives at the point $(x_0, y_0) = (2, 4)$: ✓

$$\begin{aligned} & & f(x_0,y_0) &= 1\,, \quad \checkmark \\ f'_x &= \frac{2(x+1)}{(y-1)^2}\,, & f'_x(x_0,y_0) &= \frac{2}{3}\,, \quad \checkmark \\ f'_y &= -2\frac{(x+1)^2}{(y-1)^3}\,, & f'_y(x_0,y_0) &= -\frac{2}{3}\,, \quad \checkmark \\ f''_{xx} &= \frac{2}{(y-1)^2}\,, & f''_{xx}(x_0,y_0) &= \frac{2}{9}\,, \quad \checkmark \\ f''_{yy} &= 6\frac{(x+1)^2}{(y-1)^4}\,, & f''_{yy}(x_0,y_0) &= \frac{2}{3}\,, \quad \checkmark \\ f''_{xy} &= -4\frac{(x+1)}{(y-1)^3}\,, & f''_{xy}(x_0,y_0) &= -\frac{4}{9}\,. \quad \checkmark \end{aligned}$$

The first order Taylor approximation is

$$\begin{aligned} \varphi_1 &= f(x_0,y_0) + f'_x(x_0,y_0)\cdot(x-x_0) + f'_y(x_0,y_0)\cdot(y-y_0) \\ &= 1+\frac{2}{3}\cdot 0.03 - \frac{2}{3}\cdot(-0.02) \approx 1.0333\,, \quad \checkmark \end{aligned}$$

and the second order Taylor approximation is

$$\begin{aligned} \varphi_2 &= \varphi_1 + \frac{f''_{xx}(x_0,y_0)}{2}(x-x_0)^2 + \frac{f''_{yy}(x_0,y_0)}{2}(y-y_0)^2 + f''_{xy}(x-x_0)(y-y_0) \\ &= \varphi_1 + \frac{1}{9}\cdot 0.03^2 + \frac{1}{3}\cdot 0.02^2 - \frac{4}{9}\cdot 0.03\cdot(-0.02) \\ &= \varphi_1 + 0.0005 \approx 1.0338\,. \quad \checkmark \end{aligned}$$

Absence of minor mistakes. ✓